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Rotundity of Quotient Spaces in Metric Linear Spaces

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Abstract. In this paper, we discuss the inheritance of strict convexity, uniform convexity and local uniform convexity by the quotient spaces of metric linear spaces. We also show that, as in the case of normed linear spaces, completeness is a three- space property in metric linear spaces as well.

Keywords: Metric linear space, Strict convexity, Uniform convexity, Local uniform convexity, Three-space property.

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1. Introduction

The study of special normed linear spaces, in which closed unit balls are round in the sense that unit spheres include no nontrivial line segments, was initiated independently by Clarkson [6] and Krein (see [3]) and these spaces were called *strictly convex*. Clarkson got interested in the uniform version of this property, and he initiated the study of uniformly convex normed linear spaces. Thereafter, M. M. Day explored strictly convex and uniformly convex normed linear spaces in great depth and published a series of papers on these

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spaces (see [10, page 426]. He used the terms *rotund* and *uniformly rotund* for these spaces. Lovaglia [9] introduced the local version of uniform convexity and called the normed linear spaces with this property as *locally uniformly convex*. Such spaces are strictly in-between strictly convex and uniformly convex spaces. Recently, uniform convexity and local uniform convexity have been introduced and discussed in asymmetric normed spaces (not necessarily metrizable) by Tsar'kov [16], and Alimov and Tsar'kov [5].

The three notions of convexity, namely, strict convexity, uniform convexity and local uniform convexity were extended to more general spaces viz. metric linear spaces by Albinus [4], Ahuja et al. [2],and Narang [11] respectively. Recall that a linear space X equipped with a metric d is called a *metric linear space* if both addition and scalar multiplication are continuous and the metric d is translation invariant. It is easy to see that if $(X, ||.||)$ is a normed linear space, then it is also a metric linear space with the metric function as $d(x, y) = ||x - y||$. On the other hand, there are plenty of examples of metric linear spaces that are not normed linear spaces (see e.g. [12] and [13]). In normed linear spaces, there are equivalent ways of defining strict convexity and uniform convexity (see, e.g. [10]). Sastry and Naidu [14] observed that the equivalent definitions of strict convexity remain equivalent in metric linear spaces as well, but that is not the case for uniform convexity. Therefore, they formulated three different forms of uniform convexity corresponding to their equivalent counterparts in normed linear spaces and called these properties *U.C.I* (which is same as uniform convexity introduced in [2]) ,*U.C.II and U.C. III*. They also studied the interrelationships between these three forms of uniform convexity in [14] and [15]. It is known (see [2], [11]) that every uniformly convex metric linear space is locally uniformly convex and every locally uniformly convex metric linear space is strictly convex but none of the reverse implications hold. Some other known forms of convexities in normed linear spaces lying in-between strict convexity and uniform convexity, 2R- property, compact local uniform convexity and mid-point local uniform convexity have been extended to metric linear spaces by the authors in a forthcoming paper [7].

As in the case of normed linear spaces, it is easy to see that, a metric linear space is strictly convex/U.C.I/U.C.II or U.C.III respectively if and only if each of its subspace is so. Therefore, it is natural to wonder whether the corresponding form of convexity is also inherited by the quotient spaces of the metric linear spaces that are equipped with any of the above mentioned forms of convexities. We may recall that if M is a closed linear subspace of a metric linear space (X, d), then the quotient space X/M is also a metric linear space under the metric $\bar{d}$ defined as

$$\bar{d}(x + M, y + M) = d(x - y, M) = \inf\{d(x - y, m) : m \in M\}.$$

In case of normed linear spaces, Köthe and Day (see [8]) had raised the question: To what extent is the rotundity of a normed linear space E inherited by its quotient space E/L? Klee [8] partially answered the question by showing that in general, rotundity need not be inherited by the quotient spaces of normed linear spaces. Klee [8] proved that if L is a reflexive subspace of a strictly convex normed linear space, then E/L is strictly convex and remarked that in general nothing more can be expected. However, it is known (see [10]) that if L is a proximinal subspace of a strictly convex normed linear space, then E/L is also strictly convex. In Section 2, we extend this result to metric linear spaces.

