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Geometric Studies on Inequalities of Harmonic Functions in a Complex Field Based on ξ-Generalized Hurwitz-Lerch Zeta Function

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Abstract. Authors, define and establish a new subclass of harmonic regular schlicht functions (HSF) in the open unit disc through the use of the extended generalized Noor-type integral operator associated with the ξ-generalized Hurwitz-Lerch Zeta function (GHLZF). Furthermore, some geometric properties of this subclass are also studied.

Keywords: Harmonic function, Regular function, Schlicht function, Noor integral operator, ξ-generalized Hurwitz-Lerch zeta function.

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1. Introduction

Harmonic schlicht functions (HSF) have vastly been utilized not only in applied mathematics, but also in other numerous fields such as physics, aerodynamics, electronics, operation research, engineering and medicine. The theory of HSF is categorized under the most interesting topic in Geometric Function Theory (GFT), which is a generalization of the regular schlicht functions (RSF). Since then, the study of geometrical properties of HSF is key in ongoing seek in GFT.

The first study on the theory of HSF was by Clunie and Sheil-Small [8] in 1984. In their seminal work, they studied each harmonic function φ on a simply connected domain which can be expressed in the form $\varphi = \mu + \overline{\nu}$. The function μ is called the regular part while ν is the co-regular part of φ. A necessary and sufficient condition [8] for φ to be locally univalent and sense preserving in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is $|\mu'(z)| > |\nu'(z)|$ in $\mathbb{D}$. Moreover, they introduced the class $\mathbb{H}$ consisting of HSF $\varphi = \mu + \overline{\nu}$ which are sense-preserving in $\mathbb{D}$, and normalized by the conditions $\varphi(0) = \varphi'(0) - 1 = 0$, where the regular part μ and the co-regular part ν are given as follows:

$$\mu(z) = z + \sum_{\iota=2}^{\infty} \gamma_\iota z^\iota, \quad \nu(z) = \sum_{\iota=1}^{\infty} \eta_\iota z^\iota, \quad |\eta_1| < 1. \tag{1.1}$$

Obviously, class $\mathbb{H}$ reduces to class $\mathbb{S}_{\mathbb{A}}$, which includes normalized RSF φ defined in $\mathbb{D}$, if the co-regular part ν is zero. As a result, the functions $\varphi(z)$, in this case, can be expressed as:

$$\varphi(z) = z + \sum_{\iota=2}^{\infty} \gamma_\iota z^\iota. \tag{1.2}$$

In RSF theory, there are assorted studies on convexity and starlikeness and other properties of such functions. Further, some analogous studies were done on harmonic functions investigating convexity, starlikeness, and others. Clunie and Sheil-Small [8] were the first to introduce a subclass of $\mathbb{H}$ consisting of harmonic convex functions, symbolized by $\mathbb{H}_{\mathbb{CV}}$. In addition, they investigated geometric properties of class $\mathbb{H}$ such as coefficient bounds, growth and distortion bounds and covering theorems. Then, in 1990, Sheil-Small [26] considered a subclass of $\mathbb{H}$ involving harmonic starlike functions, symbolized by $\mathbb{H}_{\mathbb{ST}}$.

In 1975, Silverman [28] presented the class $\mathbb{S}_{\mathbb{T}}$ of RSF with negative coefficients and opened new trends for studies. The subclasses of class $\mathbb{S}_{\mathbb{T}}$ have been explored by numerous researchers for different objectives with various properties. Subclasses analogous to these results have not been explored on HSF in the literature. In 1998, Silverman [29] attempted to fill this gap by considering the class $\mathbb{H}_{\mathbb{T}}$ of HSF with negative coefficients rather than positive coefficients.

Denote by $\mathbb{H}_{\mathbb{T}}$ the subclass of $\mathbb{H}$ including functions $\varphi = \mu + \overline{\nu}$, such that μ and ν are of the formula

$$\mu(z) = z - \sum_{\iota=2}^{\infty} |\gamma_\iota|\, z^\iota, \quad \nu(z) = -\sum_{\iota=1}^{\infty} |\eta_\iota|\, z^\iota, \quad |\eta_1| < 1. \tag{1.3}$$

Since then, various subclasses of HSF and theirs properties have been investigated by numerous complex analysts. One may refer to Jahangiri and Ahuja [17, 18], among many others.

Convolution (Hadamard product) is a significant tool for identifying diverse subclasses and operators. In [8], Clunie and Sheil-Small introduced the concept of convolution of two harmonic univalent functions as follows:
for two functions $\varphi_\kappa \in \mathbb{H}$ is given by $\varphi_\kappa(z) = \mu_\kappa(z) + \overline{\nu_\kappa(z)} = z + \sum_{\iota=2}^{\infty} \gamma_{\iota,\kappa} z^\iota + \overline{\sum_{\iota=1}^{\infty} \eta_{\iota,\kappa}, z^\iota}$, $\kappa = 1, 2$, $|\eta_{1,1}| < 1$, $|\eta_{1,2}| < 1$, $z \in \mathbb{D}$, the convolution is denoted by $\varphi_1 * \varphi_2$ and defined as:

$$(\varphi_1 * \varphi_2)(z) = z + \sum_{\iota=2}^{\infty} \gamma_{\iota,1}\, \gamma_{\iota,2}\, z^\iota + \overline{\sum_{\iota=1}^{\infty} \eta_{\iota,1}\, \eta_{\iota,2}\, z^\iota}. \tag{1.4}$$

Denote by $\mathbb{A}$ the class of normalized regular functions φ in $\mathbb{D}$. The first integral operator defined on $\mathbb{A}$ was proposed by Alexander [2]. Later, in 1965, Libera [19] considered another integral operator and discussed specific properties of starlike functions under this operator. These works stimulated many researchers for studying operators, such as Bernardi [5] in 1969, Miller, Mocanu and Reade [21] in 1974. The following year, Ruscheweyh [25] presented the differential operator $D^l\varphi(z)$, for $\varphi \in \mathbb{A}$, by utilizing convolution concept as follows: for a function $\varphi \in \mathbb{A}$ and $l > -1$, the Ruscheweyh differential operator $D^l\varphi(z)$ is defined by $D^l : \mathbb{A} \to \mathbb{A}$,

$$D^l\varphi(z) = \frac{z}{(1-z)^{l+1}} * \varphi(z) = z + \sum_{\iota=2}^{\infty} \frac{(l+1)_{\iota-1}}{(\iota-1)!}\, \gamma_\iota z^\iota \tag{1.5}$$

such that $D^0\varphi(z) = \varphi(z)$ and $D^1\varphi(z) = z\varphi'(z)$, $z \in \mathbb{D}$.

Corresponding to differential operator $D^l\varphi(z)$, Noor [22] in 1999 imposed an integral operator, which is denoted by $I_l\varphi(z)$ and called the Noor Integral operator of $l-$th order of φ, as follows:
for a function $\varphi \in \mathbb{A}$ and $l \in \mathbb{N}_0$, the Noor integral operator $I_l\varphi(z)$ is defined

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by $I^l : \mathbb{A} \to \mathbb{A}$,

$$I_l\varphi(z) = \varphi_l^{(-1)}(z) * \varphi(z) = \left[\frac{z}{(1-z)^{l+1}}\right]^{-1} * \varphi(z)$$
$$= z + \sum_{\iota=2}^{\infty} \frac{\iota!}{(l+1)_{\iota-1}}\, \gamma_\iota z^\iota, \tag{1.6}$$

such that

$$\varphi_l(z) * \varphi_l^{(-1)}(z) = \frac{z}{(1-z)^2}. \tag{1.7}$$

Note that $I_0\varphi(z) = z\varphi'(z)$ and $I_1\varphi(z) = \varphi(z)$, $z \in \mathbb{D}$.

Afterwards, numerous authors have introduced and studied several Noor-type integral operators by employing hypergeometric functions and theirs generalizations and extension. Some of the previous studies will be mentioned here.

