FINITE GROUPS WITH SPECIFIC NUMBER OF 2-ENGELIZERS

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Abstract. In 2016, the second and third authors introduced the notion of 2-Engelizer of the element x in a given group G and denoted the set of all 2-Engelizers in G by $E^2(G)$. They also constructed the possible values of $|E^2(G)|$ (Bull. Korean Math. Soc. **53** No. 3, (2016), 657-665).

In the present paper, we classify all non 2-Engel finite groups G, when $|E^2(G)| = 4, 5$.

1. Introduction

For an element x of a given group G, we call

$$E^2_G(x) = \{y \in G : [x, y, y] = 1, [y, x, x] = 1\}$$

to be the set of *2-Engelizer* of x in G. The family of all 2-Engelizers in G is denoted by $E^2(G)$ and $|E^2(G)|$ denotes the number of distinct 2-Engelizers in G (see [8] for more details).

As an example consider $Q_{16} = \langle a, b : a^8 = 1, a^4 = b^4, b^{-1}ab = a^{-1}\rangle$, the Quaternion group of order 16 and take the element b in Q_{16}. Then one can easily check that the 2-Engelizer set of b is as follows:

$$E^2_{Q_{16}}(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}.$$

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We remark that for the identity element e of G, we have $G = E_G^2(e)$ and hence $G \in E^2(G)$. Clearly in general, the 2-Engelizer of each non-trivial element of a group G does not form a subgroup. (see [8], Example 2.3 for more information).

In 2016, Moghaddam and Rostamyari [8] gave a condition under which the 2-Engelizer of each non-trivial element of G forms a subgroup.

Theorem 1.1. ([8], *Theorem 2.5*) *Let G be an arbitrary group. Then the set of each 2-Engelizer of a non-trivial element in G forms a subgroup if and only if the group $x^{E_G^2(x)}$ is abelian, for all non-trivial element x of G.*

They also proved that $|E^2(G)| \geqslant 4$, for any non 2-Engel group G, with abelian $x^{E_G^2(x)}$, for all $1 \neq x \in G$.

In the present article, we study the groups with such properties. One of our goals in this article is to calculate the number of 2-Engelizers of Dihedral group of order $2n$. Also, our main result is a characterization of finite groups with exactly $|E^2(G)| = 4, 5$.

2. Preliminary Results

An element x of a group G is called a *right 2-Engel* element, if for every $y \in G$, $[x, {}_2 y] = [x, y, y] = 1$, and the set of all right 2-Engel elements of G is denoted by $R_2(G)$. Many mathematicians have done interesting researches in this area (see [1, 6, 7, 9] for more information).

The following lemmas show the relationship between 2-Engelizers and the group G, even if the group is infinite. Also their results play an important role in finding lower bound for $|E^2(G)|$.

Lemma 2.1. *Let G be a group. Then $R_2(G)$ is the intersection of all 2-Engelizers in G.*

Proof. Clearly, $R_2(G) \subseteq \bigcap_{x\in G} E_G^2(x)$. Now, suppose $y \in \bigcap_{x\in G} E_G^2(x)$ then $[x, y, y] = [y, x, x] = 1$, for all $x \in G$ which gives $y \in R_2(G)$. $\square$

Lemma 2.2. *A group G is the union of 2-Engelizers of all elements in $G\backslash R_2(G)$, that is to say $G = \bigcup_{x\in G\backslash R_2(G)} E_G^2(x)$.*

Proof. Clearly, $\bigcup_{x\in G\backslash R_2(G)} E_G^2(x) \subseteq G$. By the definition, if $g \in R_2(G)$ then $g \in E_G^2(x)$, for every $x \in G$ and hence $g \in \bigcup_{x\in G\backslash R_2(G)} E_G^2(x)$. In case $g \in G\backslash R_2(G)$, then $g \in E_G^2(g)$ and so

$$g \in \bigcup_{x\in G\backslash R_2(G)} E_G^2(x).$$

Therefore $G \subseteq \bigcup_{x\in G\backslash R_2(G)} E_G^2(x)$ and the proof is complete. $\square$

Lemma 2.3. *Let $|E^2_{G/R_2(G)}(xR_2(G))| = p$, for some non right 2-Engel element x of a group G and a prime number p. For all $y \in G \backslash R_2(G)$, if $E^2_{G/R_2(G)}(xR_2(G)) = E^2_{G/R_2(G)}(yR_2(G))$, then*

$$E^2_G(x) = E^2_G(y).$$

Proof. Clearly,

$$E^2_G(x)/R_2(G) \subseteq E^2_{G/R_2(G)}(xR_2(G)).$$

Assume that $E^2_G(x)/R_2(G) < E^2_{G/R_2(G)}(xR_2(G))$. As $|E^2_{G/R_2(G)}(xR_2(G))| = p$ and $|E^2_G(x)/R_2(G)|$ divides $|E^2_{G/R_2(G)}(xR_2(G))|$, we get $|E^2_G(x)/R_2(G)| = 1$ and so $E^2_G(x) = R_2(G)$. Thus $x \in R_2(G)$, which is a contradiction. Therefore $E^2_G(x)/R_2(G) = E^2_{G/R_2(G)}(xR_2(G))$.

