

## On A Class of Soc-Injective Modules

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**ABSTRACT.** Let  $R$  be a ring. The class of  $SA$ -injective right  $R$ -modules ( $SAI_R$ ) is introduced as a class of soc-injective right  $R$ -modules. Let  $N$  be a right  $R$ -module. A right  $R$ -module  $M$  is said to be  $SA$ - $N$ -injective if every  $R$ -homomorphism from a semi-artinian submodule of  $N$  into  $M$  extends to  $N$ . A module  $M$  is called  $SA$ -injective, if  $M$  is  $SA$ - $R$ -injective. We characterize rings over which every right module is  $SA$ -injective. Conditions under which the class  $SAI_R$  is closed under quotient (resp. direct sums, pure homomorphic images) are given. The definability of the class  $SAI_R$  is studied. Finally, relations between  $SA$ -injectivity and certain generalizations of injectivity are given.

**Keywords:** Semi-artinian submodule, Definable class, Injective module, Noetherian module, Flat module.

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### 1. INTRODUCTION

Throughout  $R$  is an associative ring with identity and all modules are unitary  $R$ -modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right  $R$ -module (resp. right  $R$ -homomorphism). We use  $R\text{-Mod}$  (resp.  $\text{Mod-}R$ ) to denote the class of left (resp. right)  $R$ -modules. We will use  $M^*$  to denote the character module  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  of a right module

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$M$ . Let  $\mathcal{G}$  (resp.  $\mathcal{F}$ ) be a class of right (resp. left)  $R$ -modules. A pair  $(\mathcal{F}, \mathcal{G})$  is called almost dual pair if  $\mathcal{G}$  is closed under summands and direct products, and for any left  $R$ -module  $M$ ,  $M \in \mathcal{F}$  if and only if  $M^* \in \mathcal{G}$  [12, p. 66]. An exact sequence  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$  of right  $R$ -modules is said to be pure if the sequence  $0 \rightarrow \text{Hom}_R(N, A) \rightarrow \text{Hom}_R(N, B) \rightarrow \text{Hom}_R(N, C) \rightarrow 0$  is exact, for every finitely presented right  $R$ -module  $N$  and we called that  $\alpha(A)$  is a pure submodule of  $B$  [21]. A right  $R$ -module  $M$  is called  $FP$ -injective if every monomorphism  $\alpha: M \rightarrow N$  is pure. A right  $R$ -module  $M$  is called pure injective if  $M$  is injective with respect to all pure short exact sequences [21]. If a subclass  $\mathcal{G}$  of  $\text{Mod-}R$  is closed under pure submodules, direct limits and direct products, then it is called a definable class [16]. We denote by  $\text{Soc}(M)$  to the socle of a module  $M$ . A right  $R$ -module  $M$  is called semi-artinian if for any proper submodule  $N$  of  $M$  we have  $\text{Soc}(M/N) \neq 0$  [9, p. 238]. We will denote to the sum of all semi-artinian submodules of a right  $R$ -module  $M$  by  $\text{Sa}(M)$ . If  $N$  is a submodule of a right  $R$ -module  $M$ , the notation  $N \subseteq^{sa} M$  means that  $N$  is a semi-artinian submodule of  $M$ .

We refer the reader to [2], [9], [16], [18] and [21], for general background materials.

Injective modules have been studied extensively, and several generalizations for these modules are given, for example, soc-injective modules [1],  $\mathcal{L}$ -injective Modules [13], and  $n$ - $FP$ -injective modules [5]. If  $\text{Ext}^1(R/K, M) = 0$ , for any semisimple right ideal  $K$  of  $R$ , then a right  $R$ -module  $M$  is called soc-injective [1], where  $\text{Ext}^1(A, B)$  is defined as the first right derived functor of  $\text{Hom}_R(A, B)$ , for any two right  $R$ -modules  $A, B$  (see [4, Ch. VI] for more details).

In section 2 of this paper, we introduce the class of  $SA$ -injective modules. This class of modules lies between injective modules and soc-injective modules. We first give examples to show that the notion of  $SA$ -injectivity is distinct from that of injectivity and soc-injectivity. We characterize rings over which every module is  $SA$ -injective. We prove the equivalence of the following statements: (1) Every right  $R$ -module is  $SA$ -injective; (2) Every semi-artinian module is  $SA$ -injective; (3) Every semi-artinian right ideal of  $R$  is  $SA$ -injective; (4) Every semi-artinian right ideal of  $R$  is a direct summand of  $R$ . Conditions under which the class of  $SA$ -injective right  $R$ -modules ( $SAI_R$ ) is closed under quotient are given. For instance, we prove that the equivalence of the following: (1) The class  $SAI_R$  is closed under quotient; (2) Sums of any two  $SA$ -injective submodules of any right  $R$ -module is  $SA$ -injective; (3) All semi-artinian right ideals of  $R$  are projective. Finally, we give conditions such that the class  $SAI_R$  is closed under direct sums. For instance, we prove that the following are equivalent. (1)  $\text{Sa}(R_R)$  is noetherian; (2) Any direct sum of  $SA$ -injective right  $R$ -modules is  $SA$ -injective; (3) The class  $SAI_R$  is closed under pure submodules; (4) All  $FP$ -injective modules are  $SA$ -injective.

In section 3, we study the definability of the class  $SAI_R$ . It is shown that the following assertions are equivalent: (1)  $SAI_R$  is definable; (2) The class  $SAI_R$  is closed under pure submodules and pure homomorphic images; (3) Every semi-artinian right ideal in  $R$  is finitely presented; (4) A module  $M \in SAI_R$  iff  $M^* \in (SAI_R)^\ominus$ ; (5) A module  $M \in SAI_R$  iff  $M^{**} \in SAI_R$ . Finally, we prove that if the class  $SAI_R$  is a definable, then the class of flat left  $R$ -modules and the class  $(SAI_R)^\ominus$  are coincide iff all modules in  $SAI_R$  are  $FP$ -injective iff all pure-injective modules in  $SAI_R$  are injective.