In 2006, Noor [23] once again imposed Noor integral operator $I_l(\varrho, \varsigma, \tau)\varphi(z)$ on $\mathbb{A}$ by utilizing the well known Gauss hypergeometric function as follows:

$$I_l(\varrho, \varsigma; \tau)\varphi(z) = [zF(\varrho, \varsigma; \tau; z)]^{(-1)} * \varphi(z)$$
$$= z + \sum_{\iota=2}^{\infty} \frac{(\tau)_{\iota-1}(l+1)_{\iota-1}}{(\varrho)_{\iota-1}(\varsigma)_{\iota-1}}\, \gamma_n z^\iota, \tag{1.8}$$

such that

$$[zF(\varrho, \varsigma; \tau; z)] * [zF(\varrho, \varsigma; \tau; z)]^{(-1)} = \frac{z}{(1-z)^{l+1}}, \qquad (z \in \mathbb{D}). \tag{1.9}$$

and the function $F(\varrho, \varsigma; \tau; z)$ is the Gauss hypergeometric function defined as follows: (see, [9])

For ϱ, ς and τ be real or complex numbers with τ other than $0, -1, -2, ...$, and

$$F(\varrho, \varsigma; \tau; z) = \sum_{\iota=0}^{\infty} \frac{(\varrho)_\iota(\varsigma)_\iota}{(\tau)_\iota(1)_\iota}\, z^\iota = 1 + \frac{\varrho\varsigma}{\tau} z + \frac{\varrho(\varrho+1)\varsigma(\varsigma+1)}{\tau(\tau+1)} \frac{z^2}{2!} + ... \tag{1.10}$$

where, $(\sigma)_\iota$ is the Pochhammer symbol given by

$$(\sigma)_\iota := \frac{\Gamma(\sigma+\iota)}{\Gamma(\sigma)} = \begin{cases} 1, & (\iota = 0), \\ \sigma(\sigma+1)(\sigma+2)...(\sigma+\iota-1), & (\iota \in \mathbb{N}). \end{cases} \tag{1.11}$$

In 2015, Ibrahim *et al.* [16] defined the following generalized Noor-type integral operator $Q^{\xi}_{\ell;\kappa,\varepsilon}(\tau; \varrho, \varsigma; z; m)\varphi(z)$ on class $\mathbb{A}$ by making use of the extension

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Gauss hypergeometric functions:

$$Q^{\xi}_{\ell;\kappa,\varepsilon}(\tau;\varrho,\varsigma;z;m)\varphi(z)$$

$$= z+\sum_{\iota=2}^{\infty}\frac{(\tau)_{\iota-1}}{(\varrho)_{\iota-1}(\varsigma)_{\iota-1}}\frac{\Omega\, B(\varsigma+\iota-1,\tau-\varsigma+m)}{B^{\alpha,\beta;\kappa,\varepsilon}_{\ell}(\varsigma+\iota-1,\tau-\varsigma+m)}(\xi+1)_{\iota-1}\,\gamma_\iota\, z^{\iota}, \quad (1.12)$$

where

$$\Omega=\frac{B^{\alpha,\beta;\kappa,\varepsilon}_{\ell}(\varsigma,\tau-\varsigma+m)}{B(\varsigma,\tau-\varsigma+m)}, \quad (1.13)$$

a further extension for the extended $F_{\ell;\kappa,\varepsilon}(\varrho,\varsigma;\tau;z;m)$ Gauss hypergeometric functions is given by (see [1])

$$F_{\ell;\kappa,\varepsilon}(\varrho,\varsigma;\tau;z;m):=\sum_{\iota=0}^{\infty}\frac{(\varrho)_\iota(\varsigma)_\iota}{(\tau)_\iota}\frac{B^{\alpha,\beta;\kappa,\varepsilon}_{\ell}(\varsigma+\iota,\tau-\varsigma+m)}{B(\varsigma+\iota,\tau-\varsigma+m)}\frac{z^\iota}{\iota!}, \quad (1.14)$$

$$\Big(0\le\ell,\ 0<\Re(\kappa),\ 0<\Re(\varepsilon),\ m<\Re(\varsigma)<\Re(\tau),\ |z|<1\Big),$$

$B(\omega,\nu)$ is the familiar Beta function defined by (see, [36], p.8)

$$B(\omega,\nu)=\begin{cases}\int_0^1\rho^{\omega-1}(1-\rho)^{\nu-1}d\rho & (0<\Re(\rho);0<\Re(\nu))\\[2ex] \frac{\Gamma(\omega)\ \Gamma(\nu)}{\Gamma(\omega+\nu)} & (\omega,\nu\in\mathbb{C}\backslash\mathbb{Z}_0^-)\end{cases} \quad (1.15)$$

and the extended Beta function $B^{(\alpha,\beta;\kappa,\varepsilon)}_{\ell}(\omega,\nu)$, which is defined by (see, [38])

$$B^{(\alpha,\beta;\kappa,\varepsilon)}_{\ell}(\omega,\nu)=\int_0^1\rho^{\omega-1}(1-\rho)^{\nu-1}F\left(\alpha;\beta;-\frac{\ell}{\rho^\kappa(1-\rho)^\varepsilon}\right)d\rho, \quad (1.16)$$

$$\Big(0\le\kappa,\ 0\le\varepsilon,\ 0\le\Re(\ell),\ 0<\min\{\Re(\alpha),\Re(\beta)\},\ -\Re(\kappa\alpha)<\Re(\omega),-\Re(\varepsilon\alpha)<\Re(\nu)\Big).$$

Recently, Al-Janaby *et al.* [3] defined an extended generalized integral operator of Noor-type on the class of the harmonic multivalent functions by using Fox-Wright generalized hypergeometric function as follows:

$$GN^{a,b}_{p}[\wp_1]\varphi(z)=GN^{a,b}_{p}[\wp_1]_p[\mu_1]\mu(z)+\overline{GN^{a,b}_{p}[\wp_1]_p[\mu_1]\nu(z)}, \quad (1.17)$$

where,

$$GN^{a,b}_{p}[\wp_1]\mu(z)=z^p+\sum_{\iota=p+1}^{\infty}\frac{\Delta\prod_{l=1}^{b}\Gamma(\chi_l+(\iota-p)\beta_l)}{\prod_{l=1}^{a}\Gamma(\wp_l+(\iota-p)\alpha_l)}(\xi+p)_{\iota-p}\,\gamma_\iota\, z^\iota, \quad (1.18)$$

and

$$GN^{a,b}_{p}[\wp_1]\nu(z)=z^p+\sum_{\iota=p}^{\infty}\frac{\Delta\prod_{l=1}^{b}\Gamma(\chi_l+(\iota-p)\beta_l)}{\prod_{l=1}^{a}\Gamma(\wp_l+(\iota-p)\alpha_l)}(\xi+p)_{\iota-p}\,\eta_\iota\, z^\iota. \quad (1.19)$$

On the other hand, the study of Hurwitz-Lerch Zeta function plays a significant role in GFT. Several authors have introduced numerous linear convolution operators on various regular function classes by utilizing Hurwitz-Lerch Zeta function and various of its generalizations. One may refer to some of theirs contributions: Ghanim [13], Ghanim and Al-Janaby [14], Ghanim and Darus [15], Răducanu and Srivastava [24], Srivastava and Attiya [35], Srivastava *et.al.* [41, 42].

