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(C, C')-Controlled g-Fusion Frames in Hilbert Spaces

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Abstract. Controlled frames in Hilbert spaces have been recently introduced by P. Balazs and etc. for improving the numerical efficiency of interactive algorithms for inverting the frame operator. In this paper we develop a theory based on g-fusion frames on Hilbert spaces, which provides exactly the frameworks not only to model new frames on Hilbert spaces but also for deriving robust operators. In particular, we can define analysis, synthesis and frame operators with representation space compatible for (C, C')-Controlled g-fusion frames, which even yield a reconstruction formula. Also, some useful concepts such as Q-dual and perturbation are introduced and investigated.

Keywords: G-fusion frame, Controlled fusion frame, Controlled g-fusion frame, Q-dual.

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1. Introduction

Frames, as a generalization of the bases in Hilbert spaces, were first introduced by Duffin and Schaeffer [5] during their study of nonharmonic Fourier series in 1952. Controlled frames for spherical wavelets were introduced in [2] for obtaining a numerically more efficient approximation algorithm and the related theory for general frames were developed in [1]. Also, Controlled frames as generalization of frames, have been introduced for getting an improved solution of a linear system of the equation $Ax = B$, which can be solved by the equation $PAx = PB$, where P is a suitable matrix for getting a better duplicate algorithm [2]. Recent developments in this direction can be found in [8, 9, 10, 11, 12].

Throughout this paper H and K are separable Hilbert spaces, $\{H_j\}_{j\in\mathbb{I}}$ is a sequence of Hilbert spaces and $I \subseteq \mathbb{Z}$. We denote by $\mathcal{B}(H, K)$ the set of all the bounded and linear operators from H to K. If $H = K$, then $\mathcal{B}(H, H)$ will be denoted as $\mathcal{B}(H)$. Also, $GL(H)$ is called the set of all bounded linear operators which have bounded inverses on H. It is easy to check that if $C, C' \in GL(H)$, then C^*, C^{-1} and CC' are in $GL(H)$. Assume that Id_H is the identity operator on H and π_W is the orthogonal projection from H onto a closed subspace $V \subseteq H$.

2. Preliminaries

In this section, some necessary definitions and lemmas are introduced.

Definition 2.1. A sequence $\{f_i\}_{i\in\mathbb{I}}$ in H is a frame if there exist constants $0 < A \leqslant B < \infty$ such that for all $f \in H$

$$A\left\|f\right\|^2 \leq \sum_{i\in\mathbb{I}} \left|\langle f, f_i\rangle\right|^2 \leq B\left\|f\right\|^2.$$

The constants A, B are frame bounds; A is the lower bound and B is the upper bound. The frame is thight if $A = B$, it is called a Parseval frame if $A = B = 1$. If we only have the upper bound, We call $\{f_i\}_{i\in\mathbb{I}}$ a Bessel sequence. If $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence then the following operators are bounded:

$$T : l^2(I) \longmapsto H,$$
$$T(c_i) = \sum_{i\in\mathbb{I}} c_i f_i$$

$$T^* : H \longmapsto l^2(I),$$
$$T^* f = \{\langle f, f_i\rangle\}_{i\in\mathbb{I}}$$

$$S : H \longmapsto H,$$
$$Sf = TT^* f = \sum_{i\in\mathbb{I}} \langle f, f_i\rangle f_i.$$

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These operators are called synthesis operator; analysis operator and frame operator, respectively. The representation space employed in this setting is

$$l^2(I) = \Big\{ \{c_i\}_{i\in\mathbb{I}} : c_i \in \mathbb{C}, \sum_{i\in\mathbb{I}} \|c_i\|^2 < \infty \Big\}.$$

Definition 2.2. Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. We say $\Lambda := (W_i, \Lambda_i, v_i)$ is a g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 \leq B\|f\|^2.$$