In section 4, we give relations between  $SA$ -injectivity and certain generalizations of injectivity (in particular, quasi-injectivity and  $F$ -injectivity). Firstly, we prove that a ring  $R$  is a right semi-artinian ring iff every  $SA$ -injective right  $R$ -module is quasi-injective iff every cyclic  $SA$ -injective right  $R$ -module is quasi-injective. Then, we prove that a commutative ring  $R$  is semisimple if and only if  $R$  is a semi-artinian ring and every quasi-injective  $R$ -module is  $SA$ -injective. Also, we prove that  $Sa(R_R)$  is a noetherian right  $R$ -module if and only if every  $F$ -injective right  $R$ -module is  $SA$ -injective. Finally, we prove that a ring  $R$  is a (von Neumann) regular and every  $P$ -injective right  $R$ -module is  $SA$ -injective if and only if every  $SA$ -injective right  $R$ -module is  $P$ -injective and every semi-artinian right ideal of  $R$  is a direct summand of  $R_R$ .

## 2. $SA$ -INJECTIVE MODULES

**Definition 2.1.** Let  $N$  be a module. A module  $M$  is called  $SA$ - $N$ -injective, if for any semi-artinian submodule  $K$  of  $N$ , any homomorphism  $f : K \rightarrow M$  extends to  $N$ .  $M$  is called  $SA$ -injective if  $M$  is  $SA$ - $R$ -injective. A ring  $R$  is called  $SA$ -injective if the module  $R_R$  is  $SA$ -injective.

We will use  $SAI_R$  to denote the class of  $SA$ -injective right  $R$ -modules.

**EXAMPLES 2.2.** (1) All injective modules are  $SA$ -injective. Since 0 is the only semi-artinian right ideal in  $\mathbb{Z}$ , we have the right  $\mathbb{Z}$ -module  $\mathbb{Z}$  is a  $SA$ -injective but it is not injective. Hence  $SA$ -injectivity is a proper generalization of injectivity.

(2) Since every semisimple module is semi-artinian, we have every  $SA$ -injective module is soc-injective. The converse is not true in general, for example: let  $R = \mathbb{Z}_2[x_1, x_2, \dots]$  where  $x_i^3 = 0$  for all  $i$ ,  $x_i^2 = x_j^2 \neq 0$  for all  $i$  and  $j$  and  $x_i x_j = 0$  for all  $i \neq j$ . By [1, Example 5.7],  $R$  is a semiprimary commutative and soc-injective ring but it is not self injective. By [18, Example 1, p. 184],  $R$  is a right semi-artinian ring, so that Proposition 2.5 in [18, p. 183] implies that  $I \subseteq^{sa} R_R$  for any right ideal  $I$  in  $R$  and hence  $R$  is not  $SA$ -injective ring.

(3) Clearly, if  $\text{Soc}(N_R) = 0$ , then 0 is the only semi-artinian submodule of  $N$  and hence every module is  $SA$ - $N$ -injective. Particularly, all  $\mathbb{Z}$ -modules are  $SA$ -injective.

(4) All modules with zero socles are  $SA$ -injective, this follows from the fact that  $\text{Soc}(M) = 0$  if and only if  $\text{Sa}(M) = 0$ , for any module  $M$ .

**Proposition 2.3.** *Let  $N$  be a module. Then following statements hold:*

- (1) *The class of  $SA$ - $N$ -injective modules is closed under isomorphic copies, direct products, direct summands and finite direct sums.*
- (2) *For any submodule  $K$  of  $N$ , if  $M$  is  $SA$ - $N$ -injective module, then  $M$  is  $SA$ - $K$ -injective.*
- (3) *If  $M$  is  $SA$ - $N$ -injective module, then  $M$  is  $SA$ - $K$ -injective, for any module  $K$  isomorphic to  $N$ .*

*Proof.* Clear. □

**Corollary 2.4.** *The class of  $SA$ -injective right  $R$ -modules ( $SAI_R$ ) is closed under isomorphic copies, direct products, direct summands and finite direct sums.*

**Proposition 2.5.** *Let  $M$  be a module and  $\{N_i : i \in I\}$  be a family of modules. If  $\bigoplus_{i \in I} N_i$  is a multiplication module, then  $M$  is  $SA$ - $\bigoplus_{i \in I} N_i$ -injective iff  $M$  is  $SA$ - $N_i$ -injective, for all  $i \in I$ .*

*Proof.*  $(\Rightarrow)$  By Proposition 2.3((2),(3)).

$(\Leftarrow)$  Let  $K \subseteq^{s.a} \bigoplus_{i \in I} N_i$ . Since  $\bigoplus_{i \in I} N_i$  is a multiplication module (by hypothesis), we have from [20, Theorem 2.2, p. 3844] that  $K = \bigoplus_{i \in I} K_i$  with  $K_i$  a submodule of  $N_i$ , for all  $i \in I$ . By [9, p. 238],  $K_i \subseteq^{s.a} N_i$ . For  $i \in I$ , consider the following diagram:

$$\begin{array}{ccc}
 K_i & \xrightarrow{i_2} & N_i \\
 i_{K_i} \downarrow & & \downarrow i_{N_i} \\
 K = \bigoplus_{i \in I} K_i & \xrightarrow{i_1} & \bigoplus_{i \in I} N_i \\
 f \downarrow & & \\
 M & & 
 \end{array}$$

where  $i_{K_i}$ ,  $i_{N_i}$  are injection maps and  $i_1$ ,  $i_2$  are inclusion maps. The hypothesis implies that there exists homomorphism  $h_i : N_i \rightarrow M$  such that  $h_i \circ i_2 = f \circ i_{K_i}$ . By [9, Theorem 4.1.6(2)], there exists exactly one homomorphism  $h : \bigoplus_{i \in I} N_i \rightarrow M$  satisfying  $h_i = h \circ i_{N_i}$ . Thus  $f \circ i_{K_i} = h_i \circ i_2 = h \circ i_{N_i} \circ i_2 = h \circ i_1 \circ i_{K_i}$  for all  $i \in I$ . Let  $(a_i)_{i \in I} \in \bigoplus_{i \in I} K_i$ , thus  $a_i \in K_i$ , for all  $i \in I$  and  $f((a_i)_{i \in I}) = f(\sum_{i \in I} i_{K_i}((a_i)_{i \in I})) = (h \circ i_1)((a_i)_{i \in I})$ . Thus  $f = h \circ i_1$  and the proof is complete. □

Recall that a ring  $R$  is called a right invariant if each of its right ideals is an ideal of  $R$  [20, p. 3839].