Clearly for all $y \in G \backslash R_2(G)$,

$$E^2_G(y)/R_2(G) \subseteq E^2_{G/R_2(G)}(yR_2(G)) = E^2_{G/R_2(G)}(xR_2(G)).$$

Hence $|E^2_{G/R_2(G)}(xR_2(G))| = |E^2_G(y)/R_2(G)|$, and so $E^2_G(y)/R_2(G) = E^2_G(x)/R_2(G)$. Thus

$$\frac{E^2_G(x)}{R_2(G)} = \frac{E^2_G(y)}{R_2(G)} = \{R_2(G), x_1R_2(G), x_2R_2(G), ..., x_{p-1}R_2(G)\},$$

where $\{x_1, ..., x_{p-1}\} \subseteq \big(E^2_G(x) \cap E^2_G(y)\big) \backslash R_2(G)$. So $E^2_G(x) = E^2_G(y)$. □

In the next result we calculate the number of 2-Engelizers of Dihedral group of order $2n$, except D_8, as it is nilpotent of class 2.

Proposition 2.4. *Let D_{2n} be the Dihedral group of order $2n$. Then $|E^2(D_{2n})| = n + 2$, when n is odd and otherwise $\frac{n}{2} + 2$.*

Proof. Let $D_{2n} = \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1}\rangle = \{1, x, \cdots, x^{n-1}, y, yx, \cdots, yx^{n-1}\}$ and $n \geqslant 3$. Now $E^2_{D_{2n}}(1) = D_{2n}$. Next consider $E^2_{D_{2n}}(x^i)$, where $1 \leqslant i \leqslant n-1$. Suppose $yx^j \in E^2_{D_{2n}}(x^i)$, then

$$[yx^j, x^i, x^i] = 1, [x^i, yx^j, yx^j] = x^{4i} = 1 \Rightarrow n \mid 4i.$$

If n is odd then n divides i, a contradiction. If n is even then $i = \frac{n}{2}$ or $\frac{n}{4}$, (if $\frac{n}{4} \in \mathbb{Z}$). Therefore $E^2_{D_{2n}}(x^i) = \langle x\rangle$, if n is odd or n is even and $i \neq \frac{n}{2}, \frac{n}{4}$.

Next consider $E^2_{D_{2n}}(yx^j)$, $0 \leqslant j \leqslant n-1$. Suppose $x^i \in E^2_{D_{2n}}(yx^j)$ then by a similar argument $i = \frac{n}{2}$ or $\frac{n}{4}$. Therefore if n is odd then $x^i \notin E^2_{D_{2n}}(yx^j)$ and if n is even then $x^{\frac{n}{2}}$ and $x^{\frac{n}{4}} \in E^2_{D_{2n}}(yx^j)$. Moreover, $E^2_{D_{2n}}(x^{\frac{n}{2}}) = E^2_{D_{2n}}(x^{\frac{n}{4}}) = D_{2n}$.

Now suppose $yx^k \in E^2_{D_{2n}}(yx^j)$, where $0 \leqslant k \neq j \leqslant n-1$. Then

$$[yx^k, yx^j, yx^j] = x^{-4j+4k} = 1 \Rightarrow n \mid 4(k-j),$$

$$[yx^j, yx^k, yx^k] = x^{4j-4k} = 1 \Rightarrow n \mid 4(j-k).$$

If n is odd then n divides $k-j$ or $j-k$, a contradiction. If n is even then $k-j=n$, $k-j=-n$, $k-j=\frac{n}{2}$ or $k-j=\frac{n}{4}$. Hence if n is odd

$$E^2_{D_{2n}}(1) = D_{2n}, E^2_{D_{2n}}(x^i) = \langle x\rangle, E^2_{D_{2n}}(yx^j) = \{1, yx^j\},$$

and so $|E^2(D_{2n})| = n+2$.

Also, as $yx^{j-n} = yx^{j+n}$ for even number n

$$E^2_{D_{2n}}(x^{\frac{n}{2}}) = E^2_{D_{2n}}(x^{\frac{n}{4}}) = E^2_{D_{2n}}(1) = D_{2n}, E^2_{D_{2n}}(x^i) = \langle x\rangle (i \neq \frac{n}{2}, \frac{n}{4}),$$

$$E^2_{D_{2n}}(yx^j) = \{1, yx^j, x^{\frac{n}{2}}, x^{\frac{n}{4}}, yx^{j+\frac{n}{2}}, yx^{j+\frac{n}{4}}\}.$$

Thus $|E^2(D_{2n})| = \frac{n}{2}+2$. □

In the next remark, we discuss the important property of the elements of a given group G, which will be used in Example 3.6.