In terms of the Hurwitz-Lerch Zeta function $\Theta(z,\upsilon,\omega)$ defined by (see, for example, [34, p. 121 *et seq.*], [33] and [37, p. 194 *et seq.*])

$$\Theta(z,\upsilon,\omega) := \sum_{\iota=0}^{\infty} \frac{z^{\iota}}{(\iota+\omega)^{\upsilon}} \tag{1.20}$$

$$(\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \upsilon \in \mathbb{C} \quad \text{when} \quad |z|<1; 1<\Re(\upsilon) \quad \text{when} \quad |z|=1).$$

The following new family of the ξ-Generalized Hurwitz-Lerch Zeta function (GHLZF) was discussed and studied systematically by Srivastava [39]:

$$\Theta^{(\varrho_1,\cdots,\varrho_p,\varsigma_1,\cdots,\varsigma_q)}_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}(z,\upsilon,\omega;\kappa,\xi)$$
$$= \frac{1}{\xi\,\Gamma(\upsilon)} \sum_{\iota=0}^{\infty} \frac{\prod_{j=1}^{p}(\xi_j)_{\iota\varrho_j}}{(\omega+\iota)^{\upsilon}\cdot\prod_{j=1}^{q}(\delta_j)_{\iota\varsigma_j}}\, H^{2,0}_{0,2}\left[(\omega+\iota)\kappa^{\frac{1}{\xi}}\,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (\upsilon,1),\left(0,\frac{1}{\xi}\right)\end{matrix}\right] \frac{z^{\iota}}{\iota!} \tag{1.21}$$

$$\big(0<\min\{\Re(\omega),\Re(\upsilon)\};\ 0<\Re(\kappa);\ 0<\xi\big),$$

where

$$\Bigg(\xi_j \in \mathbb{C}\ \ (j=1,\cdots,p);\ \delta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ \ (j=1,\cdots,q);\ 0<\varrho_j\ \ (j=1,\cdots,p);$$

$$0<\varsigma_j\ \ (j=1,\cdots,q);\ \text{and}\ \ 0 \leqq 1+\sum_{j=1}^{q}\varsigma_j-\sum_{j=1}^{p}\varrho_j\Bigg)$$

and the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p}\varrho_j^{-\varrho_j}\right)\cdot\left(\prod_{j=1}^{q}\varsigma_j^{\varsigma_j}\right).$$

Definition 1.1. The H-function involved in the right-hand side of (1.21) is the well-known Fox's H-function [[20], Definition 1.1] (see also [32],[31]) defined by

$$H_{\mathfrak{p},\mathfrak{q}}^{m,n}(z) = H_{\mathfrak{p},\mathfrak{q}}^{m,n}\left[z \left| \begin{matrix} (\omega_1, A_1), \cdots, (\omega_{\mathfrak{p}}, A_{\mathfrak{p}}) \\ (\kappa_1, B_1), \cdots, (\kappa_{\mathfrak{q}}, B_{\mathfrak{q}}) \end{matrix} \right. \right] = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \Xi(\upsilon) z^{-\upsilon} \, \mathrm{d}\upsilon \qquad \big(z \in \mathbb{C} \setminus \{0\};\ |\arg(z)| < \pi\big), \tag{1.22}$$

an empty product is interpreted as 1, $m, n, \mathfrak{p}$ and $\mathfrak{q}$ are integers such that

$$1 \leqq m \leqq \mathfrak{q} \qquad \text{and} \qquad 0 \leqq n \leqq \mathfrak{p},$$

$$0 < A_j \quad (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad 0 < B_j \quad (j = 1, \cdots, \mathfrak{q}),$$

$$\omega_j \in \mathbb{C} \quad (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad \kappa_j \in \mathbb{C} \quad (j = 1, \cdots, \mathfrak{q})$$

and $\mathcal{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(\kappa_j + B_j \upsilon)\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\{\Gamma(1 - \omega_j + A_j \upsilon)\}_{j=1}^{n}.$$

It is worthy of mention here that, by using the fact that [[39], p. 1496, Remark 7]

$$\lim_{\kappa \to 0} \left\{ H_{0,2}^{2,0} \left[(\omega + \iota)\kappa^{\frac{1}{\xi}} \left| \begin{matrix} \overline{\qquad\qquad} \\ (\upsilon, 1), \left(0, \frac{1}{\xi}\right) \end{matrix} \right. \right] \right\} = \xi \, \Gamma(\upsilon) \qquad (0 < \xi),$$

the equation (1.21) reduces to the following form:

$$\Theta_{\xi_1, \cdots, \xi_p; \delta_1, \cdots, \delta_q}^{(\varrho_1, \cdots, \varrho_p, \varsigma_1, \cdots, \varsigma_q)}(z, \upsilon, \omega; 0, \xi) := \Theta_{\xi_1, \cdots, \xi_p; \delta_1, \cdots, \delta_q}^{(\varrho_1, \cdots, \varrho_p, \varsigma_1, \cdots, \varsigma_q)}(z, \upsilon, \omega) = \sum_{\iota=0}^{\infty} \frac{\prod_{j=1}^{p} (\xi_j)_{\iota \varrho_j}}{(\omega + \iota)^{\upsilon} \cdot \prod_{j=1}^{q} (\delta_j)_{\iota \varsigma_j}} \frac{z^{\iota}}{\iota!}. \tag{1.23}$$

Definition 1.2. The function $\Theta_{\xi_1, \cdots, \xi_p; \delta_1, \cdots, \delta_q}^{(\varrho_1, \cdots, \varrho_p, \varsigma_1, \cdots, \varsigma_q)}(z, \upsilon, \omega)$ involved in (1.23) is the multiparameter extension and generalization of the Hurwitz-Lerch Zeta function $\Theta(z, \upsilon, \omega)$ introduced by Srivastava *et al.* *[*[40]*, p. 503, Eq. (6.2)]* defined by

$$\Theta_{\xi_1, \cdots, \xi_p; \delta_1, \cdots, \delta_q}^{(\varrho_1, \cdots, \varrho_p, \varsigma_1, \cdots, \varsigma_q)}(z, \upsilon, \omega) = \sum_{\iota=0}^{\infty} \frac{\prod_{j=1}^{p} (\xi_j)_{\iota \varrho_j}}{(\omega + \iota)^{\upsilon} \cdot \prod_{j=1}^{q} (\delta_j)_{\iota \varsigma_j}} \frac{z^{\iota}}{\iota!}, \tag{1.24}$$

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$$\Bigg(p, q \in \mathbb{N}_0;\ \xi_j \in \mathbb{C}\ (j = 1, \cdots, p);\ \omega, \delta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\ (j = 1, \cdots, q);$$
$$\varrho_j, \varsigma_k \in \mathbb{R}^+\ (j = 1, \cdots, p;\ k = 1, \cdots, q);$$
$$-1 < \Delta \text{ when } \upsilon, z \in \mathbb{C};$$
$$\Delta = -1 \text{ and } \upsilon \in \mathbb{C} \text{ when } |z| < \nabla^*;$$
$$\Delta = -1 \text{ and } \frac{1}{2} < \Re(\Xi) \text{ when } |z| = \nabla^*\Bigg)$$

with

$$\nabla^* := \left(\prod_{j=1}^{p} \varrho_j^{-\varrho_j}\right) \cdot \left(\prod_{j=1}^{q} \varsigma_j^{\varsigma_j}\right),$$

$$\Delta := \sum_{j=1}^{q} \varsigma_j - \sum_{j=1}^{p} \varrho_j \quad \text{and} \quad \Xi := s + \sum_{j=1}^{q} \delta_j - \sum_{j=1}^{p} \xi_j + \frac{p-q}{2}.$$

Consequently, by the above works on Noor integral operator and its generalizations. In Section 2, we continue our investigations and studies in the theory of operators. Here we'll introduce a new extended generalized Noor-type integral operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon, \omega; \kappa, \xi)\, \varphi(z)$ on $\mathbb{H}$, which is defined by employing GHLZF.

In earlier investigations, various subclasses of $\mathbb{H}$ were proposed by researchers, such as Aydoğan *et al.* [4], Duman *et al.* [10], Dziok *et al.* and [11] El-Ashwah [12]. From another side, several authors introduced a sequence of classes of HSF. This line of study are presented here.

In 2003, Yalcin *et al.* [43] established a subclass $\mathbb{H}_1(\beta) = HP(\beta)$ consisting of functions $\varphi \in \mathbb{H}$ which achieve the following first-order differential inequality:

$$\Re\{\mu'(z) + \nu'(z)\} > \beta, \qquad (0 \le \beta < 1). \tag{1.25}$$

They also studied a sufficient condition $\sum_{\iota=1}^{\infty} \iota\,(|\gamma_\iota| + |\eta_\iota|) \le 2 - \beta$, where $\gamma_1 = 1$, for functions to be in the subclass $\mathbb{H}_1(\beta)$. This condition is necessary when the coefficients are negative. Moreover, they investigated distortion theorems and extreme points.