We call Λ a Parseval g-fusion frame if $A = B = 1$. When the right hand of above inequality holds, Λ is called a g-fusion Bessel sequence for H with bound B. We define the space

$$\mathscr{H}_2 := (\sum_{j\in\mathbb{J}} \bigoplus H_j)_{l_2}$$

by:

$$\mathscr{H}_2 = \big\{ \{f_j\}_{j\in\mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty \big\}.$$

with the inner product defined by

$$\langle \{f_j\}, \{g_j\} \rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j \rangle.$$

It is clear that $\mathscr{H}_2$ is a Hilbert space with pointwise operations. Suppose that Λ be a g-fusion frame for H. Then The synthesis and analysis operator are denoted by (for more details refer to [13])

$$T_{\Lambda'} : \mathscr{H}_2 \longrightarrow H,$$
$$T_{\Lambda'}(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j$$

and

$$T^*_{\Lambda'} : H \longrightarrow \mathscr{H}_2,$$
$$T^*_{\Lambda'} f : \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\mathbb{J}}.$$

Now, the g-fusion frame operator is defined by

$$S_{\Lambda'} : H \longrightarrow H,$$
$$S_{\Lambda'} f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f.$$

and

$$\langle S_{\Lambda'} f, f \rangle = \sum_{j\in\mathbb{J}} v_j^2 \left\|\Lambda_j \pi_{W_j} f\right\|^2.$$

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Therefore

$$AI \leq S_{\Lambda'} \leq BI.$$

This means that $S_{\Lambda'}$ is a bounded, positive and invertible operator . So, we have the reconstruction formula for any $f \in H$:

$$f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} S_{\Lambda'}^{-1} f = \sum_{j\in\mathbb{J}} v_j^2 S_{\Lambda'}^{-1} \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f.$$

Lemma 2.3. *([13]) Λ is a g-fusion frame for H if and only if the operator*

$$S_{\Lambda'} : f \longrightarrow \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f$$

is a well-defined, bounded and surjective.

Lemma 2.4. *([1]) Let $u : H \to H$ be a linear operator. Then the following statements are equivalent:*

1. *for some $0 < A \leq B < \infty$ we get $AI \leq u \leq BI$, ;*
2. *u is positive and $A\|f\|^2 \leq \left\|u^{\frac{1}{2}} f\right\|^2 \leq B\|f\|$, for some $0 < A \leq B < \infty$;*
3. *$u \in GL^+(H)$.*

Definition 2.5. A sequence $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i) : i \in \mathbb{I}\}$ is called generalized frame, or simply a g-frame, for H with respect to $\{H_i : i \in I\}$ if there exist constants$A > 0$ and $B < 0$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2 .$$

The numbers A and B are called g-frame bounds. $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i) : i \in \mathbb{I}\}$ is called tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$. If the second part of the above inequality holds , the sequence is called g-Bessel sequence.

Definition 2.6. Let $C, C' \in GL^+(H)$. The family $\Lambda = \{\Lambda_i \in \mathcal{B}(H, H_i) : i \in \mathbb{I}\}$ will be called a (C, C')-controlled g-frame for H, if Λ is a g-Bessel sequence and there exist constants $A > 0$ and $B < 0$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \langle \Lambda_i C' f, \Lambda_i C f\rangle \leq B\|f\|^2 .$$

A and B will be called controlled frame bounds . If the second part of the above inequality holds, it will be called (C, C')-controlled g-Bessel sequence with bound B.

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3. (C,C')-Controlled g-Fusion Frames

Definition 3.1. Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$. Let $\{H_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert spaces, $C, C' \in GL(H)$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. $\Lambda_{CC'} := (W_i, \Lambda_i, v_i)$ is a (C,C')-controlled g-fusion frame (briefly CC'-GF) for H if there exist constants $0 < A \le B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \le B\|f\|^2.$$

We call $\Lambda_{CC'}$ is a Parseval CC'-GF if $A = B = 1$. If only the second Inequality is required, We call $\Lambda_{CC'}$ is a (C,C')-Controlled Bessel g-fusion sequence (briefly CC'-GBS) with bound B.