**Corollary 2.6.** (1) Let  $M$  be a module over a right invariant ring  $R$  and  $1 = \lambda_1 + \lambda_2 + \dots + \lambda_m$  in  $R$  such that  $\lambda_j$  are orthogonal idempotent. Then  $M$  is SA-injective iff  $M$  is SA- $\lambda_j R$ -injective for every  $j = 1, 2, \dots, m$ .

(2) If  $M$  is SA- $aR$ -injective module and  $aR \cong bR$ , where  $a$  and  $b$  are idempotents of  $R$ , then  $M$  is SA- $bR$ -injective.

*Proof.* (1) By [2, Corollary 7.3],  $R = \bigoplus_{j=1}^m \lambda_j R$ . Since  $R$  is a right invariant ring, we get from [20, Proposition 3.1, p. 3855] that  $R$  is a multiplication module and hence Proposition 2.5 implies that  $M$  is SA-injective iff  $M$  is SA- $\lambda_j R$ -injective for all  $1 \leq j \leq m$ .

(2) By Proposition 2.3(3).  $\square$

**Proposition 2.7.** The following statements are equivalent for a module  $M$ .

- (1) All modules are SA- $M$ -injective.
- (2) All semi-artinian modules are SA- $M$ -injective.
- (3) All semi-artinian submodules of  $M$  are SA- $M$ -injective.
- (4) All semi-artinian submodules of  $M$  are direct summands of  $M$ .

*Proof.* Straightforward.  $\square$

Proposition 2.7 implies the next result.

**Corollary 2.8.** For a ring  $R$ , the following conditions are equivalent.

- (1)  $\text{Mod-}R = \text{SAI}_R$ .
- (2) All semi-artinian modules are SA-injective.
- (3) All semi-artinian right ideals of  $R$  are SA-injective.
- (4) If  $I \subseteq^{\text{sa}} R_R$ , then  $I$  is a direct summand of  $R_R$ .

**Corollary 2.9.** A module  $M$  is semisimple if and only if  $M$  is semi-artinian and all modules are SA- $M$ -injective.

*Proof.* ( $\Rightarrow$ ) It is obvious.

( $\Leftarrow$ ) If  $K$  is a submodule of  $M$ , then  $K$  is semi-artinian by [9, p. 238] and hence Proposition 2.7 implies that  $K$  is a direct summand of  $M$ . Thus  $M$  is a semisimple module.  $\square$

As a special case of Corollary 2.9, we have the following corollary.

**Corollary 2.10.** A ring  $R$  is a right semisimple ring if and only if it is a right semi-artinian ring and  $\text{Mod-}R = \text{SAI}_R$ .

In general, not every semi-artinian submodule of a projective module is projective, for example, if  $M = \mathbb{Z}_4$  as  $\mathbb{Z}_4$ -module and  $K = 2\mathbb{Z}_4$ , then  $K \subseteq^{\text{sa}} M$  but  $K$  is not a projective  $\mathbb{Z}_4$ -module.

**Theorem 2.11.** *The following conditions are equivalent for a projective module  $M$ .*

- (1) *The class of  $SA$ - $M$ -injective modules is closed under quotient.*
- (2) *Every quotient of an injective module is  $SA$ - $M$ -injective.*
- (3) *If  $K_1$  and  $K_2$  are two  $SA$ - $M$ -injective submodules of a module  $N$ , then  $K_1 + K_2$  is  $SA$ - $M$ -injective.*
- (4) *If  $K_1$  and  $K_2$  are two injective submodules of a module  $N$ , then  $K_1 + K_2$  is  $SA$ - $M$ -injective.*
- (5) *If  $K \subseteq^{sa} M$ , then  $K$  is projective.*

*Proof.* (1)  $\Rightarrow$  (2) and (3)  $\Rightarrow$  (4) are obvious.

(2)  $\Rightarrow$  (5) Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xhookrightarrow{i} & M \\ & & \downarrow f & & \\ E & \xrightarrow{h} & N & \longrightarrow & 0 \end{array}$$

where  $N$  and  $E$  are modules,  $K$  is a semi-artinian submodule of  $M$ ,  $h$  is an epimorphism and  $f$  is a homomorphism. We can assume that  $E$  is injective (see, e.g. [3, Proposition 5.2.10]). By  $SA$ - $M$ -injectivity of  $N$ ,  $f$  can be extended to a homomorphism  $g : M \rightarrow N$ . By projectivity of  $M$ , there is a homomorphism  $\tilde{g} : M \rightarrow E$  such that  $h \circ \tilde{g} = g$ . Let  $\tilde{f} : K \rightarrow E$  be the restriction of  $\tilde{g}$  over  $K$ . It is clear that  $h \circ \tilde{f} = f$ . Then  $K$  is projective.

(5)  $\Rightarrow$  (1) Let  $L$  and  $N$  be modules such that  $N$  is  $SA$ - $M$ -injective and  $h : N \rightarrow L$  is an epimorphism. If  $K \subseteq^{sa} M$  and  $f : K \rightarrow L$  is any homomorphism, then the hypothesis implies that  $K$  is projective and hence there is a homomorphism  $g : K \rightarrow N$  with  $h \circ g = f$ . By  $SA$ - $M$ -injectivity of  $N$ , there is a homomorphism  $\tilde{g} : M \rightarrow N$  with  $\tilde{g} \circ i = g$ . Let  $\beta = h \circ \tilde{g} : M \rightarrow L$ . Then  $\beta \circ i = h \circ \tilde{g} \circ i = h \circ g = f$ . and hence  $L$  is an  $SA$ - $M$ -injective module.

