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On Lah-Ribarič Inequality Involving Averages of Convex Functions

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Abstract. By using the integral arithmetic mean and the Lah-Ribarič inequality we give the extension of Wulbert's result from [15]. Also, we obtain inequalities with divided differences using the Lah-Ribarič inequality. As a consequence, the convexity of higher order for function defined by divided difference is proved. Further, we construct a new family of exponentially convex functions and Cauchy-type means by exploring at linear functionals with the obtained inequalities.

Keywords: Lah-Ribarič inequality, Divided differences, n-convex function, (n, m)-convex function, Exponential convexity.

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1. Introduction

Let f be a continous function on an interval I with a nonempty interior. Then, define:

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x, y \in I,\ x \neq y, \\ f(x), & x = y \in I. \end{cases} \tag{1.1}$$

Wulbert in [15], proved that the integral arithmetic mean F defined in (1.1) is convex on I^2 if f is convex on I. Zhang and Chu, in [16], rediscovered (without referring to and citing Wulbert's result) that the necessary and sufficient

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condition for the convexity of the integral arithmetic mean F is for f to be convex on I.

Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0,...,x_n \in [a,b]$, is defined recursively (see [1], [10]) by

$$f[x_i] = f(x_i),\ (i = 0,\dots,n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$. The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\dots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{1.2}$$

For divided difference the following holds:

$$f[x_0,\dots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)},\ \text{ where } \omega(x) = \prod_{j=0}^{n}(x - x_j),$$

so we have that

$$f[x_0,\dots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0,j\neq i}^{n}(x_i - x_j)}.$$

If the function f has continuous n-th derivative on $[a,b]$, the divided difference $f[x_0,\dots,x_n]$ can be represented in integral form by

$$f[x_0,\dots,x_n] = \int_{\Delta_n} f^{(n)}\left(\sum_{i=0}^{n} u_i x_i\right) du_0 \dots du_{n-1},$$

where

$$\Delta_n = \left\{(u_0,\dots,u_{n-1}) : u_i \geq 0, \sum_{i=0}^{n-1} u_i \leq 1\right\}$$

and $u_n = 1 - \sum_{i=0}^{n-1} u_i$.

The notion of n-convexity goes back to Popoviciu ([12]). We follow the definition given by Karlin ([5]):

Definition 1.1. A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, n-th order divided difference of f satisfies

$$f[x_0,...,x_n] \geq 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[a, b]$ is the uniform limit of the sequence of n-convex polynomials. Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [6].

In [3] is proved the following Jensen inequality for divided differences:

Theorem 1.2. *Let f be an $(n+2)$-convex function on (a, b) and $\mathbf{x} \in (a, b)^{n+1}$. Then*

$$G(\mathbf{x}) = f[x_0, \ldots, x_n]$$

is a convex function of the vector $\mathbf{x} = (x_0, \ldots, x_n)$. Consequently,

$$f\left[\sum_{i=0}^{m} a_i x_0^i, \ldots, \sum_{i=0}^{m} a_i x_n^i\right] \leq \sum_{i=0}^{m} a_i f[x_0^i, \ldots, x_n^i] \tag{1.3}$$

holds for all $a_i \geq 0$ such that $\sum_{i=0}^{m} a_i = 1$.

Schur polynomial in $n+1$ variables $x_0, \ldots, x_n$ of degree $d = d_0 + \ldots + d_n$ (d_j's form nonincreasing sequence non-negative integers, i.e. $d_0 \geq \ldots \geq d_n$) is defined as

$$S_{(d_0,\ldots,d_n)}(x_0, \ldots, x_n) = \frac{\det\left[x_i^{d_{n-j}+j}\right]_{i,j=0}^{n}}{\det\left[\ x_i^j\ \right]_{i,j=0}^{n}}.$$

The numerator consists of alternating polynomials (they change the sign under any transposition of the variables) and so they are all divisible by the denominator which is Vandermonde determinant. Schur polynomial is also symmetric because the numerator and denominator are both alternating.