Remark 2.5. Let $x, y \notin Z(G)$ and $xy \in Z(G)$, then for all $g \in G$

$$[xy, g] = 1 \Rightarrow g^x = g^{y^{-1}}.$$

Thus $\phi_x(g) = \phi_{y^{-1}}(g)$ implies that $\phi_{xy} = id$ and so $x = y^{-1}$.

Similarly, if $x, y \notin R_2(G)$ and $xy \in R_2(G)$, then for every $g \in G$

$$[xy, g, g] = 1 \Rightarrow g^{[x,g]^y} = g^{[y,g]^{-1}}.$$

Hence $[x, g]^y = [g, y]$ and again $x = y^{-1}$.

3. Main Results

Many authors have studied the influence of the number of centralizers on a finite group G (see [2, 3, 5]). It is clear that a group is 1-centralizer if and only if it is abelian. In [3] Belcastro and Sherman proved that there are no groups with 2 or 3 centralizers. They also proved that G has 4 centralizers if and only if $G/Z(G) \cong C_2 \times C_2$ and G has 5 centralizers if and only if $G/Z(G) \cong C_3 \times C_3$ or S_3. Ashrafi in [2] showed that if G has 6 centralizers, then $G/Z(G) \cong D_8, A_4, C_2 \times C_2 \times C_2$ or $C_2 \times C_2 \times C_2 \times C_2$.

The above results concerning the centralizers give us some motivation to study the concept of 2-Engelizers of groups. Our results in this section show that some known facts on centralizers of groups can be established for 2-Engelizers, and in some cases the results are different and more interesting.

Note that, in this section we work with the groups under the condition of Theorem 1.1, so that each 2-Engelizer of a non-trivial element of the group G must form a subgroup.

Theorem 3.1. *Let G be a group such that $G/R_2(G) \cong C_p \times C_p$, for any prime number p. Then $|E^2(G)| = p + 2$.*

Proof. Assume that $G/R_2(G) \cong C_p \times C_p$, then

$$\frac{G}{R_2(G)} = \langle xR_2(G), yR_2(G) : x^p, y^p, [x, y] \in R_2(G)\rangle.$$

Clearly any non-trivial proper subgroup $H/R_2(G)$ of $G/R_2(G)$ has order p. Therefore $H = R_2(G) \cup h_1R_2(G) \cup h_2R_2(G) \cup ... \cup h_{p-1}R_2(G)$, where $h_i \in H\backslash R_2(G)$ for all $1 \leqslant i \leqslant p-1$. Thus the proper subgroups of G properly containing $R_2(G)$ are one of the following forms:

$$R_2(G) \cup xR_2(G) \cup x^2R_2(G) \cup ... \cup x^{p-1}R_2(G);$$

$$R_2(G) \cup yR_2(G) \cup y^2R_2(G) \cup ... \cup y^{p-1}R_2(G) \text{ or}$$

$R_2(G) \cup x^iy^jR_2(G)$, for all $1 \leqslant i, j \leqslant p-1$. Note that, for all $1 \leqslant i, j \leqslant p-1$, it is easy to see that $x^iy^jR_2(G) = x^jy^iR_2(G)$ since $[x, y] \in R_2(G)$. Hence we have only $p-1$ proper subgroups of G of latest type. For simplicity, we denote all the above subgroups by $H_1, H_2, ..., H_{p+1}$, respectively. Now we show that $H_1, H_2, ..., H_{p+1}$ are the only proper 2-Engelizers of G. Let $a \in G\backslash R_2(G)$ then $aR_2(G) = bR_2(G)$, for some

$$b \in \{x, ..., x^{p-1}, y, ..., y^{p-1}, xy, xy^2, ..., xy^{p-1}, ..., x^{p-1}y, ..., x^{p-1}y^{p-1}\}.$$

Note that the order of each 2-Engelizers of $G/R_2(G)$ can not be p^2 or 1. Therefore $E^2_{G/R_2(G)}(aR_2(G)) = E^2_{G/R_2(G)}(bR_2(G))$ and Lemma 2.3 imply that $E^2_G(a) = E^2_G(b)$. Again let $b \in H_i\backslash R_2(G)$ then $E^2_G(b) \subseteq \cup_{j=1}^{p+1}H_j$, as $H_1, ..., H_{p+1}$ are the only proper subgroups of G. Also $b \in E^2_G(b)$, and hence $E^2_G(b) \neq H_j$, for $1 \leqslant i \neq j \leqslant p+1$. Therefore $E^2_G(b) = H_i$, and $H_1, H_2, ..., H_{p+1}$ are the only proper 2-Engelizers of G and so $|E^2(G)| = p + 2$. □

In 1926, Scorza [10] showed the following result, which is useful for our further investigation (see also [4]).