In 2004, Yalcin and Öztürk [44] considered a subclass $\mathbb{H}_2(\alpha) = HP(\alpha)$ composing of functions $\varphi \in \mathbb{H}$ which achieve the second-order differential inequality as:

$$\Re\{\alpha z\,(\mu''(z) + \nu''(z)) + (\mu'(z) + \nu'(z))\} > 0, \qquad (0 \le \alpha). \tag{1.26}$$

In addition, they discussed a sufficient condition $\sum_{\iota=1}^{\infty} \iota\,(1 + \alpha(\iota - 1))\,(|\gamma_\iota| + |\eta_\iota|)$ ≤ 2, where $\gamma_1 = 1$, for functions involving the aforementioned subclass $\mathbb{H}_2(\alpha)$,

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which is shown to be necessary when the coefficients are negative. They analyzed distortion theorems and extreme points as well.

In 2010, based on the study of Yalcin and Öztürk [44], Chandrashekar *et al.* [7] introduced a subclass $\mathbb{H}_2(\alpha,\beta) = HP(\alpha,\beta)$ of class $\mathbb{H}$, which achieves the following condition:

$$\Re\{\alpha z\left(\mu''(z)+\nu''(z)\right)+\left(\mu'(z)+\nu'(z)\right)\}>\beta, \qquad (0\le\alpha,\ 0\le\beta<1). \quad (1.27)$$

They also investigated a sufficient condition $\sum_{\iota=1}^{\infty}\iota\left(1+\alpha(\iota-1)\right)\left(|\gamma_\iota|+|\eta_\iota|\right)\le 2-\beta$, where $\gamma_1=1$, for functions including to above subclass $\mathbb{H}_2(\alpha,\beta)$, which is shown to be necessary when the coefficients are negative.

In 2015, Sokól *et al.* [30] imposed a subclass $\mathbb{H}_3(\alpha,\beta) = R_H(\alpha,\beta)$ of class $\mathbb{H}$, which achieves the third-order differential inequality as follows:

$$\Re\{\alpha z^2\left(\mu'''(z)+\nu'''(z)\right)+3\alpha z\left(\mu''(z)+\nu''(z)\right)+\left(\mu'(z)+\nu'(z)\right)\}>\beta, \quad (1.28)$$

$(0\le\alpha,\ 0\le\beta<1)$.
They examined a sufficient condition $\sum_{\iota=1}^{\infty}\iota\left(1+\alpha(\iota^2-1)\right)\left(|\gamma_\iota|+|\eta_\iota|\right)\le 2-\beta$, where $\gamma_1=1$, for functions belonging to above subclass $\mathbb{H}_3(\alpha,\beta)$, which is shown to be necessary when the coefficients are negative. Furthermore, distortion bounds, extreme points, convolution and convex combinations are studied.

Motivated by previous works, Section 3 imposes a new subclass $\mathbb{GH}_4(\alpha,\beta)$ of $\mathbb{H}$ defined by utilizing a new extended $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$ generalized Noor-type integral operator which is considered in Section 2 satisfying the forth-order differential inequality. Moreover, coefficient bounds, distortion bounds, extreme points, convolution, convex combinations, and closure under an integral operator are given.

2. Proposed Operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$

In this section, by employing GHLZF given in (1.21), we define a new extended generalized Noor-type integral operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$ on $\mathbb{H}$.

By setting $\varrho_1=\cdots,\varrho_p=\varsigma_1=\cdots=\varsigma_q=1,$ and $\xi_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$ $(j=1,\cdots,p)$ in (1.21), we yield that

$$\Theta_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}^{(1,\cdots,1,1,\cdots,1)}(z,\upsilon,\omega;\kappa,\xi)$$
$$=\frac{1}{\xi\,\Gamma(\upsilon)}\sum_{\iota=0}^{\infty}\frac{\prod_{j=1}^{p}(\xi_j)_\iota}{(\omega+\iota)^{\upsilon}\cdot\prod_{j=1}^{q}(\delta_j)_\iota}\,H_{0,2}^{2,0}\left[(\omega+\iota)\kappa^{\frac{1}{\xi}}\,\middle|\,\begin{matrix}\overline{\qquad\quad}\\(\upsilon,1),\left(0,\frac{1}{\xi}\right)\end{matrix}\right]\frac{z^\iota}{\iota!}$$

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$$= \frac{1}{\xi\ \Gamma(\upsilon)} \sum_{\iota=1}^{\infty} \frac{\prod_{j=1}^{p} (\xi_j)_{\iota-1}}{(\omega+(\iota-1))^{\upsilon} \cdot \prod_{j=1}^{q} (\delta_j)_{\iota-1}}\ H_{0,2}^{2,0}\left[(\omega+(\iota-1))\kappa^{\frac{1}{\xi}} \middle|\ \overline{\qquad\qquad} \atop (\upsilon,1),\left(0,\frac{1}{\xi}\right)\right]\ \frac{z^{\iota-1}}{(\iota-1)!}.$$

Next, we consider a new function $\left(z\Theta^{(1,\cdots,1,1,\cdots,1)}_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}(z,\upsilon,\omega;\kappa,\xi)\right)^{-1}$ as:

$$\left(z\Theta^{(1,\cdots,1,1,\cdots,1)}_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}(z,\upsilon,\omega;\kappa,\xi)\right)^{-1}$$

$$= \xi\ \Gamma(\upsilon) \sum_{\iota=1}^{\infty} \frac{(\omega+(\iota-1))^{\upsilon} \cdot \prod_{j=1}^{q} (\delta_j)_{\iota-1}\ \ (\zeta+1)_{\iota-1}}{\prod_{j=1}^{p} (\xi_j)_{\iota-1}\ \ H_{0,2}^{2,0}\left[(\omega+(\iota-1))\kappa^{\frac{1}{\xi}} \middle|\ \overline{\qquad\qquad} \atop (\upsilon,1),\left(0,\frac{1}{\xi}\right)\right]}\ z^{\iota}, \tag{2.1}$$

such that,

$$\left(z\Theta^{(1,\cdots,1,1,\cdots,1)}_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}(z,\upsilon,\omega;\kappa,\xi)\right) * \left(z\Theta^{(1,\cdots,1,1,\cdots,1)}_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}(z,\upsilon,\omega;\kappa,\xi)\right)^{-1}$$

$$= \frac{z}{(1-z)^{\zeta+1}} \quad = z + \sum_{\iota=2}^{\infty} \frac{(\zeta+1)_{\iota-1}}{(\iota-1)!}\ z^{\iota}, \tag{2.2}$$

where $-1 < \zeta$.

Thus, from (2.1), we impose the following operator $\Sigma^{\delta_1,\ldots,\delta_q}_{\xi_1,\ldots,\xi_p}(\upsilon,\omega;\kappa,\xi) : \mathbb{A} \longrightarrow \mathbb{A}$:

$$\Sigma^{\delta_1,\ldots,\delta_q}_{\xi_1,\ldots,\xi_p}(\upsilon,\omega;\kappa,\xi)\,\varphi(z) = \frac{\Lambda}{\xi\Gamma(\upsilon)}\left(z\Theta^{(1,\cdots,1,1,\cdots,1)}_{\xi_1,\cdots,\xi_p;\delta_1,\cdots,\delta_q}(z,\upsilon,\omega;\kappa,\xi)\right)^{-1} * \varphi(z) \tag{2.3}$$

where Λ is defined by

$$\Lambda = \frac{H_{0,2}^{2,0}\left[(\omega)\kappa^{\frac{1}{\xi}} \middle|\ \overline{\qquad\qquad} \atop (\upsilon,1),\left(0,\frac{1}{\xi}\right)\right]}{(\omega)^{\upsilon}}. \tag{2.4}$$

The calculation gives the generalized Noor-type integral operator as:

$$\Sigma^{\delta_1,\ldots,\delta_q}_{\xi_1,\ldots,\xi_p}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$$

$$= z + \sum_{\iota=2}^{\infty} \frac{\Lambda\ \ (\omega+(\iota-1))^{\upsilon}\ \ \prod_{j=1}^{q} (\delta_j)_{\iota-1}\ \ (\zeta+1)_{\iota-1}}{\prod_{j=1}^{p} (\xi_j)_{\iota-1}\ \ H_{0,2}^{2,0}\left[(\omega+(\iota-1))\kappa^{\frac{1}{\xi}} \middle|\ \overline{\qquad\qquad} \atop (\upsilon,1),\left(0,\frac{1}{\xi}\right)\right]}\ \ \gamma_{\iota}\ z^{\iota}. \tag{2.5}$$

Remark 2.1. For some suitably chosen parameters $\kappa,\ \xi,\ \upsilon,\ \omega,\ q,\ p,\ \delta_j$ and ξ_j, the operator $\Sigma^{\delta_1,\ldots,\delta_q}_{\xi_1,\ldots,\xi_p}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$ defined in (2.5) can be reduced to various operators previously mentioned. Thus, we have the following special cases:

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(1) By taking $\kappa = 0$, $q = 1$, $p = 2$, $\omega = \upsilon = \delta_1 = \xi_1 = 1$, $\xi_2 = 2$ and $\zeta = l$ in (2.5), we obtain the Ruscheweyh's differential operator defined by (1.5).
(2) For $\kappa = 0$, $q = 1$, $p = 2$, $\omega = \upsilon = \delta_1 = 1$, $\xi_1 = l + 1$ and $\xi_2 = \zeta + 1$ the operator (2.5) reduces to Noor integral operator given in (1.6).
(3) For $\kappa = 0$, $q = 1$, $p = 2$, $\omega = \upsilon = 1$, $\delta_2 = \tau$, $\xi_1 = \varrho$, $\xi_2 = \varsigma$ and $\zeta = l$ the operator (2.5) provides Noor-type integral operator defined in (1.8).