Using Schur polynomial and Vandermonde determinant (extended with logarithmic function)

$$V(\mathbf{x}; p, q) = \det\begin{bmatrix} 1 & x_0 & {x_0}^2 & \ldots & {x_0}^{n-1} & {x_0}^p \ln^q x_0 \\ 1 & x_1 & {x_1}^2 & \ldots & {x_1}^{n-1} & {x_1}^p \ln^q x_1 \\ 1 & x_2 & {x_2}^2 & \ldots & {x_2}^{n-1} & {x_2}^p \ln^q x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & {x_n}^2 & \ldots & {x_n}^{n-1} & {x_n}^p \ln^q x_n \end{bmatrix}$$

we obtain:

Proposition 1.3. *For monomial function $h(x) = x^{n+k}$, where $k \geq 1$ is an integer, holds*

$$\begin{aligned} h[x_0, \ldots, x_n] &= S_{(k,\underbrace{0,\ldots,0}_{n-times})}(x_0, \ldots, x_n) = \frac{V(\mathbf{x}; n+k, 0)}{V(\mathbf{x}; n, 0)} \\ &= \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_k}. \end{aligned}$$

For potential function $f(x) = x^p = x^{n+p-n}$, where p is a real number, holds

$$f[x_0, \ldots, x_n] = \frac{V(\mathbf{x}; p, 0)}{V(\mathbf{x}; n, 0)}.$$

Let $f(x, y)$ be a real-valued function defined on $I \times J$ $(I = [a, b], J = [c, d])$. Then the (l_1, l_2) divided difference of the function f at distinct points $x_0, \ldots, x_{l_1} \in I, y_0, \ldots, y_{l_2} \in J$, is defined by (see [10])

$$\begin{aligned} f\left[\begin{array}{c} x_0, \ldots, x_{l_1} \\ y_0, \ldots, y_{l_2} \end{array}\right] &= f([y_0, \ldots, y_{l_2}])[x_0, \ldots, x_{l_1}] \\ &= f([x_0, \ldots, x_{l_1}])[y_0, \ldots, y_{l_2}] \\ &= \sum_{i=0}^{l_1} \sum_{j=0}^{l_2} \frac{f(x_i, y_j)}{\omega'(x_i)\omega'(y_j)}, \end{aligned} \tag{1.4}$$

where $\omega(x) = \prod_{i=0}^{l_1}(x - x_i),\ \omega(y) = \prod_{j=0}^{l_2}(y - y_j)$.

Definition 1.4. A function $f : I \times J \to \mathbb{R}$ is said to be (l_1, l_2)-convex or convex of order (l_1, l_2) if for all distinct points $x_0, \ldots, x_{l_1} \in I, y_0, \ldots, y_{l_2} \in J$,

$$f\left[\begin{array}{c} x_0, \ldots, x_{l_1} \\ y_0, \ldots, y_{l_2} \end{array}\right] \geq 0. \tag{1.5}$$

If this inequality is strict, then f is said to be strictly (l_1, l_2)-convex.

Popoviciu in [13] proved the following theorem:

Theorem 1.5. *If the partial derivative $f^{(l_1+l_2)}_{x^{l_1}y^{l_2}}$ of f exists, then f is (l_1, l_2)-convex iff*

$$f^{(l_1+l_2)}_{x^{l_1}y^{l_2}} \geq 0. \tag{1.6}$$

If the inequality in (1.6) is strict, then f is strictly (l_1, l_2)-convex.

The well known Lah-Ribarič inequality is given in the following theorem (see [7]):

Theorem 1.6. *Let f be a real valued convex function on $[m, M]$. Then for $m \leq x_k \leq M,\ p_k > 0\ (1 \leq k \leq n)$ and $\sum_{k=1}^n p_k = 1$ we have*

$$\sum_{k=1}^n p_k f(x_k) \leq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M), \tag{1.7}$$

where $\bar{x} = \sum_{k=1}^n p_k x_k$.

The goal of this paper is to give the extension of Wulbert's result from [15] and also to obtain inequalities with divided differences using the Lah-Ribarič inequality. As a consequence, we will proof the convexity of higher order for function defined by divided difference. In the last section, a new family of exponentially convex functions and Cauchy-type means are constructed by looking to the linear functionals associated with the obtained inequalities.