It is known (see [10, page 455]) that in normed linear spaces, uniform rotundity is always carried over to quotient spaces. In Section 2, we show that if (X, d) is a metric linear space with U.C.I, then every quotient space of X has U.C.I as well, whereas strict convexity, U.C.II, U.C.III and local uniform convexity are inherited by the quotient spaces formed modulo the proximinal subspaces of metric linear spaces. It is not known so far whether these properties are inherited by the quotient space modulo any closed subspace of metric linear space, and we leave it as an open question.

In Section 3, we show that in metric linear spaces, completeness is a three-space property. Recall that a property P defined for metric linear spaces is called a *three-space property* if and only if for any closed subspace M of a metric linear space X, whenever two of the spaces X, M, and X/M have the property P, then the third must also have it. We observe that strict convexity is not a three-space property.

2. Rotundity properties inherited by the quotient spaces

In this section, we discuss the inheritance of the rotundity properties by the quotient spaces of metric linear spaces. We begin with strict convexity.

Definition 2.1. A metric linear space (X, d) is said to be *strictly convex* (or *rotund)* if for any $r > 0$ and $x, y \in X$, $d(x, 0) \leq r$, $d(y, 0) \leq r$ imply $d(\frac{x+y}{2}, 0) < r$ unless $x = y$.

This formulation of strict convexity is due to Ahuja et al. [1]. Vasilev ([17], [18]) named such spaces as *strongly convex* and used them to study approximative properties of sets in metric linear spaces. As mentioned earlier, it is well known (see [8]) that if $(X, ||.||)$ is a strictly convex normed linear space and M is a closed subspace of X, then the quotient space X/M need not be strictly convex, where norm on X/M is defined as $||x + M||_1 = \inf \{||x + m|| : m \in M\}$. But if $(X, ||.||)$ is a strictly convex normed linear space and M is a proximinal subspace of X, then the quotient space X/M is also strictly convex. We show that this result holds in case of metric linear spaces also. We recall that a subset M of a metric space (X, d) is called *proximinal* if

for each element $x \in X$, there is a point in M which is at a minimum distance from M. In other words, for each $x \in X$, there exists $m_0 \in M$ such that $d(x, m_0) = d(x, M)$, where $d(x, M) = \text{Inf}\ \{d(x, m) : m \in M\}$. It is known that a proximinal subset of a metric space is always closed (see [10, page 441]).

Proposition 2.2. *Let (X, d) be a strictly convex metric linear space and M be a proximinal subspace of X, then the quotient space $(X/M, \bar{d})$ is also strictly convex.*

Proof. Let $r > 0$ be any given real number and $x + M \neq y + M \in X/M$ be such that $\bar{d}(x + M, M) \leq r$ and $\bar{d}(y + M, M) \leq r$. Then $d(x, M) \leq r$ and $d(y, M) \leq r$. As M is proximinal, there exist $m_1, m_2 \in M$ such that $d(x, m_1) = d(x, M)$ and $d(y, m_2) = d(y, M)$. Then $d(x, m_1) \leq r$ and $d(y, m_2) \leq r$. By translation invariance of d, we have $d(x - m_1, 0) = d(x, m_1) \leq r$ and $d(y - m_2, 0) = d(y, m_2) \leq r$. As $x + M \neq y + M$, we have $x - m_1 \neq y - m_2$, then by the strict convexity of (X, d), we have $d\left(\frac{x-m_1+y-m_2}{2}, 0\right) < r$, that is, $d\left(\frac{x+y}{2}, \frac{m_1+m_2}{2}\right) < r$. As M is a linear subspace of X, $\frac{m_1+m_2}{2} \in M$, implying that $d\left(\frac{x+y}{2}, M\right) \leq d\left(\frac{x+y}{2}, \frac{m_1+m_2}{2}\right) < r$. Then $\bar{d}\left(\frac{(x+M)+(y+M)}{2}, M\right) = \bar{d}\left(\frac{x+y}{2} + M, M\right) = d\left(\frac{x+y}{2}, M\right) < r$. Thus, $(X/M, \bar{d})$ is strictly convex. $\square$

Before proceeding further, we recall the various forms of uniform convexity in metric linear spaces introduced in [14].