(1)  $\Rightarrow$  (3) Let  $K_1$  and  $K_2$  be two  $SA$ - $M$ -injective submodules of a module  $K$ . Thus  $K_1 + K_2$  is a homomorphic image of the direct sum  $K_1 \oplus K_2$ .  $SA$ - $M$ -injectivity of  $K_1 \oplus K_2$  and the hypothesis imply that  $K_1 + K_2$  is  $SA$ - $M$ -injective.

(4)  $\Rightarrow$  (2) Let  $F$  be an injective module with submodule  $D$ . Let  $B = F \oplus F$ ,  $L = \{(x, x) \mid x \in D\}$ ,  $\bar{B} = B/L$ ,  $K_1 = \{b + L \in \bar{B} \mid b \in F \oplus 0\}$ ,  $K_2 = \{b + L \in \bar{B} \mid b \in 0 \oplus F\}$ . Then  $\bar{B} = K_1 + K_2$ . Since  $(F \oplus 0) \cap L = 0$  and  $(0 \oplus F) \cap L = 0$ ,  $F \cong K_i$ ,  $i = 1, 2$ . Since  $K_1 \cap K_2 = \{b + L \in \bar{B} \mid b \in D \oplus 0\} = \{b + L \in \bar{B} \mid b \in 0 \oplus D\}$ ,  $K_1 \cap K_2 \cong D$  under  $b \mapsto b + L$  for all  $b \in D \oplus 0$ . By hypothesis,  $\bar{B}$  is  $SA$ - $M$ -injective. Injectivity of  $K_1$  implies that  $\bar{B} = K_1 \oplus A$  for some submodule  $A$  of  $\bar{B}$ , so  $A \cong (K_1 + K_2)/K_1 \cong K_2/K_1 \cap K_2 \cong F/D$ . By Proposition 2.3(5),  $F/D$  is  $SA$ - $M$ -injective.  $\square$

Theorem 2.11 implies the following result.

**Corollary 2.12.** *The following statements are equivalent.*

- (1) *The class  $SAI_R$  is closed under quotient.*
- (2) *Every quotient of an injective module is SA-injective.*
- (3) *For any module  $N$ , if  $N_1$  and  $N_2$  are submodules of  $N$  with  $N_1, N_2 \in SAI_R$ , then  $N_1 + N_2 \in SAI_R$ .*
- (4) *For any module  $N$ , if  $N_1$  and  $N_2$  are injective submodules of  $N$ , then  $N_1 + N_2 \in SAI_R$ .*
- (5) *If  $I \subseteq^{sa} R_R$ , then  $I$  is projective.*

**Theorem 2.13.** *If  $M$  is a finitely generated module, then the following statements are equivalent.*

- (1)  *$Sa(M)$  is noetherian.*
- (2) *The class of SA- $M$ -injective modules is closed under direct sums.*
- (3) *Direct sums of injective modules are SA- $M$ -injective.*
- (4) *If  $K$  is injective module, then  $K^{(S)}$  is SA- $M$ -injective for any index set  $S$ ,*
- (5) *If  $K$  is injective module, then  $K^{(\mathbb{N})}$  is SA- $M$ -injective.*

*Proof.* (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) Clear.

(1)  $\Rightarrow$  (2) Let  $E = \bigoplus_{i \in I} M_i$ , where  $M_i$  are SA- $M$ -injective modules and  $f : K \rightarrow E$  be a homomorphism with  $K \subseteq^{sa} M$ . Since  $Sa(M)$  is a noetherian module, we have  $K$  is finitely generated and hence  $f(K) \subseteq \bigoplus_{j \in I_1} M_j$ , for some finite subset  $I_1$  of  $I$  and hence  $\bigoplus_{j \in I_1} M_j$  is SA-injective. Then  $f$  can be extended to a homomorphism  $g : M \rightarrow E$  and so  $E$  is SA-injective.

(5)  $\Rightarrow$  (1) Let  $K_1 \subseteq K_2 \subseteq \dots$  be a chain of submodules of  $Sa(M)$ . For each  $i \geq 1$ , let  $F_i = E(M/K_i)$ ,  $F = \bigoplus_{i=1}^{\infty} F_i$  and  $M_i = \prod_{j=1}^{\infty} F_j = F_i \oplus (\prod_{j=1, j \neq i}^{\infty} F_j)$ , then  $M_i$  is injective. By hypothesis,  $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} F_i) \oplus (\bigoplus_{i=1}^{\infty} \prod_{j=1, j \neq i}^{\infty} F_j)$  is SA- $M$ -injective and hence Proposition 2.3(1) implies that  $F$  itself is SA- $M$ -injective.

Define  $f : H = \bigcup_{i=1}^{\infty} K_i \rightarrow F$  by  $f(a) = (a + K_i)_i$ . Clearly,  $f$  is a well defined homomorphism. Since  $Sa(M) \subseteq^{sa} M$  (by [9, p. 238]), we have  $\bigcup_{i=1}^{\infty} K_i \subseteq^{sa} M$  and hence  $f$  can be extended to a homomorphism  $g : M \rightarrow F$ . Since  $M$  is finitely generated, we have  $g(M) \subseteq \bigoplus_{i=1}^n E(M/K_i)$  for some  $n$  and hence  $f(\bigcup_{i=1}^{\infty} K_i) \subseteq \bigoplus_{i=1}^n E(M/K_i)$ . Since  $\pi_i f(x) = \pi_i(x + K_j)_{j \geq 1} = x + K_i$ , for all  $x \in H$  and  $i \geq 1$ , where  $\pi_i : \bigoplus_{j \geq 1} E(M/K_j) \rightarrow E(M/K_i)$  is the projection map,  $\pi_i f(H) = H/K_i$  for all  $i \geq 1$ . Since  $f(H) \subseteq \bigoplus_{i=1}^n E(M/K_i)$ ,  $H/K_i = \pi_i f(H) = 0$ ,

for all  $i \geq n+1$ , so  $H = K_i$  for all  $i \geq n+1$  and hence the chain  $K_1 \subseteq K_2 \subseteq \dots$  terminates at  $K_{n+1}$ . Thus  $\text{Sa}(M)$  is a noetherian module.  $\square$