Theorem 3.2. ([4], *Theorem 1*) *A group G is the non-trivial union of three subgroups if and only if it is homomorphic to the Klein four group.*

Now, using the above theorem we have the following result.

Theorem 3.3. *Let G be a group, then $|E^2(G)| = 4$ if and only if $G/R_2(G) \cong C_2 \times C_2$.*

Proof. Using Theorem 3.1, it is enough to show that $|E^2(G)| = 4$ implies that $G/R_2(G) \cong C_2 \times C_2$.

Suppose $|E^2(G)| = 4$, then $E^2(G) = \{G, E^2_G(x), E^2_G(y), E^2_G(z)\}$, where x, y and z are non 2-Engel elements of G. Thus $G = E^2_G(x) \cup E^2_G(y) \cup E^2_G(z)$, as G is the union of its proper 2-Engelizers. Hence, Theorem 3.2 implies that $G/(E^2_G(x) \cap E^2_G(y) \cap E^2_G(z))$ is isomorphic with Klein four group.

Now, it is enough to show that $R_2(G) = E^2_G(x) \cap E^2_G(y) \cap E^2_G(z)$. Clearly $E^2_G(xy)$ must be equal to G, $E^2_G(x)$, $E^2_G(y)$ or $E^2_G(z)$.

If $E^2_G(xy) = G$ then $xy \in R_2(G)$ and $[xy, y, y] = 1$ implies that $[x, y, y] = 1$. Also, $[y, xy, xy] = 1$ implies that $[y, x, x] = 1$ and so $y \in E^2_G(x)$. Now, for every $g \in E^2_G(x)$ we have

$$[xy, g, g] = 1 \Rightarrow [y, g, g] = 1 \textit{ and } [g, xy, xy] = 1 \Rightarrow [g, y, y] = 1.$$

Thus $g \in E^2_G(y)$ and so $E^2_G(x) \subseteq E^2_G(y)$, which is a contradiction.

By the same argument if $E^2_G(xy) = E^2_G(x)$ or $E^2_G(y)$ we obtain a contradiction. Hence, $E^2_G(xy) = E^2_G(yx) = E^2_G(z)$. Now, it is clear that $g \in E^2_G(x) \cap E^2_G(y)$ implies that $g \in E^2_G(xy)$ and $g \in E^2_G(x) \cap E^2_G(xy)$ implies that $g \in E^2_G(y)$. Also $g \in E^2_G(y) \cap E^2_G(xy)$ implies that $g \in E^2_G(x)$. Hence, the intersection of any two 2-Engelizers is $R_2(G)$, which gives the result. $\square$

To prove our main result we need the following lemma.

Lemma 3.4. *Let $|E^2(G)| = 5$ and E^2_i be the proper 2-Engelizers of the group G, for $i = 1, 2, 3, 4$. Then*
(a) none of them is contained in the union of the others;
(b) no element of G is in exactly two or three of E^2_i's, $1 \leq i \leq 4$.

Proof. (a) By the contrary, assume that E^2_1 is a subset of $E^2_2 \cup E^2_3 \cup E^2_4$, and hence $G = \bigcup_{i=2}^4 E^2_i$. Theorem 3.2 implies that $G/\bigcap_{i=2}^4 E^2_i \cong C_2 \times C_2$. Now, in this case we show that $\bigcap_{i=2}^4 E^2_i = R_2(G)$, and then we obtain a contradiction.

Choose any $x_2 \in E^2_2 \setminus (E^2_3 \cup E^2_4)$, $x_3 \in E^2_3 \setminus (E^2_2 \cup E^2_4)$, and $x_4 \in E^2_4 \setminus (E^2_2 \cup E^2_3)$. We show that $E^2_i = E^2_G(x_i)$, for $i = 2, 3, 4$. For example, assume $E^2_G(x_2) \neq E^2_2$, then we have $E^2_G(x_2) = E^2_1$. Thus $E^2_2 \setminus (E^2_3 \cup E^2_4) \subseteq E^2_1 \setminus (E^2_3 \cup E^2_4)$ and so $E^2_2 \subseteq E^2_1$. Now, we could interchange the role of E^2_1 by E^2_2. Hence $E^2_1 = E^2_2$, which is impossible and so $E^2_i = E^2_G(x_i)$, for $i = 2, 3, 4$.