If we assume $\Lambda_{CC'}$ is a CC'-GF for H and $C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'$ is a positive operator for each $i \in \mathbb{I}$, then $C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' = C'^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C$ and therefore

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \left\|(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{1/2} f\right\|^2 \le B\|f\|^2.$$

Let

$$\mathcal{K}^2_{\Lambda_j} := \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}} f \ : \ f \in H\} \subset (\bigoplus_{i\in\mathbb{I}} H)_{l^2}.$$

It is easy to check that $\mathcal{K}^2_{\Lambda_j}$ is a closed subspace. We can define the *controlled analysis operator* T^*_Λ by

$$T^*_\Lambda : H \to \mathcal{K}^2_{\Lambda_j},$$
$$T^*_\Lambda f = \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}} f\}_{i\in\mathbb{I}}.$$

It is easy to check that the controlled analysis operator is bounded linear operator. Thus, $T_\Lambda := (T^*_\Lambda)^*$ is well-defined and bounded and the *controlled synthesis operator* T_Λ can be defined by

$$T_\Lambda : \mathcal{K}^2_{\Lambda_j} \to H,$$
$$T_\Lambda\Big(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}} f\Big) = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' f.$$

Now, we can define the CC'-GF operator $S_{CC'}$ on H by

$$S_{CC'} f := T_\Lambda T^*_\Lambda f = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' f.$$

We can write for each $f \in H$

$$\begin{aligned}\langle S_{CC'} f, f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' f, f\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C' f, \Lambda_i\pi_{W_i} C f\rangle,\end{aligned}$$

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therefore, we get

$$AId_H \leq S_{CC'} \leq BId_H.$$

Theorem 3.2. *$\Lambda_{CC'}$ be a CC'-GBS for H with bound B if and only if the operator*

$$T_\Lambda : \mathcal{K}^2_{\Lambda_j} \to H,$$
$$T_\Lambda(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f) = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f.$$

is well -defined and bounded operator with $\|T_\Lambda\| \leq \sqrt{B}$.

Proof. The necessary condition follows from the definition of CC'-GBS. We only need to prove that the sufficient condition holds. Let T_Λ be well-defined and bounded operator with $\|T_\Lambda\| \leq \sqrt{B}$. For any $f \in H$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f, f\rangle \\ &= \langle T_\Lambda\big(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\big), f\rangle \\ &\leq \|T_\Lambda\| \left\|(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\right\| \|f\|.\end{aligned}$$

But

$$\left\|(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|^2 = \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle.$$

It follows that

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle \leq B\|f\|^2$$

and this means that $\Lambda_{CC'}$ is a CC'-GBS for H. □

Theorem 3.3. *Let $C \in GL^+(H)$. $\Lambda := (W_i, \Lambda_i, v_i)$ is a g-fusion frame for H if and only if Λ is a CC-GF.*

Proof. Suppose that Λ is a CC-GF with Bounds A and B for H. for each $f \in \mathbb{H}$, we obtain

$$\begin{aligned}A\|f\|^2 &= A\left\|CC^{-1}f\right\|^2 \\ &\leq A\|C\|^2 . \left\|C^{-1}f\right\|^2 \\ &\leq \|C\|^2 \sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i\pi_{W_i}CC^{-1}f\right\|^2 \\ &= \|C\|^2 \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i\pi_{W_i}f\|^2\end{aligned}$$

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Hence

$$A\|C\|^{-2}\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 .$$

On the other hand, for any $f \in H$, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 &= \sum_{i\in\mathbb{I}} v_i^2 \left\|\Lambda_i \pi_{W_i} CC^{-1} f\right\|^2 \\ &\leq B\left\|C^{-1}f\right\|^2 \\ &\leq B\left\|C^{-1}\right\|^2 . \|f\|^2\end{aligned}$$

Thus, Λ is a g-fusion frame for H with bounds $A\|C\|^{-2}$,$B\left\|C^{-1}\right\|^2$.