**Proposition 2.14.** *The following statements are equivalent.*

- (1)  $\text{Sa}(R_R)$  is noetherian.
- (2) The class  $\text{SAI}_R$  is closed under direct sums.
- (3) Any direct sum of injective modules is  $\text{SA}$ -injective.
- (4) If  $K$  is injective module, then  $K^{(S)}$  is  $\text{SA}$ -injective for any index set  $S$ .
- (5) If  $K$  is injective module, then  $K^{(\mathbb{N})}$  is  $\text{SA}$ -injective.
- (6) The class  $\text{SAI}_R$  is closed under pure submodules.
- (7) All  $\text{FP}$ -injective modules are  $\text{SA}$ -injective.

*Proof.* By applying Theorem 2.13, we have the equivalent of (1), (2), (3), (4) and (5).

(1)  $\Rightarrow$  (6). Let  $N \in \text{SAI}_R$  and  $K$  a pure submodule of  $N$ . Let  $C \subseteq^{sa} R_R$ , thus the hypothesis implies that  $C$  is finitely generated and so  $R/C$  is a finitely presented. Hence the sequence  $\text{Hom}_R(R/C, N) \rightarrow \text{Hom}_R(R/C, N/K) \rightarrow 0$  is exact. By [8, Theorem XII.4.4 (4), p. 491], the sequence  $\text{Hom}_R(R/C, N) \rightarrow \text{Hom}_R(R/C, N/K) \rightarrow \text{Ext}^1(R/C, K) \rightarrow \text{Ext}^1(R/C, N)$  is exact. Thus  $\text{Ext}^1(R/C, K) = 0$  and hence  $K \in \text{SAI}_R$ . Therefore, the class  $\text{SAI}_R$  is closed under pure submodules.

(6)  $\Rightarrow$  (7). If  $M$  is any  $\text{FP}$ -injective module, then  $M$  is a pure submodule of a  $\text{SA}$ -injective module. By hypothesis,  $M \in \text{SAI}_R$ .

(7)  $\Rightarrow$  (1). Let  $I$  be a submodule of  $\text{Sa}(R_R)$ , thus  $I \subseteq^{sa} R_R$ . Let  $\alpha : I \rightarrow M$  be a homomorphism, where  $M$  is a  $\text{FP}$ -injective module. By hypothesis,  $M$  is  $\text{SA}$ -injective and hence  $\alpha$  extends to  $R_R$ . By [6],  $I$  is finitely generated and hence  $\text{Sa}(R_R)$  is a noetherian module.  $\square$

### 3. DEFINABILITY OF THE CLASS $\text{SAI}_R$

If  $\mathcal{X} \subseteq \text{Mod-}R$ , then we write  $\mathcal{X}^\ominus = \{M \in R\text{-Mod} \mid M^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{X}\}$  and  $\mathcal{X}^+ = \{M \in \text{Mod-}R \mid M \text{ is a pure submodule of a module in } \mathcal{X}\}$ .

**Lemma 3.1.** *The pair  $((\text{SAI}_R)^\ominus, \text{SAI}_R)$  is an almost dual pair over a ring  $R$ .*

*Proof.* By Corollary 2.4 and [12, Proposition 4.2.11, p. 72].  $\square$

**Corollary 3.2.** *Consider the following conditions for the class  $\text{SAI}_R$  over a ring  $R$ .*

- (1) The class  $\text{SAI}_R$  is definable.
- (2)  $(\text{SAI}_R, (\text{SAI}_R)^\ominus)$  is an almost dual pair over a ring  $R$ .
- (3)  $(\text{SAI}_R)^* \subseteq (\text{SAI}_R)^\ominus$ .
- (4)  $(\text{SAI}_R)^{**} \subseteq \text{SAI}_R$ .
- (5) The class  $\text{SAI}_R$  is closed under pure homomorphic images.

*Then (1)  $\Leftrightarrow$  (2), (1)  $\Rightarrow$  (3), (1)  $\Rightarrow$  (5) and (3)  $\Leftrightarrow$  (4). Moreover, if  $\text{Sa}(R_R)$  is noetherian, then all five conditions are equivalent.*



*Proof.* (1)  $\Leftrightarrow$  (2). By Lemma 3.1 and [12, Proposition 4.3.8, p. 89].

(1)  $\Rightarrow$  (3). Since  $SAI_R$  is a definable class, it is closed under pure submodules and hence  $(SAI_R)^+ = SAI_R$ . Since  $((SAI_R)^\ominus, SAI_R)$  is an almost dual (by Lemma 3.1), it follows from [12, Theorem 4.3.2, p. 85], that  $(SAI_R)^* \subseteq (SAI_R)^\ominus$ .

(1)  $\Rightarrow$  (5). By [16, 3.4.8, p. 109].

(3)  $\Rightarrow$  (4). By Lemma 3.1 and [12, Theorem 4.3.2, p. 85].

(4)  $\Rightarrow$  (1) and (5)  $\Rightarrow$  (1). Suppose that  $\text{Sa}(R_R)$  is a noetherian module. By Proposition 2.14, the class  $SAI_R$  is closed under pure submodules and hence  $(SAI_R)^+ = SAI_R$ . Thus the results follow from [12, Theorem 4.3.2, p. 85].  $\square$

**Corollary 3.3.** *If every SA-injective modules is pure-injective, then the following statements are equivalent for a class  $SAI_R$  over a ring  $R$ .*

- (1)  $SAI_R$  is definable.
- (2) The class  $SAI_R$  is closed under direct sums.
- (3)  $(SAI_R)^+ = SAI_R$
- (4)  $\text{Sa}(R_R)$  is a noetherian module.