Now, let $x \in \bigcap_{i=2}^4 E^2_i \setminus R_2(G)$, then we have the following cases:
(*i*) $E^2_G(x) \neq G$, as $x \notin R_2(G)$;
(*ii*) $E^2_G(x) \neq E^2_1$, as $x \notin E^2_1$;
(*iii*) $E^2_G(x) \neq E^2_2$, as $x_3, x_4 \in E^2_G(x) \setminus E^2_2$;

(iv) $E^2_G(x) \neq E^2_3$, as $x_2, x_4 \in E^2_G(x) \setminus E^2_3$;
(v) $E^2_G(x) \neq E^2_4$, as $x_2, x_3 \in E^2_G(x) \setminus E^2_4$.

Hence $\bigcap_{i=2}^4 E^2_i \setminus R_2(G) = \emptyset$, which gives part (a).

(b) First take an element $x \in (E^2_1 \cap E^2_2) \setminus (E^2_3 \cup E^2_4)$, then clearly $x_1, x_2 \in E^2_G(x)$. But $x_1 \notin E^2_2$ and this implies that $E^2_G(x) \neq E^2_1$ or E^2_2.

Also $E^2_G(x) \neq E^2_3$ or E^2_4, as $x \notin E^2_3 \cup E^2_4$. On the other hand, $E^2_G(x) \neq G$, as $x \in G \setminus R_2(G)$. Therefore $E^2_G(x) \neq G, E^2_1, E^2_2, E^2_3$ or E^2_4, which contradicts the number of 2-Engelizers $|E^2(G)| = 5$, and so $(E^2_1 \cap E^2_2) \setminus (E^2_3 \cup E^2_4) = \emptyset$.

Now assume that $x \in (E^2_1 \cap E^2_2 \cap E^2_3) \setminus E^2_4$, then $x_1, x_2, x_3 \in E^2_G(x)$. It can be easily seen that $E^2_G(x) \neq E^2_1$, E^2_2 or E^2_3. Also $E^2_G(x) \neq E^2_4$ or G, as $x \notin E^2_4$ and $x \notin R_2(G)$. Therefore $E^2_G(x) \neq G, E^2_1, E^2_2, E^2_3, E^2_4$, which means $|E^2(G)|$ must be at least 6 and this gives a contradiction. Thus $(E^2_1 \cap E^2_2 \cap E^2_3) \setminus E^2_4 = \emptyset$. □

Remark 3.5. Note that the above lemma shows that the group G is at most a disjoint union of its four proper 2-Engelizers, when $|E^2(G)| = 5$. Also, in this case we have

$$R_2(G) = \bigcap_{i=1}^{4} E^2_i = E^2_i \cap E^2_j \cap E^2_k = E^2_i \cap E^2_j,$$

for all $1 \leq i \neq j \neq k \leq 4$.

In the following, we compute the number of 2-Engelizers of some groups, which will be used in our final result.

Example 3.6. *(i) If $G/R_2(G) \cong S_3 = \langle xR_2(G), yR_2(G) \mid x^2, y^3, yy^x \in R_2(G)\rangle$. Then it is clear that $|\frac{G/R_2(G)}{H/R_2(G)}| = 2$ or 3, for every proper subgroup $H/R_2(G)$ of $G/R_2(G)$. Thus $H = R_2(G) \cup h_1R_2(G)$ or $H = R_2(G) \cup h_2R_2(G) \cup h_3R_2(G)$, where $h_1, h_2, h_3 \in H \setminus R_2(G)$. Therefore the proper subgroups of G properly containing $R_2(G)$ are as follows:*

$$H_1 = R_2(G) \cup yR_2(G) \cup y^2R_2(G);\ \ H_2 = R_2(G) \cup xR_2(G);$$

$$H_3 = R_2(G) \cup xyR_2(G);\ \ H_4 = R_2(G) \cup xy^2R_2(G).$$

Take an element $a \in G \setminus R_2(G)$ then $aR_2(G) = hR_2(G)$, for some $h \in \{y, y^2, x, xy, xy^2\}$. Thus, $E^2_{G/R_2(G)}(aR_2(G)) = E^2_{G/R_2(G)}(hR_2(G))$ and so Lemma 2.3 implies that $E^2_G(a) = E^2_G(h)$.

Now, we show that H_i's are the only proper 2-Engelizers of G. Assume $h \in H_i \setminus R_2(G)$ and $E^2_G(h) \subseteq \bigcup_{j\neq i} H_j$, where $1 \leqslant i, j \leqslant 4$. On the other hand, $h \in E^2_G(h)$ implies that $E^2_G(h) \neq H_j$, for $1 \leqslant j \neq i \leqslant 4$. Therefore $E^2_G(h) = H_i$ gives the claim and so $|E^2(G)| = 5$.

(ii) The factor group $G/R_2(G) \cong C_2 \times C_6$, has the following presentation

$$\frac{G}{R_2(G)} = \langle xR_2(G), yR_2(G) \mid x^2, y^6, [x,y] \in R_2(G)\rangle$$
$$= \{\bar{1}, \bar{x}, \bar{y}, \bar{y^2}, \bar{y^3}, \bar{y^4}, \bar{y^5}, \bar{xy}, \bar{xy^2}, \bar{xy^3}, \bar{xy^4}, \bar{xy^5}\},$$

where $^-$ means modulo $R_2(G)$.

