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Stabilization of a Type III Thermoelastic Bresse System with Distributed Delay-time

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Abstract. In this paper, we investigate a Bresse-type system of thermoelasticity of type III in the presence of a distributed delay. We prove the well-posedness of the problem. Furthermore, an exponential stability result will be shown without the usual assumption on the wave speeds. To achieve our goals, we make use of the semigroup method and the energy method.

Keywords: Bresse system, Delay terms, Decay rate, Lyaponov methode, Thermoelastic.

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1. Introduction

Originally the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as (see [6]):

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1 \\ \rho_2\psi_{tt} = M_x - Q + F_2 \\ \rho_1 w_{tt} = N_x - IQ + F_3, \end{cases} \tag{1.1}$$

where

$$N = k_0\left(w_x - l\varphi\right), Q = k\left(\varphi_x + lw + \psi\right), M = b\psi_x$$

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We use N, Q and M to denote respectively the axial force, the shear force and the bending moment. By w, φ and ψ we are denoting respectively the longitudinal, vertical and shear angle displacements. Here $\rho_1 = \rho A = \rho I, k_0 = EA, k = k'GA$ and $l = R^{-1}$. For material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, K for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denoting external forces. System (1.1) is an undamped system and its associated energy remains constant when the time t evolves. To stabilize system (1.1), many damping terms have been considered by several authors. (see [1], [3], [11]). Messaoudi et al. [12] established an exponential stability result for the Timoshenko-type system with thermoelasticity and second sound. Apalara in [2] obtained an exponential stability result for the following linear damped Timoshenko system with second sound and internal distributed delay,

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t\left(x, t-s\right) ds = 0, \text{ in } (0,1)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \text{ in } (0,1)\times(0,\infty) \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \text{ in } (0,1)\times(0,\infty) \\ \tau q_t + \beta q + \theta_x = 0, \text{ in } (0,1)\times(0,\infty). \end{cases}$$

Mustapha and Kafini [13] added the distributed delay term in heat equation and proved the exponential decay result under a suitable assumption on the weight of delay.

In [4] Bouzettouta et al examined a Bresse system with internal distributed delay in the feedback,

$$\begin{cases} \rho_1\varphi_{tt} - Gh\left(\varphi_{xx} + lw_x + \psi_x\right) - Ehl\left(w_x - l\varphi\right) + \mu_0\varphi_t + \rho_1\int_{\tau_1}^{\tau_2} \mu(s)\varphi_t\left(x, t-s\right) ds = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh\left(\varphi_x + lw + \psi\right) = 0 \\ \rho_1 w_{tt} - Eh\left(w_{xx} - l\varphi_x\right) + lGh\left(\varphi_x + lw + \psi\right) = 0, \end{cases}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$ with the Dirichlet and initial conditions. Regarding the similar result concerning boundary distributed delay (see [2, 4, 5, 7, 8, 9, 10, 14]).

In the present paper we are concerned at the Bresse system with a distributed delay term,

$$\begin{cases} \rho_1\phi_{tt} - k\left(\phi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\phi\right) + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\phi_t\left(x, t-s\right) ds = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\phi_x + lw + \psi\right) + \beta\theta_{tx} = 0 \\ \rho_1 w_{tt} - k_0\left(w_x - l\phi\right)_x + kl\left(\phi_x + lw + \psi\right) = 0 \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \beta\psi_{ttx} - k\theta_{txx} = 0. \end{cases} \tag{1.2}$$

where $(x,t) \in (0,1)\times\mathbb{R}_+$, with the following boundary conditions:

$$\begin{aligned} \phi(0,t) &= \phi(1,t) = \psi_x(0,t) = \psi_x(1,t) = w_x(0,t) = w_x(1,t) \\ &= \theta(0,t) = \theta(1,t) = 0, t > 0, \end{aligned} \tag{1.3}$$

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and the initial conditions

$$\begin{cases} \phi(x,0)=\phi_0(x), \phi_t(x,0)=\phi_1(x), \psi(x,0)=\psi_0(x), \\ \psi_t(x,0)=\psi_1(x), w(x,0)=w_0(x), w_t(x,0)=w_1(x), \\ \theta(x,0)=\theta_0(x),\ \theta_t(x,0)=\theta_1(x) \\ \phi_t(x,-\tau)=f(x,t) \text{ in } 0<t\leq\tau_2, \\ \phi(0,t)=\psi_x(0,t)=w_x(0,t)=\theta(0,t)=0, \quad \forall t\geq 0 \\ \phi_x(1,t)=\psi(1,t)=w(1,t)=0, \quad \forall t\geq 0, \end{cases} . \tag{1.4}$$

τ_1 and τ_2 are two real numbers with $0\leq\tau_1<\tau_2$, $\mu_1>0$ is a positive constant, $\mu_2:[\tau_1,\tau_2]\longrightarrow\mathbb{R}$ is an L^∞ function, $\mu_2\geq 0$ almost everywhere, and the initial data $(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,\theta_0,\theta_1,f_0)$. belong to a suitable space (see below)

And under the assumption

$$\mu_1\geq\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds. \tag{1.5}$$

The aim of this paper is to study the well-posedness and asymptotic stability of system(1.2)-(1.4).

2. Preliminaries and Well-posedness

In this section we first prove the existence and uniqueness of regular solutions to problem (1.2)-(1.4) by using a semigroup theory as in [17], and Introduce the following new variable [16].

In order to exhibit the dissipative nature of (1.2), we differentiate the first, the second and the third equations of system (1.2) with respect to t and introduce new dependent variables $\Phi=\varphi_t, \mathbf{\Psi}=\psi_t$, $\mathbf{w}=w_t$ and $z(x,\rho,t,s)=\Phi_t(x,t-\rho s)$.

$$z(x,\rho,t,s)=\Phi_t(x,t-\rho s), x\in(0,1), \rho\in(0,1), s\in(\tau_1,\tau_2), t>0. \tag{2.1}$$

Then, we have

$$sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0 \text{ in } (0,1)\times(0,1)\times(0,\infty) \times s\in(\tau_1,\tau_2). \tag{2.2}$$

Therefore, problem (1.2) takes the form

$$\begin{cases} \rho_1\Phi_{tt}-k(\Phi_x+l\mathbf{w}+\mathbf{\Psi})_x-lk_0(\mathbf{w}_x-l\Phi)+\mu_1\Phi_t \\ +\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds=0, \\ sz_t(x,\rho,t)+z_\rho(x,\rho,t)=0, \\ \rho_2\mathbf{\Psi}_{tt}-b\mathbf{\Psi}_{xx}+k(\Phi_x+l\mathbf{w}+\mathbf{\Psi})+\beta\theta_{tx}=0, \\ \rho_1\mathbf{w}_{tt}-k_0(\mathbf{w}_x-l\Phi)_x+lk(\Phi_x+l\mathbf{w}+\mathbf{\Psi})=0, \\ \rho_3\theta_{tt}-\delta\theta_{xx}+\beta\mathbf{\Psi}_{tx}-k\theta_{txx}=0. \end{cases} \tag{2.3}$$

With the initial and boundary conditions:

$$\Phi(0,t)=\Phi(1,t)=\mathbf{\Psi}(0,t)=\mathbf{\Psi}(1,t)=\mathbf{w}(0,t)=\mathbf{w}(1,t)=0, t>0. \tag{2.4}$$