Definition 2.3. (a) **U.C.I**: A metric linear space (X, d) has *U.C.I*, if given $r > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, 0) < r + \delta, d(y, 0) < r + \delta$ and $d(x, y) \geq \epsilon$ imply $d\left(\frac{x+y}{2}, 0\right) < r$.

(b) **U.C.II**: A metric linear space (X, d) has *U.C.II*, if given $r > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, 0) \leq r, d(y, 0) \leq r$ and $d(x, y) \geq \epsilon$ imply $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta$.

(c) **U.C.III**: A metric linear space (X, d) has *U.C.III*, if given $r > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, 0) = r, d(y, 0) = r$ and $d(x, y) \geq \epsilon$ imply $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta$.

We now show that if a metric linear space has U.C.I, then its quotient space by a closed subspace also has U.C.I. We shall be needing the following result in the sequel.

Lemma 2.4. *Let M be a closed linear subspace of a metric linear space (X, d), then*
(a) if $x \in X$, then $d(x, 0) \geq d(x, M)$.
(b) if $x \in X$ and $\epsilon > 0$, then there exists $x' \in X$ such that $x' + M = x + M$ and $d(x', 0) < d(x, M) + \epsilon$.
(c) if $x, y \in X$ are such that $d(x + M, y + M) < \delta$ for some $\delta > 0$, then there is $y' \in X$ such that $(x - y') + M = (x - y) + M$ and $d(x - y', 0) < \delta$.

(d) if $a \in X$ is such that $d(a, M) < \delta$, then there exists $a' \in X$ such that $a' + M = a + M$ and $d(a', 0) < \delta$.

Proof. Part (a) follows from the definition of $d(x, M)$ and the fact that 0 is in M.

For part (b), recall that $d(x, M) = \inf\{d(x, m) : m \in M\}$. Then for given $\epsilon > 0$, as $d(x, M) + \epsilon > d(x, M)$, there exist $m' \in M$ such that $d(x, m') < d(x, M) + \epsilon$. Thus $d(x - m', 0) < d(x, M) + \epsilon$. As $x - m' + M = x + M$, result follows by taking $x' = x - m'$.

For part (c), take $\epsilon = \delta - d(x - y, M) > 0$. Then by (b), there exist $x' \in X$ such that $x' + M = x - y + M$ and $d(x', 0) < d(x - y, M) + \epsilon$, that is, $d(x', 0) < \delta$. Thus $x - y - x' \in M$ and $d(x', 0) < \delta$. If $x - y - x' = m \in M$ and $y' = y + m$, then $x - y' + M = x - y - m + M = x - y + M$ and $d(x - y', 0) = d(x - y - m, 0) = d(x', 0) < \delta$, thus (c) follows.

For part (d), note that $d(0 + M, a + M) = d(a + M, 0 + M) = d(a, M) < \delta$, therefore by taking x as 0 and y as a in part (c), we get an element $a' \in X$ such that $-a' + M = -a + M$ and $d(-a', 0) < \delta$, implying that $d(a', 0) = d(-a', 0) < \delta$ and $a' + M = a + M$. □

Proposition 2.5. *Let (X, d) be a metric linear space having property U.C.I and M a closed subspace of X, then the quotient space $(X/M, \bar{d})$ also has U.C.I.*