Conversely, assume that Λ is a g-fusion frame for H with bounds A, B. Then, for each $f \in H$ we get

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} Cf, \Lambda_i \pi_{W_i} Cf\rangle = \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} Cf\|^2 \leq B\|C\|^2\|f\|^2 .$$

For the lower bound, we can write for any $f \in H$,

$$\begin{aligned}A\|f\|^2 &= A\left\|C^{-1}Cf\right\|^2 \\ &\leq A\left\|C^{-1}\right\|^2\|Cf\|^2 \\ &\leq \left\|C^{-1}\right\|^2 \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} Cf\|^2 .\end{aligned}$$

Therefore, Λ is a CC-GF for H with bounds $A\left\|C^{-1}\right\|^{-2}, B\left\|C^{-1}\right\|^{-2}$. □

Theorem 3.4. *Let $\Lambda_{CC'}$ be a CC'-GF for H with bounds A, B. Then, $\Lambda_{CC'}$ is a g-fusion frame for H. Furthermore, if $S_{\Lambda'}$ is its g-fusion frame operator, then $C^{-1}S_{\Lambda'}C' = C'^{*}S_{\Lambda'}C$.*

Proof. We define

$$\begin{aligned}S :&H \longrightarrow H, \\ Sf :=&C^{*^{-1}} S_{CC'} C'^{^{-1}}(f) = \sum_{i\in\mathbb{I}} v_i^2 \pi_{W_i}\Lambda_i^*\Lambda_i \pi_{W_i} f.\end{aligned}$$

It is easy to check that S is well-defined. Let $f \in H$, since $GL(H)$ is a C^*-subalgebra in $\mathcal{B}(H)$, so $(C^*)^{-1} = (C^{-1})^*$, we have

$$\|S\| = \sup_{\|g\|=1} \|Sg\| \leq B\left\|C^{-1}\right\|\|C'\| ,$$

so, S is bounded and

$$\sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 = \|\langle Sf, f\rangle\| \leq B\left\|C^{-1}\right\|\|C'\|\|f\|^2 .$$

Let $g \in H$ and define $f := C' S_{CC'}^{-1} C^{-1} g$. Then S is surjective and by Lemma 2.3, $\Lambda_{CC'}$ is a g-fusion frame in H and its g-fusion frame operator is $S_{\Lambda'} := S$. So, we get

$$C^{-1} S_{\Lambda'} C' = S_{CC'} = S_{CC'}^* = C'^* S_{\Lambda'} C.$$

□

Theorem 3.5. *Let $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$ and $\Theta_{CC'} = (W_i, \Theta_i, v_i)$ be two CC'-BGF for H with bounds B_1 and B_2, respectively. Suppose that T_Λ and T_Θ be their controlled analysis operators such that $T_\Theta T_\Lambda^* = Id_H$. Then , both $\Lambda_{CC'}$ and $\Theta_{CC'}$ are CC'-GF for H.*

Proof. For each $f \in H$, we have

$$\begin{aligned}
\|f\|^4 &= \langle f, f\rangle^2 \\
&= \langle T_\Lambda^* f, T_\Theta^* f\rangle^2 \\
&\leq \|T_\Lambda^* f\|^2 . \|T_\Theta^* f\|^2 \\
&= \Big(\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle\Big)\Big(\sum_{i\in\mathbb{I}} v_i^2 \langle \Theta_i \pi_{W_i} C' f, \Theta_i \pi_{W_i} C f\rangle\Big) \\
&\leq \Big(\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle\Big) B_2 \|f\|^2 .
\end{aligned}$$

Thus

$$B_2^{-1} \|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle,$$

and $\Lambda_{CC'}$ is a CC'-GF for H. Similarly ,$\Theta_{CC'}$ is a CC'-GF with the lower bound B_1^{-1} □

Theorem 3.6. *Let $\Lambda = (W_i, \Lambda_i, v_i)$ be a g-fusion frame for H. Then $\Lambda_{CC'}$ is a Parseval CC'-GF for H if and only if There exist a operator $L : H \to H$ such that $C = UL^{-q}$ and $C' = VL^{-p}$, where U, V are two operators on H such that $VU^* = Id_H$ and $p, q \in \mathbb{R}$, $p + q = 1$.*