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$$\begin{cases} \Phi(x,0)=\Phi_0(x), \Phi_t(x,0)=\Phi_1(x), \mathbf{\Psi}(x,0)=\mathbf{\Psi}_0(x), \\ \mathbf{\Psi}_t(x,0)=\mathbf{\Psi}_1(x), \mathbf{w}(x,0)=\mathbf{w}_0(x), \mathbf{w}_t(x,0)=\mathbf{w}_1(x), x\in(0,1) \\ \theta(x,0)=\theta_0(x),\ \theta_t(x,0)=\theta_1(x) \text{ in } (0,\infty) \\ z\left(x,0,t,s\right)=\Phi_t\left(x,t\right) \text{ on } (0,1)\times(0,\infty)\times(\tau_1,\tau_2), \\ z\left(x,\rho,0,s\right)=f_0\left(x,\rho,s\right) \text{ on } (0,1)\times(0,1)\times(\tau_1,\tau_2) \\ \Phi(0,t)=\mathbf{\Psi}_x(0,t)=\mathbf{w}_x(0,t)=\theta(0,t)=\theta(1,t)=0, \quad \forall t\geq 0 \\ \Phi_x(1,t)=\mathbf{\Psi}\left(1,t\right)=\mathbf{w}\left(1,t\right)=0, \quad \forall t\geq 0 \end{cases} \tag{2.5}$$

Remark 2.1. The third equation of (2.3) and the boundary conditions yield

$$\rho_2\frac{d^2}{dt^2}\int_0^1\Psi\left(x,t\right)dx+k\int_0^1\Psi\left(x,t\right)dx=0$$

which gives

$$\int_0^1\Psi\left(x,t\right)dx=\left(\int_0^1\Psi_0\left(x\right)dx\right)\cos\left(\sqrt{\frac{k}{\rho_2}}t\right)+\left(\int_0^1\Psi_1\left(x\right)dx\right)\sqrt{\frac{\rho_2}{k}}\sin\left(\sqrt{\frac{k}{\rho_2}}t\right).$$

Consequently, if we set

$$\begin{aligned}\widetilde{\Psi}\left(x,t\right) &= \Psi\left(x,t\right)-\left(\int_0^1\Psi_0\left(x\right)dx\right)\cos\left(\sqrt{\frac{k}{\rho_2}}t\right) \\ &\quad -\sqrt{\frac{\rho_2}{k}}\left(\int_0^1\Psi_1\left(x\right)dx\right)\sin\left(\sqrt{\frac{k}{\rho_2}}t\right)\end{aligned}$$

$\left(\Phi,z,\widetilde{\Psi},\mathbf{w},\theta\right)$ satisfies (2.3) with initial conditions for given by

$$\widetilde{\Psi}\left(x,0\right)=\Psi_0\left(x\right)-\int_0^1\Psi_0\left(x\right)dx \text{ and } \widetilde{\Psi}_t\left(x,0\right)=\Psi_1\left(x\right)-\int_0^1\Psi_1\left(x\right)dx.$$

Moreover, we have

$$\int_0^1\widetilde{\Psi}\left(x,t\right)=0,$$

which justifies the application of Poincare's inequality for $\widetilde{\Psi}$.In the sequel, we work with $\widetilde{\Psi}$ but we write Ψ for simplicity.

If we set

$$U=\left(\Phi,\Phi_t,\mathbf{\Psi},\mathbf{\Psi}_t,\mathbf{w},\mathbf{w}_t,\theta,\theta_t,z\right)^T,$$

then $U_t=\left(\Phi_t,\Phi_{tt},\mathbf{\Psi}_t,\mathbf{\Psi}_{tt},\mathbf{w}_t,\mathbf{w}_{tt},\theta_t,\theta_{tt},z_t\right)^T$.

Therefore, problem (2.3)-(2.5) can be written as

$$\begin{cases} AU=U_t, \\ U\left(0\right)=\left(\Phi_0,\Phi_1,\mathbf{\Psi}_0,\mathbf{\Psi}_1,\mathbf{w}_0,\mathbf{w}_1,\theta_0,\theta_1,f\left(x,\rho s\right)\right), \end{cases} \tag{2.6}$$

where the operator A is defined by

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$$A\begin{pmatrix}\Phi\\u\\\Psi\\v\\\mathbf{w}\\\varpi\\\theta\\\vartheta\\z\end{pmatrix}=\begin{pmatrix}u\\\frac{k}{\rho_1}\left(\Phi_x+l\mathbf{w}+\Psi\right)_x+\frac{lk_0}{\rho_1}\left(\mathbf{w}_x-l\Phi\right)-\frac{\mu_1}{\rho_1}u-\frac{1}{\rho_1}\int_{\tau_1}^{\tau_2}\mu_2(s)z\left(x,1,t,s\right)ds\\v\\\frac{b}{\rho_2}\Psi_{xx}-\frac{k}{\rho_2}\left(\Phi_x+l\mathbf{w}+\Psi\right)-\frac{1}{\rho_2}\beta\vartheta_x\\\varpi\\\frac{k_0}{\rho_1}\left(\mathbf{w}_x-l\Phi\right)_x-\frac{kl}{\rho_1}\left(\Phi_x+l\mathbf{w}+\Psi\right)\\\vartheta\\\frac{\delta}{\rho_3}\theta_{xx}-\frac{\beta}{\rho_3}v_x+\frac{k}{\rho_3}\vartheta_{xx}\\\left(\frac{-1}{s}\right)z_\rho\end{pmatrix}\tag{2.7}$$

We consider the following spaces

$$\begin{aligned}H_a^1\left(0,1\right)&=\left\{h\in H^1\left(0,1\right):h\left(0\right)=0\right\},\\H_b^1\left(0,1\right)&=\left\{h\in H^1\left(0,1\right):h\left(1\right)=0\right\},\\H_a^2\left(0,1\right)&=H^2\left(0,1\right)\cap H_a^1\left(0,1\right),\\H_b^2\left(0,1\right)&=H^2\left(0,1\right)\cap H_b^1\left(0,1\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}=&\,H_a^1\left(0,1\right)\times L^2\left(0,1\right)\times H_b^1\left(0,1\right)\times L^2\left(0,1\right),\\&\times H_b^1\left(0,1\right)\times L^2\left(0,1\right)\times H_0^1\left(0,1\right)\times L^2\left(0,1\right),\\&\times L_\omega^2\left(\left(0,L\right)\times\left(0,1\right)\times\left(\tau_1,\tau_2\right)\right).\end{aligned}$$

With

$$L_\omega^2\left(\left(0,L\right)\times\left(0,1\right)\times\left(\tau_1,\tau_2\right)\right)=\left\{z\text{ measurable }/\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2\left(s\right)z^2\left(x,\rho,s\right)dsd\rho dx<\infty\right\}.$$