Proof. Let $r > 0$ and $\epsilon > 0$ be given. As (X, d) has U.C.I, for the pair (r, ϵ), there exist $\delta > 0$ such that $d(x, 0) < r + \delta, d(y, 0) < r + \delta$ and $d(x, y) \geq \epsilon$ imply $d\left(\frac{x+y}{2}, 0\right) < r$. We show that the same δ works for the pair (r, ϵ) in X/M . Let $\alpha + M, \beta + M \in \frac{X}{M}$ be any elements such that $\bar{d}(\alpha + M, M) < r + \delta$ and $\bar{d}(\beta + M, M) < r + \delta$ and $\bar{d}(\alpha + M, \beta + M) \geq \epsilon$. As $\bar{d}(\alpha + M, M) = d(\alpha, M)$ and $\bar{d}(\beta + M, M) = d(\beta, M)$, by Lemma 2.4 (d), there exist $a, b \in X$ such that $\alpha + M = a + M, \beta + M = b + M$ and $d(a, 0) < r + \delta$ and $d(b, 0) < r + \delta$ and also, $\epsilon \leq \bar{d}(\alpha + M, \beta + M) = \bar{d}(a + M, b + M) = d(a - b, M) \leq d(a - b, 0) = d(a, b)$. So, by the definition of δ, we have $d\left(\frac{a+b}{2}, 0\right) < r$. Then $d\left(\frac{a+b}{2}, M\right) \leq d\left(\frac{a+b}{2}, 0\right) < r$, implying that $\bar{d}\left(\frac{(\alpha+M)+(\beta+M)}{2}, M\right) = \bar{d}\left(\frac{(a+M)+(b+M)}{2}, M\right) = \bar{d}\left(\frac{a+b}{2} + M, M\right) = d\left(\frac{a+b}{2}, M\right) < r$. □

We next show that if M is a proximinal subspace of a metric linear space (X, d) such that X has U.C.II or U.C.III respectively, then the quotient space $(X/M, \bar{d})$ also has U.C.II or U.C.III respectively.

Proposition 2.6. *Let (X, d) be a metric linear space having U.C.II (respectively U.C.III) and M be a proximinal subspace of X, then the quotient space $(X/M, \bar{d})$ also has U.C.II (respectively U.C.III).*

Proof. Since the proofs for both the properties are similar, we only prove it for U.C.II. Let $r > 0$ and $\epsilon > 0$ be given. As (X, d) has U.C.II, there exists $\delta > 0$

such that $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta$, whenever $d(x,0) \leq r, d(y,0) \leq r$ and $d(x,y) \geq \epsilon$. We show that this δ works for the pair (r, ϵ) in the quotient space $\frac{X}{M}$ also. Let $a + M, b + M \in X/M$ be such that $\bar{d}(a + M, M) \leq r, \bar{d}(b + M, M) \leq r$ and $\bar{d}(a + M, b + M) \geq \epsilon$. Then $d(a, M) \leq r, d(b, M) \leq r$ and $d(a - b, M) \geq \epsilon$. As M is proximinal, there exist $m_1, m_2 \in M$ such that $d(a, m_1) = d(a, M)$ and $d(b, m_2) = d(b, M)$, implying that $d(a - m_1, 0) \leq r, d(b - m_2, 0) \leq r$. Also, $\epsilon \leq d(a-b, M) \leq d(a-b, m_1-m_2) = d(a-m_1, b-m_2)$. Then, by the definition of δ, we have $d\left(\frac{(a-m_1)+(b-m_2)}{2}, 0\right) \leq r-\delta$, that is, $d\left(\frac{a+b}{2}, \frac{m_1+m_2}{2}\right) \leq r-\delta$, implying further that $d\left(\frac{a+b}{2}, M\right) \leq d\left(\frac{a+b}{2}, \frac{m_1+m_2}{2}\right) \leq r-\delta$. So, $\bar{d}\left(\frac{(a+M)+(b+M)}{2}, M\right) \leq r - \delta$. □

Definition 2.7. (see [7]) A metric linear space (X, d) is called *locally uniformly convex* or *locally uniformly rotund (LUR)* if given $\epsilon > 0$ and $x \in X$, there exists $\delta(x, \epsilon)$ such that $d\left(\frac{x+y}{2}, 0\right) \leq r - \delta(\epsilon, x)$, whenever $d(x, y) \geq \epsilon$ and $d(y, 0) \leq r$, where $r = d(x, 0)$.

Observe that this property is local version of U.C.II. One can also study the local versions of U.C.I and U.C.III. But we continue with the Definition 2.7 of an LUR space and show that in an LUR metric linear space, quotient space modulo a proximinal subspace is also LUR.