*Proof.* By Proposition 2.14, Lemma 3.1 and [12, Theorem 4.5.1, p. 103].  $\square$

If  $A$  is a right  $R$ -module and  $B$  is a left  $R$ -module, then  $\text{Tor}_1(A, B)$  is defined as the first left derived functor of the tensor product  $A \otimes_R B$  (see [4, Ch. VI] for more details).

**Lemma 3.4.** *A left  $R$ -module  $M \in (SAI_R)^\ominus$  iff  $\text{Tor}_1(R/I, M) = 0$ , for any semi-artinian right ideal  $I$  of a ring  $R$ .*

*Proof.* Let  $M$  be a left  $R$ -module and  $I \subseteq^{sa} R_R$ . By [7, Theorem 3.2.1, p. 75],  $\text{Ext}^1(R/I, M^*) \cong (\text{Tor}_1(R/I, M))^*$ , so that  $\text{Tor}_1(R/I, M) = 0$  if and only if  $M^* \in SAI_R$ . Hence  $({}_R\text{SAF}, SAI_R)$  is an almost dual, where  ${}_R\text{SAF} = \{M \in R\text{-Mod} \mid \text{Tor}_1(R/I, M) = 0, \text{ for any semi-artinian right ideal } I \text{ of a ring } R\}$ . By [12, Proposition 4.2.11, p. 72],  $(SAI_R)^\ominus = {}_R\text{SAF}$ .  $\square$

A module  $M$  is called  $n$ -presented if there is an exact sequence  $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$ , with each  $F_i$  is a finitely generated free modules [5].

**Theorem 3.5.** *The following statements are equivalent for a class  $SAI_R$  over a ring  $R$ .*

- (1)  $SAI_R$  is definable.
- (2) The class  $SAI_R$  is closed under pure submodules and pure homomorphic images.
- (3) Every semi-artinian right ideal in  $R$  is finitely presented.
- (4) A module  $M \in SAI_R$  iff  $M^* \in (SAI_R)^\ominus$ .
- (5) A module  $M \in SAI_R$  iff  $M^{**} \in SAI_R$ .

*Proof.* (1)  $\Rightarrow$  (2). By [16, 3.4.8, p. 109].

(2)  $\Rightarrow$  (3). Let  $N$  be any  $FP$ -injective module, thus there is an injective module  $H$  with pure exact sequence  $0 \rightarrow N \xrightarrow{i} H \xrightarrow{\pi} H/N \rightarrow 0$ . By hypothesis,  $H/N \in SAI_R$ . Let  $K \subseteq^{sa} R_R$ , thus  $\text{Ext}^1(R/K, H/N) = 0$ . By [8, Theorem 4.4 (4), p. 491], the sequence  $0 = \text{Ext}^1(R/K, H/N) \rightarrow \text{Ext}^2(R/K, N) \rightarrow \text{Ext}^2(R/K, H) = 0$  is exact and hence  $\text{Ext}^2(R/K, N) = 0$ . By [8, Theorem 4.4 (3), p. 491], the sequence  $0 = \text{Ext}^1(R, N) \rightarrow \text{Ext}^1(K, N) \rightarrow \text{Ext}^2(R/K, N) = 0$  is exact, so that  $\text{Ext}^1(K, N) = 0$ . By hypothesis,  $SAI_R$  is closed under pure submodules, so that  $K$  is finitely generated by Proposition 2.14 and hence [6, Proposition, p. 361] implies that  $K$  is finitely presented.

(3)  $\Rightarrow$  (1). Let  $M \in SAI_R$ . Let  $K \subseteq^{sa} R_R$ , thus  $K$  is finitely presented (by hypothesis) and hence there is an exact sequence  $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} K \rightarrow 0$ , where  $F_1, F_2$  are finitely generated free modules. Let  $\beta = i\alpha_1$ , where  $i : K \rightarrow R$  is the inclusion mapping, thus the sequence  $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/K \rightarrow 0$  is exact, where  $\pi : R \rightarrow R/K$  is the natural epimorphism. Hence  $R/K$  is a 2-presented module, so that from [5, Lemma 2.7 (2)] we have  $\text{Tor}_1(R/K, M^*) \cong (\text{Ext}^1(R/K, M))^* = 0$ . By Lemma 3.4,  $M^* \in (SAI_R)^\ominus$  and hence  $(SAI_R)^* \subseteq (SAI_R)^\ominus$ . By hypothesis, every semi-artinian right ideal in  $R$  is finitely generated, so that  $\text{Sa}(R_R)$  is noetherian. By Corollary 3.2,  $SAI_R$  is a definable class.

(1)  $\Rightarrow$  (4). By Corollary 3.2,  $(SAI_R, (SAI_R)^\ominus)$  is an almost dual pair and hence a module  $M \in SAI_R$  iff  $M^* \in (SAI_R)^\ominus$ .

(4)  $\Rightarrow$  (5). By hypothesis,  $(SAI_R)^* \subseteq (SAI_R)^\ominus$ . By Corollary 3.2,  $(SAI_R)^{**} \subseteq SAI_R$ . Hence for any module  $M$ , if  $M \in SAI_R$ , then  $M^{**} \in SAI_R$ . Conversely, if  $M^{**} \in SAI_R$ , then  $M^* \in (SAI_R)^\ominus$ . By hypothesis,  $M \in SAI_R$ .

(5)  $\Rightarrow$  (1). Let  $N$  be a  $FP$ -injective module, thus there is a pure exact sequence  $0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0$ , where  $E$  is an injective module. By [21, 34.5, p. 286], the sequence  $0 \rightarrow N^{**} \rightarrow E^{**} \rightarrow (E/N)^{**} \rightarrow 0$  is split. By hypothesis,  $E^{**} \in SAI_R$  and hence  $N^{**} \in SAI_R$ . By hypothesis,  $N \in SAI_R$  so that  $\text{Sa}(R_R)$  is noetherian by Proposition 2.14. Thus  $SAI_R$  is definable class by Corollary 3.2.  $\square$

Note that if the class  $SAI_R$  is closed under pure submodules, then  $(SAI_R)^+ = SAI_R$ . Thus we have the following corollary.