Proof. Let $\Lambda_{CC'}$ be a Parseval CC'-GF for H. So $S_{CC'} = Id_H$. Therefore , for each pairs of real numbers p, q such that $p + q = 1$, we obtain

$$Id_H = S_{CC'} = C' L C^* = C' L^p L^q C^*.$$

We define $V := C' L^p$ and $U := CL^q$. So

$$VU^* = C' L^p L^q C^* = C' L C^* = S_{CC'} = Id_H.$$

Conversely, Let V, U be two operators on H such that $VU^* = Id_H$. We define $C := UL^{-q}$ and $C' := VL^{-p}$ be two operators on H where $p, q \in R$ and $p + q = 1$. So for any $f \in H$:

$$f = C' L C^*(f) = \sum_{i\in\mathbb{I}} v_i^2 C^* \pi_{W_i} \Lambda_i^* \Lambda_i \pi_{W_i} C' f$$

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Therefore , $\Lambda_{CC'}$ is a parseval CC'-GF for H. □

As a special case of this theorem we have the following well-known result which is the most basic result for generating a Parseval frame. See Lemma 1.7 in [3].

4. Q-Duality and Perturbation of (C, C')-Controlled g-fusion frame

This section is devoted to studying the behavior of the canonical dual of a g-fusion frame under perturbations. We consider perturbations of CC'-GF in analogy to the perturbations in [7] for frames, Also Q-duals are useful tools for establishing reconstruction formula. For more information we refer the reader to [6, 7].

Definition 4.1. Assume that $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$ be a CC'-GF for H. We call a CC'-GBS as $\Theta_{CC'} := (W_i, \Theta_i, v_i)$ the Q-dual CC'-GF of $\Lambda_{CC'}$, if there exist a bounded linear operator $Q : \mathcal{K}^2_{\Lambda_j} \longrightarrow \mathcal{K}^2_{\Theta_j}$ such that:

$$T_\Lambda Q^* T_\Theta^* = C.$$

Lemma 4.2. *Let $\Lambda_{cc'} = (W_i, \Lambda_i, v_i)$ and $\Theta_{cc'} = (W_i, \Theta_i, v_i)$ be CC'-GBS for H, and Let $Q : \mathcal{K}^2_{\Lambda_j} \longrightarrow \mathcal{K}^2_{\Theta_j}$, Then the following conditions are equivalent.*

1. *$T_\Theta Q T_\Lambda^* = C$;*
2. *$T_\Lambda Q^* T_\Theta^* = C^*$;*
3. *$\langle Cf, g\rangle = \langle Q T_\Lambda^* f, T_\Theta^* g\rangle = \langle Q^* T_\Theta^* f, T_\Lambda^* g\rangle$.*

Proof. Straightforward. □

Theorem 4.3. *If $\Theta_{CC'}$ be a Q-dual for $\Lambda_{CC'}$, Then $\Theta_{CC'}$ is a CC'-GF for H.*

Proof. Let $f \in H$ and B an upper bound for $\Lambda_{CC'}$. Therefore

$$\begin{aligned}
\|f\|^4 &= \|\langle f, f\rangle\|^2 \\
&= \left\|\langle f, C^*(C^*)^{-1} f\rangle\right\|^2 \\
&= \left\|\langle T_\Lambda Q^* T_\Theta^* f, (C^*)^{-1} f\rangle\right\|^2 \\
&= \left\|\langle T_\Theta^* f, Q T_\Lambda^* (C^*)^{-1} f\rangle\right\|^2 \\
&\leq \|T_\Theta^* f\|^2 \|Q\|^2 \left\|T_\Lambda^* (C^*)^{-1} f\right\|^2 \\
&\leq \|T_\Theta^* f\|^2 \|Q\|^2 B \left\|C^{-1}\right\|^2 \|f\|^2 \\
&= \|Q\|^2 B \left\|C^{-1}\right\|^2 \|f\|^2 \sum_{i\in\mathbb{I}} v_i^2 \langle \Theta_i \pi_{W_i} C' f, \Theta_i \pi_{W_i} C f\rangle.
\end{aligned}$$