2. Inequalities involving averages

The following result is an extension of Wulbert's results:

Theorem 2.1. *Let f be a real valued convex function on $[m,M]$ and F is defined in (1.1). Then for $m \leq x_k, y_k \leq M$, $p_k > 0$ $(1 \leq k \leq n)$ and $\sum_{k=1}^n p_k = 1$ we have*

$$\sum_{k=1}^{n} p_k F(x_k, y_k) \leq \frac{M - \frac{1}{2}(\bar{x}+\bar{y})}{M-m} F(m,m) + \frac{\frac{1}{2}(\bar{x}+\bar{y}) - m}{M-m} F(M,M), \quad (2.1)$$

where $\bar{x} = \sum_{k=1}^n p_k x_k$ and $\bar{y} = \sum_{k=1}^n p_k y_k$.

Consequently, for $l_1 + l_2 = 2$ the integral arithmetic mean (1.1) is (l_1, l_2)-convex on $[m,M]^2$.

Proof. By using the Lah Ribarič inequality (1.7) we get:

$$\begin{aligned}
\sum_{k=1}^{n} p_k F(x_k, y_k) &= \sum_{k=1}^{n} p_k \int_0^1 f(sy_k + (1-s)x_k) ds \\
&= \int_0^1 \sum_{k=1}^{n} p_k f(sy_k + (1-s)x_k)) ds \\
&\leq \frac{f(m)}{M-m} \int_0^1 \left[M - \sum_{k=1}^{n} p_k (sy_k + (1-s)x_k) \right] ds \\
&\quad + \frac{f(M)}{M-m} \int_0^1 \left[\sum_{k=1}^{n} p_k (sy_k + (1-s)x_k) - m \right] ds \\
&= \frac{f(m)}{M-m} M - \frac{f(m)}{M-m} \int_0^1 (s\bar{y} + (1-s)\bar{x}) ds \\
&\quad + \frac{f(M)}{M-m} \int_0^1 (s\bar{y} + (1-s)\bar{x}) ds - m \frac{f(M)}{M-m} \\
&= \frac{M - \frac{1}{2}(\bar{x}+\bar{y})}{M-m} F(m,m) + \frac{\frac{1}{2}(\bar{x}+\bar{y}) - m}{M-m} F(M,M).
\end{aligned}$$

Now, if we put $n = 2, x_1 = m, x_2 = M, y_1 = M, y_2 = m, p_1 = p_2 = \frac{1}{2}$, then the inequality (2.1) reduces to

$$F(m,M) + F(M,m) \leq F(m,m) + F(M,M).$$

Use the definition in (1.4) we get

$$(M-m)^2 (F[m,M])[M,m] \geq 0.$$

It is known that if this holds for all possible $m, M > 0$ then F is $(1,1)$-convex function (see [13]).

Wulbert in [15], proved that the integral arithmetic mean F defined in (1.1) is convex on $[m,M]^2$, so we have $F^{(2+0)}_{x^2 y^0} \geq 0$ and $F^{(0+2)}_{x^0 y^2} \geq 0$. So, by using Theorem 1.5 function F is convex of order $(2,0)$ and $(0,2)$. $\square$

Remark 2.2. Theorem 2.1 is a generalization of the Lah-Ribarič inequality. For $x_k = y_k, k = 1, \ldots, n$ the inequality (2.1) recaptures the Lah-Ribarič inequality (1.7).

The following theorem is the integral version of Theorem 2.1:

Theorem 2.3. *Let $(\Omega, \mathbf{A}, \mu)$ be a probability space, $\alpha, \beta : \Omega \to [m, M]$ be functions from $L_1(\mu)$ and let f be an convex function on $[m, M]$ and F is defined in (1.1). Then*

$$\int_\Omega F(\alpha(u), \beta(u))d\mu(u) \le \frac{M - \frac{1}{2}(\bar{\alpha} + \bar{\beta})}{M - m} F(m, m) + \frac{\frac{1}{2}(\bar{\alpha} + \bar{\beta}) - m}{M - m} F(M, M), \tag{2.2}$$

where $\bar{\alpha} = \int_\Omega \alpha(u)d\mu(u)$ and $\bar{\beta} = \int_\Omega \beta(u)d\mu(u)$.