We will show that A generates a C_0 semigroup on $\mathcal{H}$. Let us define on the Hilbert space $\mathcal{H}$ the inner product, for

$$U=\left(\Phi,u,\Psi,v,\mathbf{w},\varpi,\theta,\vartheta,z\right)^T\ \ ,\widehat{U}=\left(\widehat{\Phi},\widehat{u},\widehat{\Psi},\widehat{v},\widehat{\mathbf{w}},\widehat{\varpi},\widehat{\theta},\widehat{\vartheta},\widehat{z}\right)^T$$

$$\begin{aligned}\left\langle U,\widehat{U}\right\rangle_{\mathcal{H}}=&\,\rho_1\int_0^1u\widehat{u}dx+\rho_2\int_0^1v\widehat{v}dx+\rho_1\int_0^1\varpi\widehat{\varpi}dx+\rho_3\int_0^1\vartheta\widehat{\vartheta}dx+b\int_0^1\Psi_x\widehat{\Psi}_xdx\\&+k\int_0^1\left(\Phi_x+\Psi+l\mathbf{w}\right)\left(\widehat{\Phi}_x+\widehat{\Psi}+l\widehat{\mathbf{w}}\right)dx+k_0\int_0^1\left(\mathbf{w}_x-l\Phi\right)\left(\widehat{\mathbf{w}}_x-l\widehat{\Phi}\right)dx\\&+\delta\int_0^1\theta_x\widehat{\theta}_xdx+\int_0^1\int_{\tau_1}^{\tau_2}s\mu\left(s\right)\int_0^1z\left(x,\rho,s\right)\widehat{z}\left(x,\rho,s\right)d\rho dsdx.\end{aligned}\tag{2.8a}$$

$\mathcal{H}$ is a Hilbert space for l small enough since, in this case, the above inner product is equivalent to the natural inner product defined on $\mathcal{H}$.

The domain of A is given by

$$D\left(A\right)=\left\{\begin{array}{c}U\in\mathcal{H}/\Phi\in H_a^2\left(0,1\right);\Psi,\mathbf{w}\in H_b^2\left(0,1\right),u,\theta\in H_a^1\left(0,1\right);v,\varpi\in H_b^1\left(0,1\right)\\;z\in L_\omega^2\left(\left(0,L\right)\times\left(0,1\right)\times\left(\tau_1,\tau_2\right)\right),u\left(x\right)=\left(x,0,s\right)\ \text{in}\ \left(0,L\right)\\,\Phi_x\left(1\right)=0,\mathbf{w}_x\left(0\right)=\Psi_x\left(0\right)=0.\end{array}\right\}.\tag{2.9}$$

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Theorem 2.2. *Let* $(\Phi_0, \Phi_1, \Psi_0, \Psi_1, \mathbf{w}_0, \mathbf{w}_1, \theta_0, \theta_1, f_0) \in \mathcal{H}$. *Assume that the hypothesis* (1.5) *holds. Then, for any initial datum* $U_0 \in \mathcal{H}$ *there exists a unique solution* $U \in C([0,\infty), \mathcal{H})$ *for problem* (2.6). *Moreover, if* $U_0 \in D(A)$, *then* $U \in C([0,\infty), D(A)) \cap C^1[0,\infty), \mathcal{H})$.

Proof. To obtain the above result, we need to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective.

Step 1: In this step, we prove that the operator A is dissipative. Let $U = (\Phi, u, \Psi, v, \mathbf{w}, \varpi, \theta, \vartheta, z)^T$,

$$\langle AU, U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u \\ \frac{k}{\rho_1}(\Phi_x+\Psi+l\mathbf{w})_x + \frac{lk_0}{\rho_1}(\mathbf{w}_x - l\Phi) - \frac{\mu_1}{\rho_1}u - \frac{1}{\rho_1}\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\,ds \\ v \\ \frac{b}{\rho_2}\Psi_{xx} - \frac{k}{\rho_2}(\Phi_x+\Psi+l\mathbf{w}) - \frac{1}{\rho_2}\beta\vartheta_x \\ \varpi \\ \frac{k_0}{\rho_1}(\mathbf{w}_x - l\Phi)_x - \frac{kl}{\rho_1}(\Phi_x+\Psi+l\mathbf{w}) \\ \vartheta \\ \frac{\delta}{\rho_3}\theta_{xx} - \frac{\beta}{\rho_3}v_x + \frac{k}{\rho_3}\vartheta_{xx} \\ \left(\frac{-1}{s}\right)z_\rho \end{pmatrix}, \begin{pmatrix} \Phi \\ u \\ \Psi \\ v \\ \mathbf{w} \\ \varpi \\ \theta \\ \vartheta \\ z \end{pmatrix} \right\rangle$$

$$\begin{aligned}
&= k\int_0^1 u(\Phi_x+\Psi+l\mathbf{w})(\Phi_x+\Psi+l\mathbf{w})_x\,dx + lk_0\int_0^1(\mathbf{w}_x - l\Phi)u\,dx - \mu_1\int_0^1 u^2\,dx \\
&- \int_0^1\int_{\tau_1}^{\tau_2} u\mu_2(s)z(x,1,t,s)\,ds\,dx \\
&+ b\int_0^1 v\Psi_{xx}\,dx - k\int_0^1(\Phi_x+\Psi+l\mathbf{w})v\,dx - \beta\int_0^1 v\vartheta_x\,dx \\
&+ k_0\int_0^1\varpi(\mathbf{w}_x - l\Phi)_x\,dx - kl\int_0^1\varpi(\Phi_x+\Psi+l\mathbf{w})\,dx \\
&+ \delta\int_0^1\vartheta\theta_{xx}\,dx - \beta\int_0^1\vartheta v_x\,dx + k\int_0^1\vartheta\vartheta_{xx}\,dx + b\int_0^1 v_x\Psi_x\,dx \\
&+ k\int_0^1(\Phi_x+\Psi+l\mathbf{w})(u_x+v+l\varpi)\,dx + k_0\int_0^1(\mathbf{w}_x - l\Phi)(\varpi_x - lu)\,dx \\
&+ \delta\int_0^1\theta_x\vartheta_x\,dx - \int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\int_0^1 z(x,\rho,s)z_\rho(x,\rho,s)\,d\rho\,ds\,dx.
\end{aligned}$$

With integration by parts we obtain,

$$\begin{aligned}
\langle AU, U\rangle_{\mathcal{H}} \quad = \quad & k\int_0^1 u(\Phi_x+\Psi+l\mathbf{w})_x\,dx + lk_0\int_0^1(\mathbf{w}_x - l\Phi)u\,dx - \mu_1\int_0^1 u^2\,dx \\
& -k\int_0^1(\Phi_x+\Psi+l\mathbf{w})v\,dx - \int_0^1\int_{\tau_1}^{\tau_2} u\mu_2(s)z(x,1,t,s)\,ds\,dx \\
& +k_0\int_0^1\varpi(\mathbf{w}_x - l\Phi)_x\,dx - kl\int_0^1\varpi(\Phi_x+\Psi+l\mathbf{w})\,dx + k\int_0^1\vartheta\vartheta_{xx}\,dx \\
& +k\int_0^1(\Phi_x+\Psi+l\mathbf{w})(u_x+v+l\varpi)\,dx + k_0\int_0^1(\mathbf{w}_x - l\Phi)(\varpi_x - lu)\,dx \\
& -\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\int_0^1 z(x,\rho,s)z_\rho(x,\rho,s)\,d\rho\,ds\,dx,
\end{aligned}$$