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Hence

$$B^{-1}\left\|Q\right\|^{-2}\left\|C^{-1}\right\|^{-2}\left\|f\right\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2\langle\Theta_i\pi_{W_i}C'f, \Theta_i\pi_{W_i}Cf\rangle$$

and this completes the proof. □

Corollary 4.4. *Assume C_{op} and D_{op} are the optimal bounds of $\Theta_{CC'}$, respectively. Then*

$$C_{op} \geq B_{op}^{-1}\left\|Q\right\|^{-2}\left\|C^{-1}\right\|^{-2} \quad and \quad D_{op} \geq A_{op}^{-1}\left\|Q\right\|^{-2}\left\|C^{-1}\right\|^{-2},$$

which A_{op} and B_{op} are the optimal bounds of $\Lambda_{CC'K}$, respectively.

Suppose that Λ_C be a C^2-GF for H. Since $S_C \geq A_C$, then by Douglas theorem, [4], there exists an operator $U \in \mathcal{B}\big(H, \mathscr{K}^2_{\Lambda_j}\big)$ such that

$$T_C U = I_H. \tag{4.1}$$

Now, we define the j-th component of Uf by $U_j f = (Uf)_j$ for each $f \in H$. By this operator, we may construct some Q-duals C^2-GF for Λ_C.

Theorem 4.5. *Let Λ_C be a C^2-GF for H. If U be an operator as in (4.1) and $\widetilde{W_i} := U_i^* C^* W_i$ such that $\Theta_C := (\widetilde{W_i}, \Lambda_i, v_i)$ is a C^2-GF for H. Then Θ is a Q-dual C^2-GF for Λ_C.*

Proof. Define the mapping

$$\Psi_0 : \mathcal{R}(T^*_\Theta) \to \mathscr{K}^2_{\Lambda_j},$$
$$\Psi_0(T^*_\Theta f) = UCf.$$

Then Ψ_0 is well-defined. indeed, if $f_1, f_2 \in H$ and $T^*_\Theta f_1 = T^*_\Theta f_2$, then $\pi_{\widetilde{W_i}}\Lambda_i^*\Lambda_i\pi_{\widetilde{W_i}}C(f_1 - f_2) = 0$. Therefore, for any $i \in \mathbb{I}$,

$$\Lambda_i^*\Lambda_i\pi_{\widetilde{W_i}}C(f_1 - f_2) \in (\tilde{W}_i)^\perp = \mathcal{R}(U_i^*)^\perp = \ker U_i.$$

Thus,

$$U_i\Lambda_i^*\Lambda_i\pi_{\widetilde{W_i}}C(f_1 - f_2) = 0,$$

and so, $C(f_1 - f_2) \in \pi_{\widetilde{W_i}}$. Hence $U_iCf_1 = U_iCf_2$, for all $i \in \mathbb{I}$. Moreover

$$\begin{aligned}\|\Psi_0\| &= \sup_{f\neq 0}\frac{\|\Psi_0 f\|}{T^*_\Theta f}\\ &\leq \sup_{f\neq 0}\frac{\|UCf\|}{\sqrt{A_\Theta}\,\|f\|}\\ &= \frac{\|U\|\,\|C\|}{\sqrt{A_\Theta}} < \infty,\end{aligned}$$

where, A_Θ is a lower frame bound of Θ_C. Therefore, Ψ_0 is a bounded operator. So, it has a unique linear extension (also denoted Ψ_0) to $\overline{\mathcal{R}(T^*_\Theta)}$. Define

$$\Psi = \begin{cases}\Psi_0, & \text{on } \overline{\mathcal{R}(T^*_\Theta)},\\ 0, & \text{on } \overline{\mathcal{R}(T^*_\Theta)}^\perp\end{cases}$$

and let $Q := \Psi^*$. This implies that $Q^* \in \mathcal{B}(\mathscr{K}^2_\Theta, \mathscr{K}^2_{\Lambda_j})$ and

$$T_C Q^* T^*_\Theta = T_C \Psi T^*_\Theta = T_C U C = C.$$

□

Definition 4.6. Let $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$ and $\Theta_{CC'} = (W_i, \Theta_i, v_i)$ be two CC'-GBS for H and $0 \leq \lambda_1, \lambda_2 < 1$. Let a sequence of positive numbers $\{c_i\}_{i\in\mathbb{I}}$ such that $\beta := \{c_i\}_{i\in\mathbb{I}} \in l^2(I)$. If

$$\left\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 \leq \lambda_1 \left\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 + \lambda_2 \left\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 + \|\beta\|_2 \|f\|$$

Then we say that $\Theta_{CC'}$ is a $(\lambda_1, \lambda_2, \beta, C, C')$-perturbation of $\Lambda_{CC'}$.