Proof. By using the integral version of Lah-Ribarič inequality we get:

$$\begin{aligned} & \int_\Omega F(\alpha(u), \beta(u))d\mu(u) = \int_0^1 \int_\Omega f(s\beta(u) + (1 - s)\alpha(u))d\mu(u)ds \\ \le \; & \frac{f(m)}{M - m} \int_0^1 \left[M - \int_\Omega (s\beta(u) - (1 - s)\alpha(u))d\mu(u) \right] ds \\ & + \frac{f(M)}{M - m} \int_0^1 \left[\int_\Omega (s\beta(u) - (1 - s)\alpha(u))d\mu(u) - m \right] ds \\ = \; & \frac{M - \frac{1}{2}(\bar{\alpha} + \bar{\beta})}{M - m} F(m, m) + \frac{\frac{1}{2}(\bar{\alpha} + \bar{\beta}) - m}{M - m} F(M, M). \end{aligned}$$

□

3. Inequalities for divided differences

In the following theorem we proof the Lah-Ribarič inequality for divided differences:

Theorem 3.1. *Let f be an $(n+2)$-convex function on $[m, M]$ and $\mathbf{x} \in [m, M]^{n+1}$. Then*

$$\begin{aligned} & \sum_{i=0}^{l} a_i f[x_0^i, \ldots, x_n^i] \\ \le \; & \frac{f^{(n)}(m)}{n!(M - m)} \left[M - \frac{1}{n+1} \sum_{j=0}^{n} \bar{x}_j \right] + \frac{f^{(n)}(M)}{n!(M - m)} \left[\frac{1}{n+1} \sum_{j=0}^{n} \bar{x}_j - m \right] \end{aligned} \tag{3.1}$$

holds for all $a_i \ge 0$ such that $\sum_{i=0}^{l} a_i = 1$ and $\bar{x}_j = \sum_{i=0}^{l} a_i x_j^i$. Consequently, for $n = 1$

$$G(\mathbf{x}) = f[x_0, x_1]$$

is a (l_1, l_2)-convex function of the vector $\mathbf{x} = (x_0, x_1)$, when $l_1 + l_2 = 2$.

Proof. Using the Lah-Ribarič inequality for convex function $f^{(n)}$, we have

$$\begin{aligned}
&\sum_{i=0}^{l} a_i f[x_0^i,\dots,x_n^i] = \sum_{i=0}^{l} a_i \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\\
= &\int_{\Delta_n} \sum_{i=0}^{l} a_i f^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\\
\leq &\frac{f^{(n)}(m)}{M-m}\int_{\Delta_n}\left(M-\sum_{i=0}^{l} a_i \sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\\
&+\frac{f^{(n)}(M)}{M-m}\int_{\Delta_n}\left(\sum_{i=0}^{l} a_i \sum_{j=0}^{n} u_j x_j^i - m\right) du_0\dots du_{n-1}\\
&\left(\text{for } g(x)=\frac{x^n}{n!} \text{ and } h(x)=\frac{x^{n+1}}{(n+1)!}\right)\\
= &\frac{f^{(n)}(m)}{M-m}\left[M\int_{\Delta_n} g^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\right.\\
&\left.-\sum_{i=0}^{l} a_i \int_{\Delta_n} h^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\right]\\
+ &\frac{f^{(n)}(M)}{M-m}\left[\sum_{i=0}^{l} a_i \int_{\Delta_n} h^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\right.\\
&\left.-m\int_{\Delta_n} g^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0\dots du_{n-1}\right]\\
= &\frac{f^{(n)}(m)}{M-m}\left[M\cdot g[x_0^i,x_1^i,\dots,x_n^i]-\sum_{i=0}^{l} a_i\cdot h[x_0^i,x_1^i,\dots,x_n^i]\right]\\
&+\frac{f^{(n)}(m)}{M-m}\left[\sum_{i=0}^{l} a_i\cdot h[x_0^i,x_1^i,\dots,x_n^i]-m\cdot g[x_0^i,x_1^i,\dots,x_n^i]\right]\\
= &\frac{f^{(n)}(m)}{n!(M-m)}\left[M-\frac{1}{n+1}\sum_{j=0}^{n}\bar{x}_j\right]+\frac{f^{(n)}(M)}{n!(M-m)}\left[\frac{1}{n+1}\sum_{j=0}^{n}\bar{x}_j - m\right].
\end{aligned}$$

Now, If we put $n=1, x_0^0=m, x_1^0=M, x_0^1=M, x_1^1=m, a_1=a_2=\frac{1}{2}$ and fact that $f'(x)=f[x,x]=G(x,x)$, similarly as in Theorem 2.1, we can proof that function G is convex function of order $(1,1)$.