Therefore, the generalized Noor-type integral operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$ when extended to HSF $\varphi = \mu + \bar{\nu}$ is defined by

$$\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z) = \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu(z) + \overline{\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu(z)}, \tag{2.6}$$

where

$$\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu(z)$$

$$= z + \sum_{\iota=2}^{\infty} \frac{\Lambda \quad (\omega + (\iota - 1))^{\upsilon} \quad \prod_{j=1}^{q} (\delta_j)_{\iota-1} \quad (\zeta + 1)_{\iota-1}}{\prod_{j=1}^{p} (\xi_j)_{\iota-1} \quad H_{0,2}^{2,0}\left[(\omega + (\iota - 1))\kappa^{\frac{1}{\xi}} \,\middle|\, \overline{\quad\quad} \atop (\upsilon, 1), \left(0, \frac{1}{\xi}\right) \right]} \; \gamma_{\iota}\, z^{\iota}, \tag{2.7}$$

and

$$\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu(z)$$

$$= \sum_{\iota=1}^{\infty} \frac{\Lambda \quad (\omega + (\iota - 1))^{\upsilon} \quad \prod_{j=1}^{q} (\delta_j)_{\iota-1} \quad (\zeta + 1)_{\iota-1}}{\prod_{j=1}^{p} (\xi_j)_{\iota-1} \quad H_{0,2}^{2,0}\left[(\omega + (\iota - 1))\kappa^{\frac{1}{\xi}} \,\middle|\, \overline{\quad\quad} \atop (\upsilon, 1), \left(0, \frac{1}{\xi}\right) \right]} \; \eta_{\iota}\, z^{\iota}. \tag{2.8}$$

3. Geometric outcomes

This section is composed of two subsections. New subclass $\mathbb{GH}_4(\alpha,\beta)$ of HSF associated with a new extended generalized Noor-type integral operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$ given in (2.6) is introduced and discussed. This study aims to determine the upper bounds for the coefficients of functions included in this considered subclass. Moreover, several geometric properties are discussed.

3.1. **Subclass $\mathbb{GH}_4(\alpha,\beta)$.** This subsection presents subclasses $\mathbb{GH}_4(\alpha,\beta)$ and $\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$ of HSF with positive and negative coefficients, respectively, which include a new integral operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\varphi(z)$ written in (2.6) and achieves the forth-order differential inequity.

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Definition 3.1. A function $\varphi \in \mathbb{H}$ is said to be in subclass $\mathbb{GH}_4(\alpha, \beta)$ if it satisfies the following inequality:

$$\begin{aligned}
&\Re\{\alpha z^3 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''''(z)\right)\\
&\quad + 6\alpha z^2 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'''(z)\right)\\
&\quad + 7\alpha z \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''(z)\right)\\
&\quad + \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'(z)\right)\} > \beta,
\end{aligned} \tag{3.1}$$

where the extended generalized Noor-type integral operator $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu(z)$ and $\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu(z)$ are given in (2.7) and (2.8), respectively, $0 \leq \alpha,\ 0 \leq \beta < 1$, and $z \in \mathbb{D}$.

Also let $\mathbb{GH}_{\mathbb{T}4}(\alpha, \beta) = \mathbb{GH}_4(\alpha, \beta) \bigcap \mathbb{H}_{\mathbb{T}}$.

Remark 3.2. We note that

(1) For $\kappa = \alpha = 0,\ p = 2,\ q = \xi = \upsilon = \omega = \delta_1 = 1,\ \xi_1 = \zeta + 1$, and $\xi_2 = 2$ in (2.5), the subclass $\mathbb{GH}_4(\alpha, \beta)$ reduces to the earlier subclass $\mathbb{H}_1(\beta)$ introduced in (1.25).

(2) For $\kappa = 0,\ p = 2,\ q = \xi = \upsilon = \omega = \delta_1 = 1,\ \xi_1 = \zeta + 1$, and $\xi_2 = 2$ in (2.5), the subclass $\mathbb{GH}_4(\alpha, \beta) = \mathbb{H}_4(\alpha, \beta)$, where $\mathbb{H}_4(\alpha, \beta)$ represents the subclass of functions $\varphi = \mu + \bar{\nu}$ be of the form (1.1) satisfying the inequality

$$\begin{aligned}
&\Re\{\alpha z^3 \left(\mu''''(z) + \nu''''(z)\right) + 6\alpha z^2 \left(\mu'''(z) + \nu'''(z)\right)\\
&+7\alpha z \left(\mu''(z) + \nu''(z)\right) + \left(\mu'(z) + \nu'(z)\right)\} > \beta,
\end{aligned} \tag{3.2}$$

where $0 \leq \alpha$, and $0 \leq \beta < 1$. Moreover, let $\mathbb{H}_{\mathbb{T}4}(\alpha, \beta) = \mathbb{H}_4(\alpha, \beta) \bigcap \mathbb{H}_{\mathbb{T}}$.

3.2. **Some Properties of** $\mathbb{GH}_4(\alpha, \beta)$**.** In this subsection, a basic result is gained by involving coefficient condition for HSF with positive coefficients in $\mathbb{GH}_4(\alpha, \beta)$ and showing the prominence of this condition for negative coefficients in $\mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$. Results related to functions included in $\mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$ are also obtained, such as growth bounds, extreme points, convolution, convex combinations, and closure under an integral operator.

3.2.1. *Coefficient Condition.* A sufficient coefficient condition for harmonic schlicht functions in $\mathbb{GH}_4(\alpha, \beta)$ is provided in the first theorem.

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Theorem 3.3. *Let* $\varphi = \mu + \bar{\nu}$ *be of the form* (1.1). *If*

$$\sum_{\iota=2}^{\infty} P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)]|\gamma_\iota| + \sum_{\iota=1}^{\infty} P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)]|\eta_\iota| \leq 1-\beta, \qquad (3.3)$$

or

$$\sum_{\iota=1}^{\infty} P_{\iota-1} \, \iota[1+\alpha(\iota^3-1)](|\gamma_\iota| + |\eta_\iota|) \leq 2-\beta, \qquad (3.4)$$

where

$$P_{\iota-1} = \frac{\Lambda \quad (\omega+(\iota-1))^{\upsilon} \quad \prod_{j=1}^{q} (\delta_j)_{\iota-1} \ \ (\zeta+1)_{\iota-1}}{\prod_{j=1}^{p} (\xi_j)_{\iota-1} \ \ H_{0,2}^{2,0}\left[(\omega+(\iota-1))\kappa^{\frac{1}{\xi}} \middle| \ \overline{\qquad\qquad} \atop (\upsilon,1),\left(0,\frac{1}{\xi}\right)\right]}, \qquad (3.5)$$

$\gamma_1 = 1$, $0 \leq \alpha$, *and* $0 \leq \beta < 1$, *then* φ *is harmonic schlicht, sense-preserving in* $\mathbb{D}$, *and* $\varphi \in \mathbb{GH}_4(\alpha, \beta)$.