Clearly, non-trivial proper subgroups of $G/R_2(G)$, which properly containing $R_2(G)$ are as follows:

$$H_1 = R_2(G) \cup xR_2(G), H_2 = R_2(G) \cup xy^3R_2(G), H_3 = R_2(G) \cup y^3R_2(G)$$
$$H_4 = R_2(G) \cup y^2R_2(G) \cup y^4R_2(G),$$
$$H_5 = R_2(G) \cup yR_2(G) \cup y^2R_2(G) \cup y^3R_2(G) \cup y^4R_2(G) \cup y^5R_2(G),$$
$$H_6 = R_2(G) \cup xyR_2(G) \cup y^2R_2(G) \cup xy^3R_2(G) \cup y^4R_2(G) \cup xy^5R_2(G),$$
$$H_7 = R_2(G) \cup xy^2R_2(G) \cup y^4R_2(G) \cup xR_2(G) \cup y^2R_2(G) \cup xy^4R_2(G).$$

Lemma 2.3 implies that H_i's are the proper 2-Engelizers of $G/R_2(G)$, for $1 \leqslant i \leqslant 4$.

Now, in the subgroups H_5, H_6 and H_7, if $aR_2(G) = bR_2(G)$, for $a \neq b$, then $a^{-1}b \in R_2(G)$. Remark 2.5 implies that $a = b$, which is a contradiction and so $|E^2(G)| = 8$.

(iii) Let $G/R_2(G) \cong A_4$ be the alternating group of degree 4. Then by a similar argument as part (i), non-trivial proper subgroups of $G/R_2(G)$ which properly containing $R_2(G)$ are as follows:

$$H_1 = R_2(G) \cup (1,2)(3,4)R_2(G), H_2 = R_2(G) \cup (1,3)(2,4)R_2(G),$$
$$H_3 = R_2(G) \cup (1,4)(2,3)R_2(G), H_4 = R_2(G) \cup (1,2,3)R_2(G) \cup (1,3,2)R_2(G),$$
$$H_5 = R_2(G) \cup (1,2,4)R_2(G) \cup (1,4,2)R_2(G),$$
$$H_6 = R_2(G) \cup (1,3,4)R_2(G) \cup (1,4,3)R_2(G),$$
$$H_7 = R_2(G) \cup (2,3,4)R_2(G) \cup (2,4,3)R_2(G).$$

Lemma 2.3 implies that H_i's are the only proper 2-Engelizers of $G/R_2(G)$, for $1 \leqslant i \leqslant 7$ and hence $|E^2(G)| = 8$.

(iv) Let $G/R_2(G)$ be a semidirect product of cyclic groups of order 3 by the one of order 4, i.e.

$$\frac{G}{R_2(G)} \cong C_3 \rtimes C_4 = \langle xR_2(G), yR_2(G) \mid x^3, y^4, x^y x \in R_2(G)\rangle$$
$$= \{1, x, x^2, y, y^2, y^3, xy, xy^2, xy^3, x^2y, x^2y^2, x^2y^3\}.$$

Then by a similar argument as in the previous parts, non-trivial proper subgroups of $G/R_2(G)$, which properly containing $R_2(G)$ are as following

$$H_1 = R_2(G) \cup y^2R_2(G), H_2 = R_2(G) \cup xR_2(G) \cup x^2R_2(G),$$

$$H_3 = R_2(G) \cup yR_2(G) \cup y^2R_2(G) \cup y^3R_2(G),$$
$$H_4 = R_2(G) \cup xyR_2(G) \cup y^2R_2(G) \cup xy^3R_2(G),$$
$$H_5 = R_2(G) \cup x^2yR_2(G) \cup y^2R_2(G) \cup x^2y^3R_2(G),$$
$$H_6 = R_2(G) \cup xy^2R_2(G) \cup x^2R_2(G) \cup y^2R_2(G) \cup xR_2(G) \cup x^2y^2R_2(G).$$

By a similar argument as used in part (i), we conclude that $|E^2(G)| = 7$.

The following result characterizes the factor group $G/R_2(G)$, when the group G is five 2-Engelizers.