By using Theorem 1.2 and similarly as in Theorem 2.1, we also can proof that function G is convex function of order $(2,0)$ and $(0,2)$. □

The integral version of Lah-Ribarič inequality for divided differences is given with following theorem:

Theorem 3.2. *Let $p, g_i : \Omega \to [m, M], (i = 0, \dots, n)$ be functions from $L_1(\mu)$ and let f be an $(n+2)$-convex function on $[m, M]$. Then*

$$\begin{aligned} & \int_\Omega p(x) f[g_0(x), \dots, g_n(x)] d\mu(x) \\ \leq \; & \frac{f^{(n)}(m)}{n!(M-m)} \left[M - \frac{1}{n+1} \sum_{i=0}^n \bar{g}_i \right] + \frac{f^{(n)}(M)}{n!(M-m)} \left[\frac{1}{n+1} \sum_{i=0}^n \bar{g}_i - m \right] \end{aligned} \tag{3.2}$$

holds for all $p(x) \geq 0$ such that $\int_\Omega p(x) d\mu(x) = 1$ and $\bar{g}_i = \int_\Omega p(x) g_i(u) d\mu(u)$.

Proof. Using the integral Lah-Ribarič inequality for convex function $f^{(n)}$, we have the following conclusion

$$\begin{aligned} & \int_\Omega p(x) f[g_0(x), \dots, g_n(x)] d\mu(x) \\ = \; & \int_{\Delta_n} \left(\int_\Omega p(x) f^{(n)} \left(\sum_{i=0}^n u_i g_i(x) \right) d\mu(x) \right) du_0 \dots du_{n-1} \\ \leq \; & \frac{f^{(n)}(m)}{M-m} \int_{\Delta_n} \left(M - \int_\Omega p(x) \left(\sum_{i=0}^n u_i g_i(x) \right) d\mu(x) \right) du_0 \dots du_{n-1} \\ & + \frac{f^{(n)}(M)}{M-m} \int_{\Delta_n} \left(\int_\Omega p(x) \left(\sum_{i=0}^n u_i g_i(x) \right) d\mu(x) - m \right) du_0 \dots du_{n-1} \\ = \; & \frac{f^{(n)}(m)}{n!(M-m)} \left[M - \frac{1}{n+1} \sum_{i=0}^n \bar{g}_i \right] + \frac{f^{(n)}(M)}{n!(M-m)} \left[\frac{1}{n+1} \sum_{i=0}^n \bar{g}_i - m \right]. \end{aligned}$$

□

4. Applications to exponential convexity

Motivated by inequalities (2.1), (2.2), (3.1) and (3.2), under the same assumptions, we define following functionals:

$$\Phi_1(f) = \frac{M - \frac{1}{2}(\bar{x} + \bar{y})}{M-m} f(m) + \frac{\frac{1}{2}(\bar{x} + \bar{y}) - m}{M-m} f(M) - \sum_{k=1}^n \frac{p_k}{y_k - x_k} \int_{x_k}^{y_k} f(t) dt, \tag{4.1}$$

$$\begin{aligned} \Phi_2(f) \; = \; & \frac{M - \frac{1}{2}(\bar{\alpha} + \bar{\beta})}{M-m} f(m) + \frac{\frac{1}{2}(\bar{\alpha} + \bar{\beta}) - m}{M-m} f(M) \\ & - \int_\Omega \left(\frac{1}{\beta(u) - \alpha(u)} \int_{\alpha(u)}^{\beta(u)} f(t) dt \right) d\mu(u), \end{aligned} \tag{4.2}$$

$$\begin{aligned}\Phi_3(f) &= \frac{f^{(n)}(m)}{n!(M-m)}\left[M-\frac{1}{n+1}\sum_{j=0}^{n}\bar{x}_j\right]+\frac{f^{(n)}(M)}{n!(M-m)}\left[\frac{1}{n+1}\sum_{j=0}^{n}\bar{x}_j-m\right]\\ &- \sum_{i=0}^{l}a_i f[x_0^i,\ldots,x_n^i] \qquad (4.3)\end{aligned}$$

and

$$\begin{aligned}\Phi_4(f) &= \frac{f^{(n)}(m)}{n!(M-m)}\left[M-\frac{1}{n+1}\sum_{i=0}^{n}\bar{g}_i\right]+\frac{f^{(n)}(M)}{n!(M-m)}\left[\frac{1}{n+1}\sum_{i=0}^{n}\bar{g}_i-m\right]\\ &- \int_{\Omega}p(x)f[g_0(x),\ldots,g_n(x)]d\mu(x). \qquad (4.4)\end{aligned}$$

Similarly as in [11] we can construct new families of exponentially convex function and Cauchy type means by looking at these linear functionals. Also, we can proof the monotonicity property of the generalized Cauchy means obtained via these functionals.