Proof. First, φ is shown to be schlicht in $\mathbb{D}$. Suppose $z_1, z_2 \in \mathbb{D}$ such that $z_1 \neq z_2$, then

$$\left|\frac{\varphi(z_1)-\varphi(z_2)}{\mu(z_1)-\mu(z_2)}\right| \geq 1 - \left|\frac{\nu(z_1)-\nu(z_2)}{\mu(z_1)-\mu(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{\iota=1}^{\infty} \eta_\iota (z_1^\iota - z_2^\iota)}{(z_1-z_2) - \sum_{\iota=2}^{\infty} \gamma_\iota (z_1^\iota - z_2^\iota)}\right| > 1 - \frac{\sum_{\iota=1}^{\infty} \iota|\eta_\iota|}{1-\sum_{\iota=2}^{\infty} \iota|\gamma_\iota|} \qquad (3.6)$$

$$\geq 1 - \frac{\sum_{\iota=1}^{\infty} \ P_{\iota-1} \frac{\iota[1+\alpha(\iota^3-1)]}{1-\beta}|\eta_\iota|}{1-\sum_{\iota=2}^{\infty} \ P_{\iota-1} \frac{\iota[1+\alpha(\iota^3-1)]}{1-\beta}|\gamma_\iota|} \geq 0.$$

Hence $|\varphi(z_1) - \varphi(z_2)| > 0$ and so φ is schlicht in $\mathbb{D}$.

To show that φ is locally schlicht and sense-preserving in $\mathbb{D}$. It suffices to show that $|\mu'(z)| > |\nu'(z)|$. By using ng the condition (3.3), we have

$$|\mu'(z)| \geq 1 - \sum_{\iota=2}^{\infty} \iota|\gamma_\iota||z|^{\iota-1} > 1 - \sum_{\iota=2}^{\infty} \iota|\gamma_\iota| \geq 1-\beta - \sum_{\iota=2}^{\infty} \ P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)]|\gamma_\iota| \qquad (3.7)$$

$$\geq \sum_{\iota=1}^{\infty} \ P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)]|\eta_\iota| > \sum_{\iota=1}^{\infty} \iota|\eta_\iota||z|^{\iota-1} = |\nu'(z)|.$$

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Utilizing the fact that $\Re(w) \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it is sufficient to obtain

$$\begin{aligned}
&\Big|(1-\beta) + \alpha z^3 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''''(z)\right) \\
&\quad + 6\alpha z^2 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'''(z)\right) \\
&\quad + 7\alpha z \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''(z)\right) \\
&\quad + \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'(z)\right)\Big| \\
-&\Big|(1+\beta) - \alpha z^3 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''''(z)\right) \\
&\quad - 6\alpha z^2 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'''(z)\right) \\
&\quad - 7\alpha z \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''(z)\right) \\
&\quad - \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'(z)\right)\Big| > 0
\end{aligned} \tag{3.8}$$

in proving $\varphi \in \mathbb{GH}_4(\alpha,\beta)$. Substituting for $\mu(z)$ and $\nu(z)$ in (3.8) yields

$$\begin{aligned}
&\left|(1-\beta) + 1 + \sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\gamma_\iota z^{\iota-1} + \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\eta_\iota z^{\iota-1}\right| \\
&- \left|(1+\beta) - 1 - \sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\gamma_\iota z^{\iota-1} - \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\eta_\iota z^{\iota-1}\right| \\
&\geq 2\left[(1-\beta) - \Big[\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\gamma_\iota||z|^{\iota-1} + \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\eta_\iota||z|^{\iota-1}\Big]\right] \\
&\geq 2\left[(1-\beta) - |z|\Big[\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\gamma_\iota| + \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\eta_\iota|\Big]\right] \\
&\geq 2(1-\beta)(1-|z|) > 0,
\end{aligned} \tag{3.9}$$

by the condition (3.3). The harmonic function

$$\varphi(z) = z + \sum_{\iota=2}^{\infty} \frac{1-\beta}{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)}\ x_\iota z^\iota + \sum_{\iota=1}^{\infty} \frac{1-\beta}{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)}\ \bar{y}_\iota \bar{z}^\iota \tag{3.10}$$

where $\sum_{\iota=2}^{\infty}|x_\iota| + \sum_{\iota=1}^{\infty}|y_\iota| = 1$, shows that the coefficient bound written by (3.3) is sharp. The functions of the form (3.10) are in $\mathbb{GH}_4(\alpha,\beta)$ because the condition (3.3) can be achieved as follows:

$$\begin{aligned}
&\sum_{\iota=2}^{\infty} \frac{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ |\gamma_\iota| + \sum_{\iota=1}^{\infty} \frac{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ |\eta_\iota| \\
&= \sum_{\iota=2}^{\infty}|x_\iota| + \sum_{\iota=1}^{\infty}|y_\iota| = 1.
\end{aligned} \tag{3.11}$$

This completes the proof of Theorem 3.3. □

Corollary 3.4. *Let $\varphi = \mu + \bar{\nu}$ be of the form* (1.1)*. If*

$$\sum_{\iota=2}^{\infty} \iota[1+\alpha(\iota^3-1)]|\gamma_\iota| + \sum_{\iota=1}^{\infty} \iota[1+\alpha(\iota^3-1)]|\eta_\iota| \leq 1-\beta, \tag{3.12}$$

or

$$\sum_{\iota=1}^{\infty} \iota[1+\alpha(\iota^3-1)](|\gamma_\iota| + |\eta_\iota|) \leq 2-\beta, \tag{3.13}$$

$\gamma_1 = 1$, $0 \leq \alpha$, *and* $0 \leq \beta < 1$, *then φ is harmonic schlicht, sense-preserving in $\mathbb{D}$, and $\varphi \in \mathbb{H}_4(\alpha, \beta)$.*

Proof. By part (2) of Remark 3.2 and Theorem 3.3, we have the required assertion. □

We proceed to show that, the condition (3.4) is also necessary for harmonic functions $\varphi = \mu + \bar{\nu}$, where μ and ν are of the form (1.3).

Theorem 3.5. *Let $\varphi = \mu + \bar{\nu}$ be of the form* (1.3)*. Then $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$ if and only if the condition* (3.3) *or* (3.4) *is achieved and it is as follows:*

$$\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\gamma_\iota| + \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\eta_\iota| \leq 1-\beta,$$

or

$$\sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)](|\gamma_\iota| + |\eta_\iota|) \leq 2-\beta,$$

where $P_{\iota-1}$ defined in (3.5)*,* $\gamma_1 = 1$, $0 \leq \alpha$, *and* $0 \leq \beta < 1$.

Proof. Since $\mathbb{GH}_{\mathbb{T}4}(\alpha, \beta) \subset \mathbb{GH}_4(\alpha, \beta)$. We only need to prove the "only if" part of this theorem. For function $\varphi(z)$ of the form (1.3), we have the conditions (3.1) as follows:

$$\begin{aligned}
&\Re\{\alpha z^3 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''''(z)\right)\\
&\quad + 6\alpha z^2 \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'''(z)\right)\\
&\quad + 7\alpha z \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu''(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu''(z)\right)\\
&\quad + \left(\Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\mu'(z) + \Sigma_{\xi_1,\ldots,\xi_p}^{\delta_1,\ldots,\delta_q}(\upsilon,\omega;\kappa,\xi)\,\nu'(z)\right)\} > \beta.
\end{aligned}$$

Consequently, we get

$$\Re\Big\{(1-\beta) - \sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\gamma_\iota|\ z^{\iota-1} - \sum_{n=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\eta_\iota|\ z^{\iota-1}\Big\}$$
$$\geq 0. \tag{3.14}$$

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The above required condition must hold for all values of z in $\mathbb{D}$. Upon choosing the values of z on the positive real axis where $0<|z|=r<1$, we must have

$$(1-\beta)-\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\gamma_\iota|\ r^{\iota-1}-\sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\eta_\iota|\ r^{\iota-1} \geq 0. \quad (3.15)$$

Letting $r\to 1^-$ through real values, it follows that

$$(1-\beta)-\left[\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\gamma_\iota|+\sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\eta_\iota|\right]\geq 0 \quad (3.16)$$

So, we have

$$\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\gamma_\iota|+\sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\eta_\iota|\leq 1-\beta. \quad (3.17)$$

□

Corollary 3.6. *Let $\varphi=\mu+\bar{\nu}$ be of the form* (1.3). *Then $\varphi\in\mathbb{H}_{\mathbb{T}4}(\alpha,\beta)$ if and only if the condition* (3.12) *or* (3.13) *holds.*

Proof. By part (2) of Remark 3.2 and Theorem 3.5, we obtain the required result. □

3.2.2. *Growth Bounds.* The following theorem considers the growth bounds for function in $\varphi\in\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$.