Theorem 3.7. *Let G be a finite group with $|E^2(G)| = 5$, then $G/R_2(G) \cong C_3 \times C_3, D_{12}, C_2 \times C_6, C_3 \rtimes C_4, A_4$ or S_3.*

Proof. Assume that $|E^2(G)| = 5$ then using Lemma 3.4 and Remark 3.5, there exist only four distinct 2-Engelizers such that $G = \bigcup_{i=1}^{4} E_i^2$. Hence

$$|G| = |E_1^2 \cup E_2^2 \cup E_3^2 \cup E_4^2| = |E_1^2| + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|.$$

Now, for computing the value of $|R_2(G)|$, we show that if E_i^2 and E_j^2 are arbitrary distinct proper 2-Engelizers of G, for $1 \leqslant i \neq j \leqslant 4$, then

$$\frac{|E_i^2||E_j^2|}{|G|} \leqslant |R_2(G)| \leqslant \frac{|G|}{6}. \qquad (*)$$

Clearly, $\frac{|E_i^2||E_j^2|}{|E_i^2E_j^2|} = |E_i^2 \bigcap E_j^2|$, and since $E_i^2E_j^2 \subseteq G$, we have $\frac{1}{|E_i^2E_j^2|} \geqslant \frac{1}{|G|}$. Therefore $|E_i^2 \bigcap E_j^2| \geqslant \frac{|E_i^2||E_j^2|}{|G|}$ implies that $|R_2(G)| \geqslant \frac{|E_i^2||E_j^2|}{|G|}$. On the other hand, one observes that

$$\begin{aligned}|G| =&|E_1^2| + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|\\ \geqslant&2|R_2(G)| + 2|R_2(G)| + 2|R_2(G)| + 2|R_2(G)| - 3|R_2(G)| = 5|R_2(G)|,\end{aligned}$$

and hence $\frac{|G|}{|R_2(G)|} \geqslant 5$. Assume $\frac{|G|}{|R_2(G)|} = 5$, then $\frac{G}{R_2(G)}$ is cyclic and so G is 2-Engel group, which implies that $\frac{|G|}{|R_2(G)|} \geqslant 6$ and proves the claim of $(*)$.

Now without loss of generality, we may assume that $|E_1^2| \geqslant |E_2^2| \geqslant |E_3^2| \geqslant |E_4^2|$. Suppose $|E_1^2| \leqslant \frac{|G|}{4}$, then we have

$$\begin{aligned}|G| =&|E_1^2| + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|\\ \leqslant&\frac{|G|}{4} + \frac{|G|}{4} + \frac{|G|}{4} + \frac{|G|}{4} - 3|R_2(G)| = |G| - 3|R_2(G)|,\end{aligned}$$

which is a contradiction. Hence $|E_1^2| = \frac{|G|}{2}$ or $\frac{|G|}{3}$. If $|E_1^2| = \frac{|G|}{2}$, we get

$$\begin{aligned}|G| =&|E_1^2| + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|\\ =&\frac{|G|}{2} + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|.\end{aligned}$$

One can easily calculate that

$$\frac{|G|}{2} < |E_2^2| + |E_3^2| + |E_4^2| \leqslant 3|E_2^2|,$$

and so $\frac{|G|}{6} < |E_2^2|$.

Now applying $(*)$ to E_1^2 and E_2^2, we have $\frac{|E_1^2||E_2^2|}{|G|} \leqslant \frac{|G|}{6}$ and hence $|E_2^2| \leqslant \frac{2|G|}{6}$. That is $\frac{|G|}{6} < |E_2^2| \leqslant \frac{|G|}{3}$, so $|E_2^2| = \frac{|G|}{5}, \frac{|G|}{4}$ or $\frac{|G|}{3}$. The property $\frac{|E_1^2||E_2^2|}{|G|} \leqslant |R_2(G)| \leqslant \frac{|G|}{6}$ implies that $\frac{|G|}{10} \leqslant |R_2(G)| \leqslant \frac{|G|}{6}$. Therefore the value of $|R_2(G)|$ must be one of $\frac{|G|}{6}, \frac{|G|}{7}, \frac{|G|}{8}, \frac{|G|}{9}$ or $\frac{|G|}{10}$.

Now if $|R_2(G)| = \frac{|G|}{7}$, then $|R_2(G)|$ divides $|E_1^2|$, and hence $2 \mid 7$, which is impossible. Similarly $|R_2(G)| \neq \frac{|G|}{9}$. Assume $|R_2(G)| = \frac{|G|}{6}$ then $|\frac{G}{R_2(G)}| = 6$ and as $\frac{G}{R_2(G)}$ can not be cyclic, hence $\frac{G}{R_2(G)} \cong S_3$.