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then,

$$\langle AU,U\rangle_{\mathcal{H}} = -\mu_1 \int_0^1 u^2 dx - k\int_0^1 \vartheta_x^2 dx$$
$$- \int_0^1 \int_{\tau_1}^{\tau_2} u\mu_2(s) z\left(x,1,t,s\right) dsdx$$
$$- \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) \int_0^1 z\left(x,\rho,s\right) z_\rho\left(x,\rho,s\right) d\rho dsdx, \tag{2.10}$$

and Integrating by parts in ρ , we have

$$\int_0^1 z_\rho\left(x,\rho,s\right) z\left(x,\rho,s\right) d\rho = \frac{1}{2}\int_0^1 \frac{\partial}{\partial\rho} z^2\left(x,\rho,s\right) d\rho$$
$$= \frac{1}{2}\left[z^2\left(x,1,s\right) - z^2\left(x,0,s\right)\right],$$

then

$$\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) \int_0^1 z_\rho\left(x,\rho,s\right) z\left(x,\rho,s\right) d\rho dsdx$$
$$= \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2\left(s\right)\left[z^2\left(x,1,s\right) - z^2\left(x,0,s\right)\right]. \tag{2.11}$$

Therefore, from (2.10) and (2.11),

$$\langle AU,U\rangle = -\mu_1 \int_0^1 u^2\left(x\right) dx - k\int_0^1 \vartheta_x^2 dx$$
$$- \int_0^1 \int_{\tau_1}^{\tau_2} u\left(x\right) \mu_2\left(s\right) z\left(x,1,s\right) dsdx$$
$$- \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) \int_0^1 z_\rho\left(x,\rho,s\right) z\left(x,\rho,s\right) d\rho dsdx$$
$$= -\mu_1 \int_0^1 u^2\left(x\right) dx - k\int_0^1 \vartheta_x^2 dx$$
$$- \int_0^1 u\left(x\right)\left(\int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z\left(x,1,s\right) ds\right) dx$$
$$- \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z^2\left(x,1,s\right) dsdx$$
$$+ \frac{1}{2}\int_{\tau_1}^{\tau_2} \mu_2\left(s\right) \int_0^1 u^2\left(x\right) dx.$$

Now, by using Cauchy-Schwarz's inequality, we can estimate,

$$\left|\int_0^1 u\left(x\right)\left(\int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z\left(x,1,s\right) ds\right) dx\right| \le \frac{1}{2}\int_0^1 u^2\left(x\right)\left(\int_{\tau_1}^{\tau_2} \mu_2\left(s\right)\right) dx$$
$$+ \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2\left(s\right) z^2\left(x,1,s\right) dsdx$$

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Therefore, from the assumption (1.5) we have,

$$\langle AU,U\rangle \leq -\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)\right)\int_0^1 u^2\left(x\right)dx-k\int_0^1\vartheta_x^2dx\leq 0 \qquad (2.12)$$

that is, the operator $\mathcal{A}$ is dissipative.

Step 2: To prove that the operator $Id-\mathcal{A}$ is surjective, that is, for any let $G=(g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8,g_9)\in\mathcal{H}$. We seek $U=(\Phi,u,\Psi,v,\mathbf{w},\varpi,\theta,\vartheta,z)^T\in D\left(A\right)$ satisfying

$$\left(Id-\mathcal{A}\right)U=G,$$

which is equivalent to

$$\begin{cases}\lambda\Phi-u=g_1\\ \lambda\rho_1u-k\left(\Phi_x+\Psi+l\mathbf{w}\right)_x-lk_0\left(\mathbf{w}_x-l\Phi\right)+\mu_1u\\ +\int_{\tau_1}^{\tau_2}\mu_2(s)z\left(x,1,t,s\right)ds=\rho_1g_2\\ \lambda\Psi-v=g_3\\ \lambda\rho_2v-b\Psi_{xx}+k\left(\Phi_x+\Psi+l\mathbf{w}\right)+\beta\vartheta_x=\rho_2g_4\\ \lambda\mathbf{w}-\varpi=g_5\\ \lambda\rho_1\varpi-k_0\left(\mathbf{w}_x-l\Phi\right)_x+kl\left(\Phi_x+\Psi+l\mathbf{w}\right)=\rho_1g_6\\ \lambda\theta-\vartheta=g_7\\ \lambda\rho_3\vartheta-\delta\theta_{xx}+\beta v_x-k\vartheta_{xx}=\rho_3g_8\\ \lambda z+s^{-1}z_\rho=g_9.\end{cases} \qquad (2.13)$$

Suppose that we have found Φ, $\Psi,\mathbf{w}$ and θ. Therefore, the first, the third and the fifth equation in (2.13) give

$$\begin{cases}u=\lambda\Phi-g_1\\ v=\lambda\Psi-g_3\\ \varpi=\lambda\mathbf{w}-g_5\\ \vartheta=\lambda\theta-g_7,\end{cases} \qquad (2.14)$$

It is clear that $u\in H_0^1\left(0,1\right),v\in H_0^1\left(0,1\right),\varpi\in H_0^1\left(0,1\right)$ and $\vartheta\in H_0^1\left(0,1\right).$ And we can find,

$$z(x,0,s)=u(x),\ \text{for } x\in\left(0,L\right),s\in\left(\tau_1,\tau_2\right). \qquad (2.15)$$

Following the same approach as in [15], we obtain, by using equations for z in (2.14)

$$\lambda z\left(x,\rho,s\right)+s^{-1}z_\rho\left(x,\rho,s\right)=f_9\left(x,\rho,s\right),\ \text{for } x\in\left(0,L\right),s\in\left(\tau_1,\tau_2\right). \qquad (2.16)$$

Then by (2.14) and (2.15)

$$z\left(x,\rho,s\right)=e^{-\lambda\rho s}u(x)+se^{-\lambda\rho s}\int_0^\rho f_9\left(x,\sigma,s\right)e^{\lambda\sigma s}d\sigma.$$

So, from (2.13) on $(0,L)\times(0,1)\times\left(\tau_1,\tau_2\right),$

$$z\left(x,\rho,s\right)=\lambda\Phi(x)e^{-\lambda\rho s}-f_1(x)e^{-\lambda\rho s}+se^{-\lambda\rho s}\int_0^\rho f_9\left(x,\sigma,s\right)e^{\lambda\sigma s}d\sigma. \qquad (2.17)$$

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By using (1.5) and (2.13) the functions Φ, Ψ, $\mathbf{w}$ and θ satisfying the following system,

$$\begin{cases} \lambda^2\rho_1\Phi - k\left(\Phi_x + l\mathbf{w} + \Psi\right)_x - lk_0\left(\mathbf{w}_x - l\Phi\right) + \mu_1 u_t \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) z\left(x, 1, t, s\right) ds = \rho_1\left(\lambda g_1 + g_2\right) \\ \lambda^2\rho_2\Psi - b\Psi_{xx} + k\left(\Phi_x + l\mathbf{w} + \Psi\right) + \beta\vartheta_x = \rho_2\left(\lambda g_3 + g_{\lambda 4}\right) \\ \lambda^2\rho_1\mathbf{w} - k_0\left(\mathbf{w}_x - l\Phi\right)_x + kl\left(\Phi_x + l\mathbf{w} + \Psi\right) = \rho_1\left(\lambda g_5 + g_6\right) \\ \lambda^2\rho_3\theta - \delta\theta_{xx} + \beta v_x - k\vartheta_{xx} = \rho_3\left(\lambda g_7 + g_8\right), \end{cases} \tag{2.18}$$