Theorem 3.7. *Let $\varphi\in\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$. Then, $r=|z|<1$*

$$|\varphi(z)|\leq(1+|\eta_1|)r+\left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right)\left[1-\frac{1}{1-\beta}\ z|\eta_1|\right]r^2, \quad (3.18)$$

and

$$|\varphi(z)|\geq(1+|\eta_1|)r-\left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right)\left[1-\frac{1}{1-\beta}\ |\eta_1|\right]r^2, \quad (3.19)$$

where,

$$P_1=\frac{\Lambda\quad(\omega+1)^{\upsilon}\quad\prod_{j=1}^{q}\delta_j\quad(\zeta+1)}{\prod_{j=1}^{p}\xi_j\quad H_{0,2}^{2,0}\left[(\omega+1)\kappa^{\frac{1}{\xi}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (\upsilon,1),\left(0,\frac{1}{\xi}\right)\end{matrix}\right]},$$

$0\leq\alpha,$ *and* $0\leq\beta<1$.

Proof. Let $\varphi\in\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$. Taking the absolute value of φ, we have

$$|\varphi(z)|\quad\leq(1+|\eta_1|)r+\sum_{\iota=2}^{\infty}\left(|\gamma_\iota|+|\eta_\iota|\right)r^{\iota}$$

$$\leq(1+|\eta_1|)r+r^2\sum_{\iota=2}^{\infty}\left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right)\left(\frac{2P_1[1+7\alpha]}{(1-\beta)}|\gamma_\iota|+\frac{2P_1[1+7\alpha]}{(1-\beta)}|\eta_\iota|\right)$$

$$\leq (1+|\eta_1|)r + r^2 \sum_{\iota=2}^{\infty} \left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right) \left(\frac{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}{(1-\beta)}|\gamma_\iota| + \frac{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}{(1-\beta)}|\eta_\iota|\right)$$

$$\leq (1+|\eta_1|)r + \left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right)\left[1 - \frac{1}{1-\beta}\ |\eta_1|\right] r^2 \tag{3.20}$$

and

$$|\varphi(z)| \quad \geq (1+|\eta_1|)r - \sum_{\iota=2}^{\infty} (|\gamma_\iota| + |\eta_\iota|)\, r^\iota$$

$$\geq (1+|\eta_1|)r - r^2 \sum_{\iota=2}^{\infty} \left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right) \left(\frac{2P_1[1+7\alpha]}{(1-\beta)}|\gamma_\iota| + \frac{2P_1[1+7\alpha]}{(1-\beta)}|\eta_\iota|\right)$$

$$\geq (1+|\eta_1|)r - r^2 \left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right) \sum_{\iota=2}^{\infty} \left(\frac{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}{(1-\beta)}|\gamma_\iota| + \frac{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}{(1-\beta)}|\eta_\iota|\right)$$

$$\geq (1+|\eta_1|)r - \left(\frac{(1-\beta)}{2P_1[1+7\alpha]}\right)\left[1 - \frac{1}{1-\beta}\ |\eta_1|\right] r^2. \tag{3.21}$$

□

Corollary 3.8. *Let* $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$*. Then* $r = |z| < 1$

$$|\varphi(z)| \leq (1+|\eta_1|)r + \left(\frac{(1-\beta)}{2[1+7\alpha]}\right)\left[1 - \frac{1}{1-\beta}\ |\eta_1|\right] r^2, \tag{3.22}$$

and

$$|\varphi(z)| \geq (1+|\eta_1|)r - \left(\frac{(1-\beta)}{2[1+7\alpha]}\right)\left[1 - \frac{1}{1-\beta}\ |\eta_1|\right] r^2, \tag{3.23}$$

where $0 \leq \alpha$*, and* $0 \leq \beta < 1$*.*

Proof. By part (2) of Remark 3.2 and Theorem 3.7, we derive the required assertion. □

3.2.3. *Extreme Points.* We determine the extreme points of closed convex hulls of $\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$ denoted by $\overline{co}\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$.

Theorem 3.9. *Let* $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$*. A function* $\varphi \in \overline{co}\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$ *if and only if*

$$\varphi(z) = \sum_{\iota=1}^{\infty} (\phi_\iota \mu_\iota(z) + \psi_\iota \nu_k(z)) \tag{3.24}$$

where

$$\begin{aligned} &\mu_1(z) = z, \\ &\mu_\iota(z) = z - \frac{1-\beta}{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}\ z^\iota, \qquad (\iota = 2,3,...), \\ &\nu_\iota(z) = z - \frac{1-\beta}{P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]}\ \overline{z^\iota}, \qquad (\iota = 1,2,...), \end{aligned} \tag{3.25}$$

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$\sum_{\iota=1}^{\infty}(\phi_\iota+\psi_\iota)=1,\ 0\le\phi_\iota$ *and* $0\le\psi_\iota$.

Proof. For a function φ of the form (3.24), we have

$$\varphi(z) \quad =\sum_{\iota=1}^{\infty}(\phi_\iota\mu_\iota(z)+\psi_\iota\nu_\iota(z))$$

$$=\sum_{\iota=1}^{\infty}(\phi_\iota+\psi_\iota)\,z-\sum_{\iota=2}^{\infty}\frac{1-\beta}{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}\ \phi_\iota z^\iota-\sum_{\iota=1}^{\infty}\frac{1-\beta}{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}\ \psi_\iota\overline{z^\iota}$$

$$=z-\sum_{\iota=2}^{\infty}\frac{1-\beta}{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}\ \phi_\iota z^\iota-\sum_{\iota=1}^{\infty}\frac{1-\beta}{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}\ \psi_\iota\overline{z^\iota}. \tag{3.26}$$

Therefore, in view of Theorem 3.5, we acquire

$$\begin{aligned}&\sum_{\iota=2}^{\infty}\frac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ |\gamma_\iota|+\sum_{\iota=1}^{\infty}\frac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ |\eta_\iota|\\&\le\sum_{\iota=2}^{\infty}\phi_\iota+\sum_{\iota=1}^{\infty}\psi_\iota=1-\phi_1\le 1.\end{aligned} \tag{3.27}$$

Therefore, $\varphi\in\overline{co}\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$.

Conversely, suppose that $\varphi\in\overline{co}\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$. Set

$$\phi_\iota=\frac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ \ |\gamma_\iota| \qquad (\iota=2,3,...)\,, \tag{3.28}$$

and

$$\psi_\iota=\frac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ \ |\eta_\iota| \qquad (\iota=1,2,...)\,. \tag{3.29}$$

On the basis of Theorem 3.5, we notice that $0\le\phi_\iota\le 1$, $(\iota=2,3,...)$ and $0\le\psi_\iota\le 1$, $(\iota=1,2,...)$. Let $\phi_1=1-\sum_{\iota=2}^{\infty}\phi_\iota-\sum_{\iota=1}^{\infty}\psi_\iota$ and notice that by Theorem 3.5, $\phi_1\ge 0$. Consequently, $\varphi(z)=\sum_{\iota=1}^{\infty}(\phi_\iota\mu_\iota(z)+\psi_\iota\nu_\iota(z))$ is obtained as required. □

Corollary 3.10. *Let* $\varphi\in\mathbb{H}_{\mathbb{T}4}(\alpha,\beta)$. *A function* $\varphi\in\overline{co}\mathbb{H}_{\mathbb{T}4}(\alpha,\beta)$ *if and only if*

$$\varphi(z)=\sum_{\iota=1}^{\infty}(\phi_\iota\mu_\iota(z)+\psi_\iota\nu_\iota(z)) \tag{3.30}$$

where

$$\begin{aligned}&\mu_1(z)=z,\\&\mu_\iota(z)=z-\frac{1-\beta}{\iota[1+\alpha(\iota^3-1)]}\ z^\iota, \qquad (\iota=2,3,...),\\&\nu_\iota(z)=z-\frac{1-\beta}{\iota[1+\alpha(\iota^3-1)]}\ \overline{z^\iota}, \qquad (\iota=1,2,...),\end{aligned} \tag{3.31}$$

$\sum_{\iota=1}^{\infty}(\phi_\iota+\psi_\iota)=1,\ 0\le\phi_\iota$ *and* $0\le\psi_\iota$.