Let $|R_2(G)| = \frac{|G|}{8}$, then as $|R_2(G)|$ divides $|E_2^2|$, if $|E_2^2| = \frac{|G|}{3}$, then $3 \mid 8$ and if $|E_2^2| = \frac{|G|}{5}$ then $5 \mid 8$, which both give contradictions. Therefore $|E_2^2| = \frac{|G|}{4}$. Also, the property $|G| = |E_1^2|+|E_2^2|+|E_3^2|+|E_4^2|-3|R_2(G)|$ implies that $\frac{|G|}{4} = |E_3^2| + |E_4^2| - 3\frac{|G|}{8}$, and hence $\frac{5|G|}{8} = |E_3^2| + |E_4^2|$. As $|E_3^2|, |E_4^2| \leqslant \frac{|G|}{4}$, we obtain $\frac{5|G|}{8} = |E_3^2| + |E_4^2| \leqslant \frac{|G|}{2}$, which is again a contradiction. So $|R_2(G)|$ can not be equal to $\frac{|G|}{8}$.

Finally, assume that $|R_2(G)| = \frac{|G|}{10}$ and $|R_2(G)|$ divides $|E_2^2|$. If $|E_2^2| = \frac{|G|}{3}$ then $3 \mid 10$, and if $|E_2^2| = \frac{|G|}{4}$ then $4 \mid 10$, which are both impossible. Therefore $|E_2^2| = \frac{|G|}{5}$. Now, again $|G| = |E_1^2|+|E_2^2|+|E_3^2|+|E_4^2| - 3|R_2|$ implies that $|E_3^2| + |E_4^2| = \frac{6|G|}{10}$. Also, $|E_2^2| \geqslant |E_3^2| \geqslant |E_4^2|$ implies that $\frac{6|G|}{10} = |E_3^2| + |E_4^2| \leqslant \frac{2|G|}{5}$, which is a contradiction. Hence $|R_2(G)| \neq \frac{|G|}{10}$.

Now, assume that $|E_1^2| = \frac{|G|}{3}$. In this case, using

$$|G| = |E_1^2| + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|,$$

we have $\frac{2|G|}{3} < |E_2^2| + |E_3^2| + |E_4^2| \leqslant 3|E_2^2|$. Thus $|E_2^2| > \frac{2|G|}{9}$. On the other hand, $|E_1^2| \geqslant |E_2^2|$ and so $\frac{2|G|}{9} < |E_2^2| \leqslant \frac{|G|}{3}$. Therefore $|E_2^2| = \frac{|G|}{3}$

or $\frac{|G|}{4}$. Again applying $(*)$ on E_1^2 and E_2^2 we get,

$$\frac{|E_1^2||E_2^2|}{|G|} \leqslant |R_2(G)| \leqslant \frac{|G|}{6}.$$

Thus $\frac{|G|}{12} \leqslant |R_2(G)| \leqslant \frac{|G|}{6}$, and hence $|R_2(G)| = \frac{|G|}{6}, \frac{|G|}{7}, \frac{|G|}{8}, \frac{|G|}{9}, \frac{|G|}{10}, \frac{|G|}{11}$ or $\frac{|G|}{12}$.

Assume that $|R_2(G)| = \frac{|G|}{7}$, and as $|R_2(G)|$ divides $|E_1^2|$ we must have $3 \mid 7$, which is impossible. Similarly $|R_2(G)| \neq \frac{|G|}{8}, \frac{|G|}{10}$ and $\frac{|G|}{11}$. Also, assume that $|R_2(G)| = \frac{|G|}{6}$, $|E_1^2| = \frac{|G|}{3}$, and $|E_2^2| = \frac{|G|}{4}$ or $\frac{|G|}{3}$, then

$$|G| = |E_1^2| + |E_2^2| + |E_3^2| + |E_4^2| - 3|R_2(G)|,$$

again implies that $\frac{11|G|}{12} = |E_3^2| + |E_4^2| \leqslant \frac{|G|}{2}$ or $\frac{5|G|}{6} = |E_3^2| + |E_4^2| \leqslant \frac{2|G|}{3}$, respectively, which are both impossible. Hence $|R_2(G)| \neq \frac{|G|}{6}$, and so we have one of the following cases:

$$|R_2(G)| = \frac{|G|}{12} \Rightarrow \frac{|G|}{|R_2(G)|} = 12 \Rightarrow \frac{G}{R_2(G)} \cong C_{12}, A_4, D_{12}, C_3 \rtimes C_4, C_2 \times C_6,$$

or

$$|R_2(G)| = \frac{|G|}{9} \Longrightarrow \frac{|G|}{|R_2(G)|} = 9 \Longrightarrow \frac{G}{R_2(G)} \cong C_9, C_3 \times C_3.$$

On the other hand, $\frac{G}{R_2(G)}$ can not be cyclic, as G is not 2-Engel group. Thus $\frac{G}{R_2(G)} \cong D_{12}, C_2 \times C_6, A_4, C_3 \rtimes C_4$ or $C_3 \times C_3$. □

Note that, Proposition 2.4, Theorem 3.1 and Example 3.6 imply that the converse of the above result is not true in general. One can easily see that, if $G/R_2(G) \cong D_{12}, C_3 \times C_3$ or S_3, then $|E^2(G)| = 5$.

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