Here we present an example for such a family of functions:

Example 4.1. Consider a family of functions

$$\Omega=\{f_s:(0,\infty)\to\mathbb{R}:s\in\mathbb{R}\}$$

defined by

$$f_s(x)=\begin{cases}\frac{x^s}{s(s-1)}, & s\notin\{0,1\},\\ \frac{x^j\ln x}{(-1)^{1-j}j!(1-j)!}, & s=j\in\{0,1\}.\end{cases}$$

Here, $\frac{d^2f_s}{dx^2}(x)=x^{s-2}=e^{(s-2)\ln x}>0$ which shows that f_s is convex for $x>0$ and $s\mapsto\frac{d^2f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in [11] we get that the mappings $s\mapsto\Phi_i(f_s), i=1,2$ are exponentially convex. Now we get:

$$\mu_{s,q}(\Phi_i,\Omega)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_i(f_0f_s)}{\Phi_i(f_s)}+\frac{1-2s}{s^2-s}\right), & s=q\notin\{0,1\},\\ \exp\left(-\frac{\Phi_i(f_0^2)}{2\Phi_i(f_0)}+1\right), & s=q=0,\\ \exp\left(-\frac{\Phi_i(f_0f_1)}{2\Phi_i(f_1)}-1\right), & s=q=1,\end{cases}$$

where for $s\neq-1,0,1$

$$\begin{aligned}\Phi_1(f_s) &= \frac{M-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\cdot\frac{m^s}{s(s-1)}+\frac{\frac{1}{2}(\bar{x}+\bar{y})-m}{M-m}\cdot\frac{M^s}{s(s-1)}\\ &- \frac{1}{s^3-s}\sum_{k=1}^{n}p_k\frac{y_k^{s+1}-x_k^{s+1}}{y_k-x_k},\end{aligned}$$

$$\begin{aligned}\Phi_1(f_{-1}) &= \frac{M-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\cdot\frac{1}{2m}+\frac{\frac{1}{2}(\bar{x}+\bar{y})-m}{M-m}\cdot\frac{1}{2M}\\ &- \frac{1}{2}\sum_{k=1}^{n}p_k\frac{\ln y_k-\ln x_k}{y_k-x_k},\end{aligned}$$

$$\begin{aligned}\Phi_1(f_0) &= \frac{\frac{1}{2}(\bar{x}+\bar{y})-M}{M-m}\cdot\ln m+\frac{m-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\cdot\ln M\\ &+ \sum_{k=1}^{n}p_k\frac{y_k\ln y_k-x_k\ln x_k}{y_k-x_k}-1,\end{aligned}$$

$$\begin{aligned}\Phi_1(f_1) &= \frac{M-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\cdot m\ln m+\frac{\frac{1}{2}(\bar{x}+\bar{y})-m}{M-m}\cdot M\ln M\\ &- \frac{1}{2}\sum_{k=1}^{n}p_k\frac{y_k^2\ln y_k-x_k^2\ln x_k}{y_k-x_k}+\frac{1}{4}(\bar{x}+\bar{y}).\end{aligned}$$

For similarly results for Jensen's inequality involving averages of convex functions see [2] and [4].