Solving system (2.18) is equivalent to finding,

$$\left(\Phi, \Psi, \mathbf{w}, \theta\right) \in H_a^2\left(0,1\right) \times H_b^2\left(0,1\right) \times H_b^2\left(0,1\right) \times H_a^2\left(0,1\right),$$

such that

$$\begin{cases} \int_0^1 \left[\lambda^2\rho_1\Phi\eta - k\left(\Phi_x + l\mathbf{w} + \Psi\right)\eta_x - lk_0\left(\mathbf{w}_x - l\Phi\right)\eta + \mu_1 u\eta \right. \\ \left. +\eta\int_{\tau_1}^{\tau_2} \mu_2(s) z\left(x, 1, t, s\right) ds\right] dx = \int_0^1 \rho_1\eta\left(\lambda g_1 + g_2\right) dx \\ \int_0^1 \left[\lambda^2\rho_2\Psi\zeta - b\Psi_x\zeta_x + k\left(\Phi_x + l\mathbf{w} + \Psi\right)\zeta + \beta\zeta\vartheta_x\right] dx = \int_0^1 \rho_2\zeta\left(\lambda g_3 + g_{\lambda 4}\right) dx \\ \int_0^1 \left[\lambda^2\rho_1\mathbf{w}\xi - k_0\left(\mathbf{w}_x - l\Phi\right)\xi_x + kl\left(\Phi_x + l\mathbf{w} + \Psi\right)\xi\right] dx = \int_0^1 \rho_1\xi\left(\lambda g_5 + g_6\right) dx \\ \int_0^1 \left[\lambda^2\rho_3\theta\chi - \delta\theta_x\chi_x + \beta\chi v_x - k\vartheta_x\chi_x\right] dx = \int_0^1 \rho_3\chi\left(\lambda g_7 + g_8\right) dx, \end{cases} \tag{2.19}$$

for all $\left(\eta, \zeta, \xi, \chi\right) \in H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right)$. From (2.17) we have,

$$z\left(x, 1, s\right) = \lambda\Phi(x)e^{-\lambda s} - f_1(x)e^{-\lambda s} + se^{-\lambda s}\int_0^1 f_9\left(x, \sigma, s\right)e^{\lambda\sigma s}d\sigma.$$

Consequently, problem (2.19) is equivalent to the problem

$$a\left(\left(\Phi, \Psi, \mathbf{w}, \theta\right), \left(\eta, \zeta, \xi, \chi\right)\right) = L\left(\eta, \zeta, \xi, \chi\right), \tag{2.20}$$

where the bilinear form

$$a : \left[H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right)\right]^2 \longrightarrow \mathbb{R},$$

and the linear form

$$L : H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \times H_0^1\left(0,1\right) \longrightarrow \mathbb{R},$$

are defined by

$$\begin{aligned} & a\left(\left(\Phi, \Psi, \mathbf{w}, \theta\right), \left(\eta, \zeta, \xi, \chi\right)\right) \\ &= \int_0^1 \left[\lambda^2\left(\rho_1\Phi\eta + \rho_2\Psi\zeta + \rho_1\mathbf{w}\xi + \rho_3\theta\chi\right) + \mu_1 u\eta - b\Psi_x\zeta_x \right. \\ &+ k\left(\Phi_x + l\mathbf{w} + \Psi\right)\left(\eta_x + l\xi + \zeta\right) - k_0\left(\mathbf{w}_x - l\Phi\right)\left(\xi_x - l\eta\right) \\ &+ \beta\left(\zeta\vartheta_x + \chi v_x\right) - \left(\delta\theta_x + k\vartheta_x\right)\chi_x + \eta\int_{\tau_1}^{\tau_2} \mu_2(s)\lambda\phi\left(x\right)e^{-\lambda s}dsdx, \end{aligned}$$

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and

$$L(\eta,\zeta,\xi,\chi)=\int_0^1\left[\rho_1\eta\left(\lambda g_1+g_2\right)+\rho_2\zeta\left(\lambda g_3+g_{\lambda 4}\right)+\rho_1\xi\left(\lambda g_5+g_6\right)\right.$$
$$\left.+\rho_3\chi\left(\lambda g_7+g_8\right)+\left(-f_1(x)e^{-\lambda s}+se^{-\lambda s}\int_0^1 f_9\left(x,\sigma,s\right)e^{\lambda\sigma s}d\sigma\right)\right]dx.$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\eta,\zeta,\xi,\chi)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$ problem (2.20) admits a unique solution $(\Phi,\Psi,\mathbf{w},\theta)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$. Applying the classical elliptic regularity, it follows from (2.19) that $(\Phi,\Psi,\mathbf{w},\theta)\in H_a^2(0,1)\times H_b^2(0,1)\times H_b^2(0,1)\times H_a^1(0,1)$ Therefore, the operator $\lambda I-A$ is surjective for any $\lambda>0$. Consequently, the existence result of theorem 2.2 follows from the Hille-Yosida theorem. □

3. Stability results

To state our decay result to the system (2.3)–(2.5), we introduce the energy functional

$$E(t)=\frac{1}{2}\int_0^1\left[\rho_1\Phi_t^2+\rho_2\psi_t^2+\rho_1\mathbf{w}_t^2+b\psi_x^2+\rho_3\theta_t^2+\delta\theta_x^2\right.$$
$$\left.+k\left(\Phi_x+\psi+l\mathbf{w}\right)^2+k_0\left(\mathbf{w}_x-l\Phi\right)^2\right]dx$$
$$+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2(s)\,z^2(x,\rho,s,t)\,dsd\rho dx. \tag{3.1}$$

We can prove that the energy is decreasing. More precisely, we have the following result.

Theorem 3.1. *Let $(\Phi,\Psi,\mathbf{w},\theta,z)$ be the solution of (2.3)–(2.5). Then there exist two positive constants α and γ such that*

$$E(t)\leq\alpha E(0)\,e^{-\gamma t},t\geq 0 \tag{3.2}$$

Lemma 3.2. *Let $(\Phi,\Psi,\mathbf{w},\theta,z)$ be the solution of (2.3)–(2.5) and assume (1.5) holds. Then the energy functional, defined by (3.1) satisfies,*

$$\frac{d}{dt}E(t)\leq -r_0\int_0^1\Phi_t^2dx-k\int_0^1\theta_{tx}^2dx, \tag{3.3}$$

with

$$r_0=\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds.$$

Proof. Multiplying $(2.3)_1$, $(2.3)_2$, $(2.3)_3$, and $(2.3)_4$ by $\Phi_t,\Psi_t,\mathbf{w}_t$ and θ_t, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, and adding the results, we obtain
(3.3). □