**Corollary 3.6.** *The class  $SAI_R$  is a definable if and only if it is closed under pure submodules and the class  $(SAI_R)^+$  is a definable.*

**Corollary 3.7.** *If the class  $SAI_R$  is a definable, then the following are equivalent.*

- (1) *The class of flat left  $R$ -modules and the class  $(SAI_R)^\ominus$  are coincide.*
- (2) *Every module in  $SAI_R$  is  $FP$ -injective.*
- (3) *Every pure-injective module in  $SAI_R$  is injective.*

*Proof.* (1)  $\Rightarrow$  (2). Let  $M \in \text{SAI}_R$ , thus  $M^* \in (\text{SAI}_R)^\ominus$  by Corollary 3.2. By hypothesis,  $M^*$  is a flat left  $R$ -module and hence [10, Theorem, p. 239] implies that  $M^{**}$  is injective. Since  $M$  is a pure submodule in  $M^{**}$ , we have  $M$  is  $FP$ -injective by [21, 35.8, p. 301].

(2)  $\Rightarrow$  (3). Let  $M$  be any pure-injective module in  $\text{SAI}_R$ . Let  $\mathcal{E} : 0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$  be an exact sequence. By hypothesis,  $M$  is  $FP$ -injective. By [17, Proposition 2.6], the sequence  $\mathcal{E}$  is pure and hence pure-injectivity of  $M$  implies that the sequence  $\mathcal{E}$  is split by [21, 33.7, p. 279]. Therefore,  $M$  is injective.

(3)  $\Rightarrow$  (1). Let  $M$  be a flat left  $R$ -module, thus  $\text{Tor}_1(N, M) = 0$ , for any right  $R$ -module  $N$ . By Lemma 3.4,  $M \in (\text{SAI}_R)^\ominus$ . Conversely, if  $M \in (\text{SAI}_R)^\ominus$ , then  $M^* \in \text{SAI}_R$ . By [16, Proposition 4.3.29, p. 149],  $M^*$  is a pure injective module. By hypothesis,  $M^*$  is injective and hence  $M$  is flat by [10, Theorem, p. 239].  $\square$

#### 4. RELATIONS BETWEEN $SA$ -INJECTIVITY AND CERTAIN GENERALIZATIONS OF INJECTIVITY

A right  $R$ -module  $M$  is called quasi-injective if, for every submodule  $N$  of  $M$ , every right  $R$ -homomorphism from  $N$  to  $M$  can be extended to a right  $R$ -endomorphism of  $M$  [3, p. 169].

In general, if  $M$  is  $SA$ -injective right  $R$ -module, then  $M$  need not be quasi-injective, for example  $\mathbb{Z}$  as  $\mathbb{Z}$ -module is  $SA$ -injective (by Example 2.2(1)) but it is not quasi-injective. Also, the converse is not true in general, for example in the ring  $\mathbb{Z}_4$ , the ideal  $I = \langle 2 \rangle$  is a quasi-injective  $\mathbb{Z}_4$ -module but it is not  $SA$ -injective  $\mathbb{Z}_4$ -module.

The following theorem gives a relation between  $SA$ -injective modules and quasi-injective modules.

**Theorem 4.1.** *The following statements are equivalent for a ring  $R$ .*

- (1)  $R$  is a right semi-artinian ring.
- (2) Every  $SA$ -injective right  $R$ -module is injective.
- (3) Every  $SA$ -injective right  $R$ -module is quasi-injective.
- (4) Every cyclic  $SA$ -injective right  $R$ -module is quasi-injective.

*Proof.* (1)  $\Rightarrow$  (2) Let  $M$  be any  $SA$ -injective right  $R$ -module. Let  $I$  be any right ideal of a ring  $R$  and  $f : I \rightarrow M$  be any right  $R$ -homomorphism. Since  $R$  is a right semi-artinian ring (by hypothesis), it follows from [9, Exercise 7(8), p. 238] that  $I$  is a semi-artinian right ideal of  $R$ . Since  $M$  is an  $SA$ -injective right  $R$ -module (by hypothesis),  $f$  extends to  $R$  and hence  $M$  is an injective right  $R$ -module.

(2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (4) are clear.

(4)  $\Rightarrow$  (1) Let  $M$  be any nonzero cyclic right  $R$ -module. We will prove that  $\text{Soc}(M) \neq 0$ . Assume that  $\text{Soc}(M) = 0$ . Let  $N$  be a nonzero submodule of  $M$ . Thus  $\text{Soc}(N) = 0$  and hence from Example 2.2(4) that  $M$  and  $N$  are  $SA$ -injective right  $R$ -modules. By Corollary 2.4,  $N \oplus M$  is an  $SA$ -injective right  $R$ -module. By hypothesis,  $N \oplus M$  is a quasi-injective right  $R$ -module. By [15, Proposition 1.17, p. 8],  $N$  is an  $M$ -injective right  $R$ -module and hence  $N$  is a direct summand of  $M$ . Thus  $M$  is semisimple and hence  $M = \text{Soc}(M) = 0$  and this is a contradiction. Thus  $\text{Soc}(M) \neq 0$  for any nonzero cyclic right  $R$ -module  $M$  and hence from [18, p. 183] we have that  $R$  is a right semi-artinian ring.  $\square$

Since every left perfect ring is right semi-artinian [9, Theorem 11.6.3, p. 294], we have the following corollary immediately from Theorem 4.1.

**Corollary 4.2.** *If  $R$  is a left perfect ring, then every  $SA$ -injective right  $R$ -module is injective (quasi-injective).*

In the following proposition, we give another connection between  $SA$ -injective modules and quasi-injective modules.