Theorem 4.7. *Let $\Lambda_{CC'}$ be a CC'-GF for H with frame bounds A, B and $\Theta_{CC'}$ be a $(\lambda_1, \lambda_2, \beta, C, C')$-perturbation of $\Lambda_{CC'}$. Then $\Theta_{CC'}$ is a CC'-GF for H with bounds:*

$$\left(\frac{(1-\lambda_1)\sqrt{A} - \|\beta\|_2}{1+\lambda_2}\right)^2 \quad , \quad \left(\frac{(1+\lambda_1)\sqrt{B} + \|\beta\|_2}{1-\lambda_2}\right)^2$$

Proof. Let $f \in H$, We have

$$\begin{aligned}
&\left\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 \\
&= \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f + v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\
&\leq \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}c')^{\frac{1}{2}}f\| + \|v_i(C^*\pi_{w_i}\Lambda_i^*\Lambda_i\pi_{W_i}c')^{\frac{1}{2}}f\|_2 \\
&\leq \lambda_1 \left\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f\right\| + \lambda_2 \left\|v_i(C^*\pi_{w_i}\Theta_i^*\Theta_i\pi_{\widetilde{W}_i}C')^{\frac{1}{2}}f\right\| \\
&\quad + \|\beta\|_2 \|f\| + \left\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1-\lambda_2)\left\|(v_i(C\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 \\
&\quad \leq (1+\lambda_1)\left\|v_i(C\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{w_i}C')^{\frac{1}{2}}f\right\|_2 + \|\beta\|_2 \|f\|.
\end{aligned}$$

Since $\Lambda_{CC'}$ is a CC'-GF with bounds A, B and analysis operator T^*_Λ and synthesis operator T_Λ , we have

$$\langle \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f, f\rangle = \langle S_{CC'}f, f\rangle = \langle T_\Lambda T^*_\Lambda f, f\rangle = \langle T^*_\Lambda f, T^*_\Lambda f\rangle$$

thus,

$$\|T^*_\Lambda f\|^2 = \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle \leq B\|f\|^2.$$

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So

$$\left\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 \leq \frac{(1+\lambda_1)\left\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\right\| + \|\beta\|_2\|f\|}{1-\lambda_2}$$
$$\leq \frac{\left((1+\lambda_1)\sqrt{B}\,\|f\| + \|\beta\|_2\|f\|\right)}{1-\lambda_2}.$$

Now, for the lower bound, we have

$$\begin{aligned}
&\left\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\right\|_2 \\
&= \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f - v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\
&\geq \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C' - C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\
&\geq \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \lambda_1\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\
&\quad - \lambda_2\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \|\beta\|_2\|f\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(1+\lambda_2)\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \\
&\quad \geq (1-\lambda_1)\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 - \|\beta\|_2\|f\|,
\end{aligned}$$

or

$$\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \geq \frac{(1-\lambda_1)\|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')f\| - \|\beta\|_2\|f\|}{1+\lambda_2}.$$

Since $\Lambda_{cc'}$ is a CC'-GF with bounds A, B and analysis operator T_Λ^* ,we have

$$\|T_\Lambda^*f\|^2 = \|v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\|^2 = \sum_{i\in\mathbb{I}} v_i^2\langle\Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle \geq A\|f\|^2.$$

Thus

$$\|v_i(C^*\pi_{W_i}\Theta_i^*\Theta_i\pi_{W_i}C')^{\frac{1}{2}}f\|_2 \geq \frac{(1-\lambda_1)\sqrt{A}\|f\| - \|\beta\|_2\|f\|}{1+\lambda_2}$$

and the proof is completed. □

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