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Lemma 3.3. *Let* $(\Phi, \Psi, \mathbf{w}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional*

$$F_1(t) := \rho_2 \int_0^1 \Psi \Psi_t dx \tag{3.4}$$

satisfies, for $\ell_1 > 0$ *and* $\ell_2 > 0$, *the estimate*

$$\begin{aligned} F_1'(t) \leq \rho_2 \int_0^1 \Psi_t^2 dx + \left(-b + \frac{\beta}{2\ell_1} + kd\ell_2\right) \int_0^1 \Psi_x^2 dx \\ + \frac{k}{2\ell_2} \int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2 dx + \beta\ell_1 \int_0^1 \theta_t^2 dx \end{aligned} \tag{3.5}$$

Proof. Taking the derivative of (3.4), using the third equation in (2.3) and performing integration by parts, we get

$$\begin{aligned} F_1'(t) &= \rho_2 \int_0^1 \left(\Psi \Psi_{tt} + \Psi_t^2\right) dx \\ &= \rho_2 \int_0^1 \Psi_t^2 dx + \int_0^1 \Psi \left(b\Psi_{xx} - k\left(\Phi_x + l\mathbf{w} + \Psi\right) - \beta\theta_{tx}\right) dx \\ &= \rho_2 \int_0^1 \Psi_t^2 dx - b \int_0^1 \Psi_x^2 dx - k \int_0^1 \Psi \left(\Phi_x + l\mathbf{w} + \Psi\right) dx - \beta \int_0^1 \Psi\theta_{tx} dx \end{aligned}$$

Using Young's and Poincaré's inequalities, for estimate (3.5)

$$\begin{aligned} F_1'(t) \leq \rho_2 \int_0^1 \Psi_t^2 dx + \left(-b + \frac{\beta}{2\ell_1} + kd\ell_2\right) \int_0^1 \Psi_x^2 dx \\ + \frac{k}{2\ell_2} \int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2 dx + \beta\ell_1 \int_0^1 \theta_t^2 dx \end{aligned}$$

□

Lemma 3.4. *Let* $(\Phi, \Psi, \mathbf{w}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional*

$$F_2(t) := \rho_1 \int_0^1 \Phi_t \left(\Phi + \int_0^x \Psi(y,t)\, dy\right) dx, \tag{3.7}$$

satisfies, for any $\varepsilon > 0$, *the estimate*

$$\begin{aligned} F_2'(t) \leq -\frac{k}{2} \int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2 dx - \frac{lk_0}{2} \int_0^1 (\mathbf{w}_x - l\Phi)^2 dx \\ + c\left(1 + \frac{1}{\varepsilon}\right) \int_0^1 \Phi_t^2 dx + \varepsilon \int_0^1 \Psi_t^2 dx \\ + c \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s)\, z^2(x,1,s,t)\, ds dx \end{aligned} \tag{3.8}$$

Proof. Taking the derivative of (3.7), and using that,

$$z(x,\rho,s,0) = f_0(x,\rho,s) \text{ in } (0,1) \times (0,1) \times (0,\tau_2)$$

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and integration by parts, we obtain

$$F_2^{'}(t) = \rho_1 \int_0^1 \Phi_t \int_0^x \Psi_y(y)\,dydx - \int_0^1 \left(\Phi + \int_0^x \Psi(y)\,dy\right) \int_{\tau_1}^{\tau_2} \mu_2(s)\,z(x,1,s,t)\,dsdx$$
$$- k \int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2\,dx + \rho_1 \int_0^1 \Phi_t^2 dx - lk_0 \int_0^1 (\mathbf{w}_x - l\Phi)^2\,dx$$
$$- \mu_1 \int_0^1 \Phi_t \left(\Phi + \int_0^x \Psi(y)\,dy\right) dx \tag{3.9}$$

Using Young's, Poincaré's, and Cauchy-Schwarz inequalities, for estimate the terms in the right hand side of (3.9)

$$\rho_1 \int_0^1 \Phi_t \int_0^x \Psi_y(y)\,dydx \le \varepsilon \int_0^1 \Psi_t^2 dx + \frac{c}{\varepsilon} \int_0^1 \Phi_t^2 dx$$

□

Lemma 3.5. *Let* $(\Phi, \Psi, \mathbf{w}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional*

$$F_3(t) = -\rho_2\rho_3 \int_0^1 \theta_t \left(\int_0^x \Psi_t(y,t)\,dy\right) dx - \delta\rho_2 \int_0^1 \theta_x \Psi dx \tag{3.10}$$

satisfies, for any $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \varepsilon_4 > 0$*; the estimate*

$$F_3^{'}(t) \le -\rho_2 \left(\varsigma_1 - \frac{\delta}{2\varepsilon_1} - \frac{k}{2\varepsilon_2}\right) \int_0^1 \Psi_t^2 dx - \rho_3 (\beta - b\varepsilon_3 - k\varepsilon_4) \int_0^1 \theta_t^2 dx$$
$$+ \rho_2\delta\varepsilon_1 \int_0^1 \theta_x^2 dx + \rho_2 k\varepsilon_2 \int_0^1 \theta_{tx}^2 dx \tag{3.11}$$
$$+ \frac{\rho_3 b}{2\varepsilon_3} \int_0^1 \Psi_x^2 dx + \frac{\rho_3 C}{2\varepsilon_4} \int_0^1 \Phi_x^2 dx$$

Lemma 3.6. *Let* $(\Phi, \Psi, \mathbf{w}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional*

$$F_4(t) = \rho_3 \int_0^1 \theta\theta_t dx + \frac{k}{2} \int_0^1 \theta_x^2 dx + \beta \int_0^1 \Psi_x \theta dx \tag{3.12}$$

satisfies, for any $\varepsilon_5 > 0$*; the estimate*

$$F_4^{'}(t) \le \left(\rho_3 + \frac{\beta}{2\varepsilon_5}\right) \int_0^1 \theta_t^2 dx + \beta\varepsilon_5 \int_0^1 \Psi_x^2 dx - \delta \int_0^1 \theta_x^2 dx \tag{3.13}$$

Proof. By differentiating (3.12) we obtain,

$$F_4^{'}(t) = \rho_3 \int_0^1 \theta_t^2 dx + \rho_3 \int_0^1 \theta\theta_{tt} dx + k\rho_3 \int_0^1 \theta_{tx}\theta_x dx$$
$$+ \beta \int_0^1 \Psi_{tx}\theta dx + \beta \int_0^1 \Psi_x \theta_t dx$$

Using Young's inequality, and integration by parts for obtain (3.13). □

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Lemma 3.7. *Let* $(\Phi, \Psi, \mathbf{w}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional*

$$F_5(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} \mu_2(s) z^2(x,\rho,s,t)\, dsd\rho dx \tag{3.14}$$

satisfies the estimate

$$\begin{aligned} F_5^{'}(t) \leq & -e^{-\tau_2} \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, dsdx \\ & - e^{-\tau_2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s) z^2(x,\rho,s,t)\, dsd\rho dx + \mu_1 \int_0^1 \int \Phi_t^2 dx. \end{aligned} \tag{3.15}$$

Proof. Differentiating $F_5(t)$, we obtain,

$$F_5^{'}(t) = 2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} \mu_2(s) z(x,\rho,s,t) z_t(x,\rho,s,t)\, dsd\rho dx$$