Proposition 2.8. *If (X, d) is an LUR metric linear space and M a proximinal subspace of X, then the quotient space $(X/M, \bar{d})$ is also LUR.*

Proof. Let $\epsilon > 0$ be given and $x + M \in X/M$ be any element. Let $r = \bar{d}(x + M, M) = d(x, M)$. As M is a proximinal subspace of X, there exists $m \in M$ such that $r = d(x, M) = d(x, m) = d(x - m, 0)$. Since (X, d) is LUR, there exists $\delta(x - m, \epsilon)$ such that $d\left(\frac{(x-m)+z}{2}, 0\right) \leq r - \delta(x - m, \epsilon)$, whenever $d(x-m, z) \geq \epsilon$ and $d(z, 0) \leq r$. We show that $\delta(x-m, \epsilon)$ works for ϵ and $x+M$ in $(X/M, \bar{d})$. Let $y + M \in X/M$ be any element such that $\bar{d}(y + M, M) \leq r$ and $\bar{d}(x + M, y + M) \geq \epsilon$. Then $d(y, M) \leq r$ and $d(x - y, M) \geq \epsilon$. Again as M is proximinal, there exists $m_1 \in M$ such that $d(y, M) = d(y, m_1)$, implying that $d(y - m_1, 0)] \leq r$ and also, $\epsilon \leq d(x - y, M) \leq d(x - y, m - m_1)$. Thus $d(y-m_1, 0) \leq r$ and $d(x-m, y-m_1) \geq \epsilon$. Then by the definition of $\delta(x-m, \epsilon)$, we have $d\left(\frac{(x-m)+(y-m_1)}{2}, 0\right) \leq r - \delta(x - m, \epsilon)$, that is, $d\left(\frac{x+y}{2}, \frac{m+m_1}{2}\right) \leq r - \delta(x - m, \epsilon)$. Then, $d\left(\frac{x+y}{2}, M\right) \leq d\left(\frac{x+y}{2}, \frac{m+m_1}{2}\right) \leq r - \delta(x - m, \epsilon)$, that is, $\bar{d}\left(\frac{(x+M)+(y+M)}{2}, M\right) \leq r - \delta(x - m, \epsilon)$. □

As mentioned earlier, there are many properties in- between strict convexity and uniform convexity, for instance, 2R-property, compact local uniform convexity and mid-point local uniform convexity etc. So far, we do not know whether these properties are inherited by the quotient spaces in case of normed linear spaces or metric linear spaces.

3. Three-space property

For normed linear spaces, it is known that completeness is a three-space property, that is, if X is a normed linear space and M any closed subspace of X, then completeness of any two of X, M and $\frac{X}{M}$ implies the completeness of the third too. We now show that this is true in case of metric linear spaces too.

Theorem 3.1. *Completeness is a three-space property for metric linear spaces.*

Proof. Let M be a closed subspace of a metric linear space (X, d). If X is complete, then clearly M is also complete and it is known that X/M is also complete (see [13, page 29]), so it is enough to show that completeness of both M and X/M implies the completeness of X. Let $\{x_n\}$ be a Cauchy sequence in X. As $\bar{d}(x_n + M, x_m + M) = d(x_n - x_m, M) \leq d(x_n - x_m, 0) = d(x_n, x_m) \to 0$ as $n, m \to \infty$, we see that $\{x_n + M\}$ is a Cauchy sequence in X/M. Since X/M is complete, $\{x_n + M\} \to y + M$ for some $y + M \in X/M$. By Lemma 2.4(b), for each $n \in \mathbb{N}$, there exists $y_n \in X$ such that $y_n + M = (x_n - y) + M$ and $d(y_n, 0) < d(x_n - y, M) + \frac{1}{n}$, implying that $d(y_n, 0)$ converges to zero, that is, $\{y_n\}$ converges to zero. Also, $x_n - y - y_n \in M$ for each n implies that $\{x_n - y - y_n\}$ is a Cauchy sequence in M. As M is complete, $\{x_n - y - y_n\} \to z \in M$, implying that $x_n = (x_n - y - y_n) + y_n + y \to z + y$. Hence X is complete. □

Since quotient space of a strictly convex normed linear space may not be strictly convex and subspace of a strictly convex normed linear space is strictly convex, it follows that strict convexity is not a three-space property even in the case of normed linear spaces. It will be interesting to study which of the other rotundity properties are three-space properties when the underlying spaces are normed linear spaces or metric linear spaces.

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