**Proposition 4.3.** *A commutative ring  $R$  is semisimple if and only if  $R$  is a semi-artinian ring and every quasi-injective  $R$ -module is  $SA$ -injective.*

*Proof.* ( $\Rightarrow$ ) By Corollary 2.10.

( $\Leftarrow$ ) Let  $M$  be any quasi-injective  $R$ -module. By hypothesis,  $M$  is  $SA$ -injective. Since  $R$  is a semi-artinian ring (by hypothesis), it follows from Theorem 4.1 that  $M$  is injective and hence from [19, Corollary 2.2] we get that  $R$  is a semisimple ring.  $\square$

The following corollary is immediately from Theorem 4.1 and Proposition 4.3.

**Corollary 4.4.** *The following statements are equivalent for a commutative ring  $R$ .*

- (1)  $R$  is semisimple.
- (2) For each  $R$ -module  $M$ ,  $M$  is  $SA$ -injective if and only if it is quasi-injective.

A right  $R$ -module  $M$  is called  $P$ -injective (resp.  $F$ -injective) if, for every principally (resp. finitely generated) right ideal  $I$  of  $R$ , every right  $R$ -homomorphism from  $I$  to  $M$  can be extended to a right  $R$ -homomorphism from  $R$  into  $M$  (see, for example [11] and [22]).

If  $M$  is  $SA$ -injective right  $R$ -module, then  $M$  need not be  $P$ -injective (resp.  $F$ -injective) in general, for example  $\mathbb{Z}$  as  $\mathbb{Z}$ -module is  $SA$ -injective (by Example 2.2(1)) but it is not  $P$ -injective (resp.  $F$ -injective). Also, the converse is not true in general, for example: let  $F = \mathbb{Z}_2$  be the field of two elements,  $F_n = F$  for  $n = 1, 2, \dots$ ,  $Q = \prod_{i=1}^{\infty} F_i$ ,  $S = \bigoplus_{i=1}^{\infty} F_i$ . If  $R$  is the subring of  $Q$  generated

by 1 and  $S$ , then  $R$  is a F-injective right  $R$ -module (by [1, Example 4.5]) and hence  $R_R$  is a P-injective module. Thus Example 4.5 in [1] implies that  $R$  is not a soc-injective right  $R$ -module and so  $R$  is not a  $SA$ -injective module. Thus  $R$  is F-injective (P-injective) right  $R$ -module but it is not  $SA$ -injective.

The following proposition gives a condition under which every F-injective right  $R$ -module is  $SA$ -injective.

**Proposition 4.5.** *Let  $R$  be a ring. Then  $Sa(R_R)$  is a noetherian right  $R$ -module if and only if every F-injective right  $R$ -module is  $SA$ -injective.*

*Proof.* ( $\Rightarrow$ ) Let  $M$  be any F-injective right  $R$ -module. Let  $I$  be a semi-artinian right ideal of  $R$  and let  $f : I \rightarrow M$  be any right  $R$ -homomorphism. Since  $Sa(R_R)$  is noetherian and  $I \subseteq Sa(R_R)$ , it follows that  $I$  is a finitely generated right ideal. By F-injectivity of  $M$ ,  $f$  extends to a right  $R$ -homomorphism from  $R$  into  $M$  and hence  $M$  is  $SA$ -injective.

( $\Leftarrow$ ) Let  $\{M_i\}_{i \in I}$  be a family of injective right  $R$ -modules. Thus  $M_i$  are F-injective modules. By [22, Proposition 2.1(c)],  $\bigoplus_{i \in I} M_i$  is an F-injective module. By hypothesis,  $\bigoplus_{i \in I} M_i$  is a  $SA$ -injective module and hence from Proposition 2.14 we get that  $Sa(R_R)$  is a noetherian right  $R$ -module.  $\square$

Directly from Proposition 4.5 and Proposition 2.14, we have the following corollary.

**Corollary 4.6.** *Let  $R$  be a ring. Then every F-injective right  $R$ -module is  $SA$ -injective if and only if every FP-injective right  $R$ -module is  $SA$ -injective.*

A ring  $R$  is called (von Neumann) regular if for any  $a \in R$ , there is  $b \in R$  such that  $a = aba$  [9, p. 38].

**Proposition 4.7.** *The following statements are equivalent.*

- (1)  $R$  is a (von Neumann) regular ring and every P-injective right  $R$ -module is  $SA$ -injective.
- (2)  $R$  is a (von Neumann) regular ring and  $Sa(R_R)$  is a noetherian right  $R$ -module.
- (3) Every  $SA$ -injective right  $R$ -module is P-injective and every semi-artinian right ideal of  $R$  is a direct summand of  $R_R$ .

*Proof.* (1)  $\Rightarrow$  (2) Since every F-injective right  $R$ -module is P-injective, we have from hypothesis that every F-injective right  $R$ -module is  $SA$ -injective. By Proposition 4.5,  $Sa(R_R)$  is a noetherian right  $R$ -module.

(2)  $\Rightarrow$  (3) Since  $R$  is a (von Neumann) regular ring, it follows from [14, Lemma 2] that every  $SA$ -injective right  $R$ -module is P-injective. Let  $I$  be any semi-artinian right ideal of  $R$ . Thus  $I \subseteq Sa(R_R)$ . Since  $Sa(R_R)$  is a noetherian right  $R$ -module (by hypothesis), we have that  $I$  is a finitely generated right ideal. By [9, Exercise 13, p. 38],  $I$  is a direct summand of  $R_R$ .

(3)  $\Rightarrow$  (1) Since every semi-artinian right ideal of  $R$  is a direct summand of  $R_R$  (by hypothesis), it follows that from Corollary 2.8 that every right  $R$ -module is  $SA$ -injective and hence every  $P$ -injective right  $R$ -module is  $SA$ -injective. Since every  $SA$ -injective right  $R$ -module is  $P$ -injective (by hypothesis), we have that every right  $R$ -module is  $P$ -injective. By [14, Lemma 2],  $R$  is a (von Neumann) regular ring.  $\square$

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