Using the second equation (2.2), we arrive at

$$\begin{aligned} F_5^{'}(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} \mu_2(s) z(x,\rho,s,t) z_\rho(x,\rho,s,t)\, dsd\rho dx \\ &= - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} \mu_2(s) \frac{d}{d\rho} z^2(x,\rho,s,t)\, dsd\rho dx. \end{aligned}$$

Integration by parts gives,

$$\begin{aligned} F_5^{'}(t) &= - \int_0^1 \int_0^1 \frac{d}{d\rho} \int_{\tau_1}^{\tau_2} \mu_2(s) \left(e^{-s\rho} z^2(x,\rho,s,t)\right) dsd\rho dx \\ & \quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s) e^{-s\rho} z^2(x,\rho,s,t)\, dsd\rho dx \\ &= \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) \left(z^2(x,0,s,t) - e^{-s\rho} z^2(x,1,s,t)\right) dsd\rho dx \\ & \quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s) e^{-s\rho} z^2(x,\rho,s,t)\, dsd\rho. \end{aligned}$$

Therefore,

$$\begin{aligned} F_5^{'}(t) \leq & -e^{-\tau_2} \int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, dsdx + \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \right) \int_0^1 \Phi_t^2 dx \\ & - e^{-\tau_2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu_2(s) z^2(x,\rho,s,t)\, dsd\rho dx. \end{aligned}$$

We, then, obtain (3.15) by virtue of (1.5). □

Lemma 3.8. *Let* $(\Phi, \Psi, \mathbf{W}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional,*

$$F_6(t) := -\rho_1 \int_0^1 \Phi_t (\mathbf{w}_x - l\Phi)\, dx - \rho_1 \int_0^1 \mathbf{w}_t (\Phi_x + l\mathbf{w} + \mathbf{\Psi})\, dx \tag{3.16}$$

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satisfies the estimate for any $\varsigma_2 > 0$,

$$F_6^{'}(t) \leq -\left(lk_0 - \frac{m_0}{2}\right)\int_0^1 (\mathbf{w}_x - l\Phi)^2\, dx - l\rho_1 \int_0^1 \mathbf{w}_t^2 dx + l\rho_1 \int_0^1 \mathbf{\Phi}_t^2 dx$$
$$+ lk \int_0^1 (\Phi_x + \mathbf{w}l + \Psi)^2\, dx + \varsigma_2 \int_0^1 \mathbf{\Psi}_t^2 dx$$
$$+ \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s)\, z^2(x,1,s,t)\, dsdx. \tag{3.17}$$

with $m_0 = \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds\right)$.

Proof. By differentiating (3.16) we obtain,

$$F_6^{'}(t) \leq -lk_0 \int_0^1 (\mathbf{w}_x - l\Phi)^2\, dx - l\rho_1 \int_0^1 \mathbf{w}_t^2 dx + l\rho_1 \int_0^1 \mathbf{\Phi}_t^2 dx$$
$$+ lk \int_0^1 (\Phi_x + \mathbf{w}l + \Psi)\, dx + \varsigma_2 \int_0^1 \mathbf{\Psi}_t^2 dx$$
$$+ \int_0^1 \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\right)(\mathbf{w}_x - l\Phi)\, dx.$$

Estimate (2.3) follows thanks to Cauchy-Schwarz inequality. □

Lemma 3.9. *Let* $(\Phi, \Psi, \mathbf{w}, \theta, z)$ *be the solution of* (2.3)–(2.5). *Then the functional*

$$F_7(t) = -\rho_1 \int_0^1 (\Phi\Phi_t + \mathbf{w}\mathbf{w}_t)\, dx \tag{3.18}$$

satisfies the estimate

$$F_7^{'}(t) \leq -\rho_1 \int_0^1 \Phi_t^2 dx - \rho_1 \int_0^1 \mathbf{w}_t^2 dx + c_1 \int_0^1 \Psi_x^2 + k_0 \int_0^1 (\mathbf{w}_x - l\Phi)^2\, dx$$
$$+ c_2 \int_0^1 (\Phi_x + l\mathbf{w}+\Psi)^2\, dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s)\, z^2(x,1,s,t)\, dsdx \tag{3.19}$$

with $m_0 = \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds\right)$.

Proof. By differentiating (3.18) we obtain,

$$F_7(t) = -\int_0^1 \Phi\left(k(\Phi_x + l\mathbf{w} + \mathbf{\Psi})_x + lk_0(\mathbf{w}_x - l\Phi) - \mu_0\Phi_t - \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, ds\right) dx$$
$$- \int_0^1 \Phi_t^2 dx - \int_0^1 \mathbf{w}\left(k_0(\mathbf{w}_x - l\Phi)_x - kl(\Phi_x + l\mathbf{w} + \Psi)\right) dx - \int_0^1 \mathbf{w}_t^2.$$

Using Young's, Poincaré's, and Cauchy-Schwarz inequalities, for estimate the terms in the right hand side of (3.19). □

Lemma 3.10. *We have*

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \forall t \geq 0$$

for two positive constants c_1 *and* c_2.

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Proof. For $N, N_i > 0$, let

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^{i=7} N_i F_i(t) \tag{3.20}$$

$$\begin{aligned}
\mathfrak{L}^{'}(t) \leq{}& N\left(-r_0 \int_0^1 \Phi_t^2 dx - k\int_0^1 \theta_{tx}^2 dx\right) + N_1\left(-b + \frac{\beta}{2\ell_1} + kd\ell_2\right)\int_0^1 \Psi_x^2 dx \\
&+ N_1\left[\rho_2 \int_0^1 \Psi_t^2 dx + \beta\ell_1 \int_0^1 \theta_t^2 dx + \frac{k}{2\ell_2}\int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2 dx\right] \\
&+ N_2\left(-\frac{k}{2}\int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2 dx - \frac{lk_0}{2}\int_0^1 (\mathbf{w}_x - l\Phi)^2 dx\right) \\
&+ N_2\left(c\left(1+\frac{1}{\varepsilon}\right)\int_0^1 \Phi_t^2 dx + \varepsilon\int_0^1 \Psi_t^2 dx + c\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, dsdx\right) \\
&+ N_3\left(-\rho_2\left(\varsigma_1 - \frac{\delta}{2\varepsilon_1} - \frac{k}{2\varepsilon_2}\right)\int_0^1 \Psi_t^2 dx - \rho_3(\beta - b\varepsilon_3 - k\varepsilon_4)\int_0^1 \theta_t^2 dx\right) \\
&+ N_3\left(\rho_2\delta\varepsilon_1\int_0^1 \theta_x^2 dx + \rho_2 k\varepsilon_2 \int_0^1 \theta_{tx}^2 dx + \frac{\rho_3 b}{2\varepsilon_3}\int_0^1 \Psi_x^2 dx + \frac{\rho_3 C}{2\varepsilon_4}\int_0^1 \Phi_x^2 dx\right) \\
&+ N_4\left(\left(\rho_3 + \frac{\beta}{2\varepsilon_5}\right)\int_0^1 \theta_t^2 dx + \beta\varepsilon_5\int_0^1 \Psi_x^2 dx - \delta\int_0^1 \theta_x^2 dx\right) \\
&+ N_5\left(-e^{-\tau_2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, dsdx\right) \\
&+ N_5\left(-e^{-\tau_2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s) z^2(x,\rho,s,t)\, dsd\rho dx + \mu_1\int_0^1 \Phi_t^2 dx\right) \\
&+ N_6\left(-\left(lk_0 - \frac{m_0}{2}\right)\int_0^1 (\mathbf{w}_x - l\Phi)^2 dx - l\rho_1\int_0^1 \mathbf{w}_t^2 dx + l\rho_1\int_0^1 \mathbf{\Phi}_t^2 dx\right) \\
&+ N_6\left(+lk\int_0^1 (\Phi_x + \mathbf{w}l + \Psi)^2 dx + \varsigma_2\int_0^1 \mathbf{\Psi}_t^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, dsdx\right) \\
&+ N_7\left(-\rho_1\int_0^1 \Phi_t^2 dx - \rho_1\int_0^1 \mathbf{w}_t^2 dx + c_1\int_0^1 \Psi_x^2 + k_0\int_0^1 (\mathbf{w}_x - l\Phi)^2 dx\right) \\
&+ N_7\left(c_2\int_0^1 (\Phi_x + l\mathbf{w} + \Psi)^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, dsdx\right)
\end{aligned}$$