Proof. By part (2) of Remark 3.2 and Theorem 3.9, the required assertion is obtained. □

3.2.4. *Closure Property.* Now we show that $\varphi \in \mathbb{H}_{\mathbb{T}4}(\alpha,\beta)$ is closed under convex combinations of its members.

Theorem 3.11. *The subclass $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$ is closed under convex combinations.*

Proof. For $j = 1, 2, ...,$ let $\varphi_i \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$, where

$$\varphi_i(z) = z - \sum_{\iota=2}^{\infty} |\gamma_{i,\iota}|\, z^{\iota} - \sum_{n=2}^{\infty} |\eta_{i,\iota}|\, \overline{z^{\iota}}. \tag{3.32}$$

Then, by Theorem 3.5, we have

$$\sum_{\iota=2}^{\infty} \frac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ |\gamma_\iota| + \sum_{\iota=1}^{\infty} \frac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\ |\eta_\iota| \le 1. \tag{3.33}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of φ_i may be written as

$$\sum_{i=1}^{\infty} t_i\varphi_i = z - \sum_{\iota=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i|\gamma_{i,\iota}|\right) z^{\iota} - \sum_{\iota=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i|\eta_{i,\iota}|\right)\overline{z^{\iota}}. \tag{3.34}$$

Then, by (3.33), we have

$$\sum_{\iota=2}^{\infty} \tfrac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\left(\sum_{i=1}^{\infty} t_i|\gamma_{i,\iota}|\right) + \sum_{\iota=1}^{\infty} \tfrac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}\left(\sum_{i=1}^{\infty} t_i|\eta_{i,\iota}|\right)$$

$$= \sum_{i=1}^{\infty} t_i \left(\sum_{\iota=2}^{\infty} \tfrac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}|\gamma_{i,\iota}| + \sum_{\iota=1}^{\infty} \tfrac{P_{\iota-1}\ \ \iota[1+\alpha(\iota^3-1)]}{1-\beta}|\eta_{i,\iota}|\right)$$

$$\le \sum_{i=1}^{\infty} t_i = 1. \tag{3.35}$$

Therefore, by Theorem 3.5, $\sum_{i=1}^{\infty} t_i\varphi_i \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$. □

Corollary 3.12. *The subclass $\mathbb{H}_{\mathbb{T}4}(\alpha,\beta)$ is closed under convex combinations.*

Proof. By part (2) of Remark 3.2 and Theorem 3.11, we get the required result. □

3.2.5. *Convolution Property.* The next theorem shows that the subclass $\mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$ is closed under convolution.

Theorem 3.13. *For $0 \le \varepsilon \le \beta < 1$, let $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\beta)$ and $\vartheta \in \mathbb{GH}_{\mathbb{T}4}(\alpha,\varepsilon)$. Then $(\varphi * \vartheta) \in \mathbb{H}_{\mathbb{T}4}(\alpha,\beta) \subset \mathbb{GH}_{\mathbb{T}4}(\alpha,\varepsilon)$.*

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Proof. Let the harmonic function $\varphi(z) = z - \sum_{\iota=2}^{\infty} |\gamma_\iota| z^\iota - \sum_{\iota=1}^{\infty} |\eta_\iota| \overline{z^\iota}$ and $\vartheta(z) = z - \sum_{\iota=2}^{\infty} |a_\iota| z^\iota - \sum_{\iota=1}^{\infty} |b_\iota| \overline{z^\iota}$. Then the convolution of φ and ϑ is defined as follows:

$$(\varphi * \vartheta)(z) = z - \sum_{\iota=2}^{\infty} |\gamma_\iota a_\iota| z^\iota - \sum_{\iota=1}^{\infty} |\eta_\iota b_\iota| \overline{z^\iota}. \tag{3.36}$$

In Theorem 3.5, since $\vartheta \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \varepsilon)$, we conclude that $|a_\iota| \leq 1$ and $|b_\iota| \leq 1$. However, $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$, then we yield

$$\begin{aligned} &\sum_{\iota=2}^{\infty} P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)] \ |\gamma_\iota \ a_\iota| + \sum_{n=1}^{\infty} P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)] \ |\eta_\iota \ b_\iota| \\ &\leq \sum_{\iota=2}^{\infty} P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)] \ |\gamma_\iota| + \sum_{\iota=1}^{\infty} P_{\iota-1} \ \iota[1+\alpha(\iota^3-1)] \ |\eta_\iota| \leq 1-\beta \leq 1-\varepsilon. \end{aligned} \tag{3.37}$$

So, $(\varphi * \vartheta) \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta) \subset \mathbb{GH}_{\mathbb{T}4}(\alpha, \varepsilon)$. □

Corollary 3.14. *For* $0 \leq \varepsilon \leq \beta < 1$, *let* $\varphi \in \mathbb{H}_{\mathbb{T}4}(\alpha, \beta)$ *and* $\vartheta \in \mathbb{H}_{\mathbb{T}4}(\alpha, \varepsilon)$. *Then* $(\varphi * \vartheta) \in \mathbb{H}_{\mathbb{T}4}(\alpha, \beta) \subset \mathbb{H}_{\mathbb{T}4}(\alpha, \varepsilon)$.

Proof. By part (2) of Remark 3.2 and Theorem 3.13, we have the required result. □

3.2.6. *A Family of Integral Operators.* Finally, a closure property of subclass $\mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$ is discussed under the generalized Bernardi-Libera-Livingston integral operator $B(z)$, which is considered as follows: (see [27])

$$B(z) = (\rho+1) \int_0^1 t^{\rho-1} \varphi(tz) dt \qquad (\rho > -1). \tag{3.38}$$

Theorem 3.15. *If* $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$. *Then* $B \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$.

Proof. Let

$$\varphi(z) = z - \sum_{\iota=2}^{\infty} |\gamma_\iota| z^\iota - \sum_{\iota=1}^{\infty} |\eta_\iota| \overline{z^\iota}. \tag{3.39}$$

Then, we have

$$\begin{aligned} B(z) &= (\rho+1) \int_0^1 t^{\rho-1} \left((tz) - \sum_{\iota=2}^{\infty} |\gamma_\iota| (tz)^\iota - \sum_{\iota=1}^{\infty} |\eta_\iota| \left(\overline{tz}\right)^\iota \right) dt \\ &= z - \sum_{\iota=2}^{\infty} |A_\iota| z^\iota - \sum_{\iota=1}^{\infty} |B_\iota| \bar{z}^\iota, \end{aligned} \tag{3.40}$$

where

$$A_\iota = \frac{\rho+1}{\rho+\iota} \ |\gamma_\iota| \qquad \text{and} \qquad B_\iota = \frac{\rho+1}{\rho+\iota} \ |\eta_\iota|.$$

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Thus, since $\varphi \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$,

$$\sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^2-1)]\left(\frac{\rho+1}{\rho+\iota}\ |\gamma_\iota|\right) + \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^\iota-1)]\left(\frac{\rho+1}{\rho+\iota}\ |\eta_\iota|\right)$$
$$\leq \sum_{\iota=2}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]\ |\gamma_\iota| + \sum_{\iota=1}^{\infty} P_{\iota-1}\ \iota[1+\alpha(\iota^3-1)]|\ \eta_\iota| \leq 1-\beta. \tag{3.41}$$

In virtue of Theorem 3.5, we obtain $B \in \mathbb{GH}_{\mathbb{T}4}(\alpha, \beta)$. □

Corollary 3.16. *If* $\varphi \in \mathbb{H}_{\mathbb{T}4}(\alpha, \beta)$. *Then* $B \in \mathbb{H}_{\mathbb{T}4}(\alpha, \beta)$.

Proof. By part (2) of Remark 3.2 and Theorem 3.15, we have the required assertion. □

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