then

$$\begin{aligned}
\mathfrak{L}^{'}(t) \leq{}& \left[-r_0 N + c\left(1+\frac{1}{\varepsilon}\right)N_2 + \frac{\rho_3 c}{2\varepsilon_4}N_3 + \mu_1 N_5 + l\rho_1 N_6 - \rho_1 N_7\right]\int_0^1 \Phi_t^2 dx \\
&+ \left[\rho_2 N_1 + \varepsilon N_2 - \rho_2\left(\varsigma_1 - \frac{\delta}{2\varepsilon_1} - \frac{k}{2\varepsilon_2}\right)N_3 + \varsigma_2 N_6\right]\int_0^1 \Psi_t^2 dx \\
&+ \left[\left(-b + \frac{\beta}{2\ell_1} + kd\ell_2\right)N_1 + \frac{\rho_3 b}{2\varepsilon_3}N_3 + \beta\varepsilon_5 N_4 + c_1 N_7\right]\int_0^1 \Psi_x^2 dx \\
&+ \left[\frac{kl}{2\epsilon}N_1 - l\rho_1 N_6 - \rho_1 N_7\right]\int_0^1 \mathbf{w}_t^2 dx
\end{aligned}$$

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$$+\left[\beta\ell_1 N_1-\rho_3\left(\beta-b\varepsilon_3-k\varepsilon_4\right)N_3+\left(\rho_3+\frac{\beta}{2\varepsilon_5}\right)N_4\right]\int_0^1\theta_t^2dx$$
$$+\left[-\delta N_4+\rho_2\delta\varepsilon_1 N_3\right]\int_0^1\theta_x^2dx$$
$$+\left[-kN+\rho_2 k\varepsilon_2 N_3\right]\int_0^1\theta_{tx}^2dx$$
$$+\left[\frac{k}{2\ell_2}N_1-\frac{k}{2}N_2+lkN_6+c_2N_7\right]\int_0^1\left(\Phi_x+l\mathbf{w}+\Psi\right)^2dx$$
$$+\left[-\frac{lk_0}{2}N_2+k_0N_7-\left(lk_0-\frac{m_0}{2}\right)N_6\right]\int_0^1\left(\mathbf{w}_x-l\Phi\right)^2dx$$
$$+\left[-e^{-\tau_2}N_5+cN_2+\frac{N_6}{2}+\frac{N_7}{2}\right]\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^2(x,1,s,t)\,dsdx$$
$$+\left[-e^{-\tau_2}N_5\right]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2(s)\,z^2(x,\rho,s,t)\,dsd\rho dx.$$

At this point we choose $\varepsilon_i\ i=1,...,5$ small enough, and N_7 large enough so that

$$\max\{\frac{k}{2\ell_2}N_1-\frac{k}{2}N_2+lkN_6+c_2N_7,-\frac{lk_0}{2}N_2+k_0N_7$$
$$-\left(lk_0-\frac{m_0}{2}\right)N_6,-e^{-\tau_2}N_5+cN_2+\frac{N_6}{2}+\frac{N_7}{2}0\}\ <0\ , \tag{3.21}$$

$$-r_0N+c\left(1+\frac{1}{\varepsilon}\right)N_2+\frac{\rho_3 c}{2\varepsilon_4}N_3+\mu_1N_5+l\rho_1N_6-\rho_1N_7\ <0\ .$$

Once N_7 is fixed, we then choose N_1 large enough such that

$$\left(-b+\frac{\beta}{2\ell_1}+kd\ell_2\right)N_1+\frac{\rho_3 b}{2\varepsilon_3}N_3+\beta\varepsilon_5N_4+c_1N_7<0,$$

we choose N_3,N_4,N_6 large enough such that

$$\left(-b+\frac{\beta}{2\ell_1}+kd\ell_2\right)N_1+\frac{\rho_3 b}{2\varepsilon_3}N_3+\beta\varepsilon_5N_4+c_1N_7<0,$$

$$\frac{kl}{2\epsilon}N_1-l\rho_1N_6-\rho_1N_7<0,$$

$$-e^{-\tau_2}N_5+cN_2+\frac{N_6}{2}+\frac{N_7}{2}<0,$$

$$\max\{-\delta N_4+\rho_2\delta\varepsilon_1N_3,\ \ -kN+\rho_2k\varepsilon_2N_3\}<0,$$

finally, we choose N large enough such that,

$$-r_0N+c\left(1+\frac{1}{\varepsilon}\right)N_2+\frac{\rho_3 c}{2\varepsilon_4}N_3+\mu_1N_5+l\rho_1N_6-\rho_1N_7<0.$$

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Exploiting Young's, Poincaré's and Cauchy-Schwarz inequalities, and the fact that $e^{-s\rho} \leq 1$ for all $\rho \in [0,1]$, we obtain

$$\begin{aligned}\mathfrak{L}(t)| &\leq c\int_0^1 \left[\Phi_t^2 + \Psi_t^2 + \Psi_x^2 + \mathbf{w}_t^2 + \theta_t^2 + \theta_x^2 + \theta_{tx}^2, (\Phi_x + l\mathbf{w} + \Psi)^2 + (\mathbf{w}_x - l\Phi)^2\right] dx \\ &+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)|z^2(x,\rho,s,t)dsd\rho dx \\ &\leq cE(t).\end{aligned}$$

Consequently, $|\mathcal{L}(t) - NE(t)| \leq cE(t)$ which yields

$$(N-c)E(t) \leq \mathfrak{L}(t) \leq (N+c)E(t), \tag{3.22}$$

choosing N such that $N - c > 0$

$$\mathcal{L}'(t) \leq -\alpha_0 E(t), \ \forall t \geq 0,$$

for some $\alpha_0 > 0$. A combination of lemma(3.10) gives

$$\mathcal{L}'(t) \leq -k_1 \mathcal{L}(t), \ \forall t \geq 0, \tag{3.23}$$

where $k_1 = \frac{\alpha_0}{c_2}$.

Finally, a simple integration of (3.23) we obtain (3.2), which complete the proof. $\square$

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