For a family of functions

$$\tilde{\Omega}=\left\{\tilde{f}_s:(0,\infty)\to\mathbb{R}:s\in\mathbb{R}\right\}$$

defined by

$$\tilde{f}_s(x)=\begin{cases}\frac{x^s}{s(s-1)\dots(s-(n+1))}, & s\notin\{0,1,\dots,n+1\},\\ \frac{x^j\ln x}{(-1)^{n+1-j}j!(n+1-j)!}, & s=j\in\{0,1,\dots,n+1\},\end{cases}$$

analogous as above it is easy to prove that $s\mapsto\Phi_i\left(\tilde{f}_s\right)$ $(i=3,4)$ are exponentially convex. In this case, we get $\tilde{\mu}_{s,q}\left(\Phi_i,\tilde{\Omega}\right)$ $(i=3,4)$ as follows

$$\tilde{\mu}_{s,q}\left(\Phi_i,\tilde{\Omega}\right)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{(-1)^{n+1}(n+1)!\Phi_i(f_0f_s)}{\Phi_i(f_s)}+\sum_{k=0}^{n+1}\frac{1}{k-s}\right), & s=q\notin\{0,1,\dots,n+1\},\\ \exp\left(\frac{(-1)^{n+1}(n+1)!\Phi_i(f_0f_s)}{2\Phi_i(f_s)}+\sum_{\substack{k=0\\k\neq s}}^{n+1}\frac{1}{k-s}\right), & s=q\in\{0,1,\dots,n+1\}.\end{cases}$$

For $s\notin\{0,1,\dots,n+1\}$

$$\begin{aligned}\Phi_3(\tilde{f}_s) &= \frac{M-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\frac{m^{s-n}}{n!(s-n)(s-(n+1))}+\frac{\frac{1}{2}(\bar{x}+\bar{y})-m}{M-m}\frac{M^{s-n}}{n!(s-n)(s-(n+1))}\\ &- \prod_{k=0}^{n+1}\frac{1}{s-k}\sum_{i=0}^{m}a_i\frac{V(\mathbf{x}^i,s,0)}{V(\mathbf{x}^i,n,0)}.\end{aligned}$$

For $s = j \in \{0, 1, \ldots, n+1\}$ we have

$$\tilde{f}_s^{(n)}(x) = \frac{1}{(-1)^{n+1-j}\, j!\,(n+1-j)!}\left(\sum_{i=0}^{n-1}\prod_{\substack{k=0\\k\neq i}}^{n-1}(j-k)x^{j-n} + \prod_{i=0}^{n-1}(j-i)x^{j-n}\ln x\right).$$

So, for $s = j \in \{0, 1, \ldots, n-1\}$

$$\Phi_3(\tilde{f}_j) = \frac{M-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\cdot\frac{m^{j-n}}{n!} + \frac{\frac{1}{2}(\bar{x}+\bar{y})-m}{M-m}\cdot\frac{M^{j-n}}{n!} - \prod_{k=0}^{n+1}\frac{1}{j-k}\sum_{i=0}^{m}a_i\frac{V(\mathbf{x}^i, j, 1)}{V(\mathbf{x}^i, n, 0)}$$

and for $l \in \{0, 1\}$

$$\begin{aligned}\Phi_3(\tilde{f}_{n+l}) \quad = \quad & (-1)^{l+1}\left[\frac{M-\frac{1}{2}(\bar{x}+\bar{y})}{M-m}\cdot\frac{m^l\left[\sum_{i=0}^{n-1}\frac{1}{n+l-1}+\ln m\right]}{n!}\right.\\ & \left.+\frac{\frac{1}{2}(\bar{x}+\bar{y})-m}{M-m}\cdot\frac{M^l\left[\sum_{i=0}^{n-1}\frac{1}{n+l-1}+\ln M\right]}{n!}\right]\\ & -\prod_{k=0}^{n+1}\frac{1}{n+l-k}\sum_{i=0}^{m}a_i\frac{V(\mathbf{x}^i, n+l, 1)}{V(\mathbf{x}^i, n, 0)}.\end{aligned}$$

For similarly results for Jensen's inequality for divided differences see [8] and [14]. See also [9].

Φ_i $(i = 1, 2, 3, 4)$ are positive, so then there exists $\xi_i, \tilde{\xi}_i \in [m, M]$ such that

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)},\ i = 1, 2, \qquad \tilde{\xi}_i^{s-q} = \frac{\Phi_i\left(\tilde{f}_s\right)}{\Phi_i\left(\tilde{f}_q\right)},\ i = 3, 4.$$

Since the function $\xi_i \mapsto \xi_i^{s-q}$ and $\tilde{\xi}_i \mapsto \tilde{\xi}_i^{s-q}$ are invertible for $s \neq q$, we have

$$m \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq M,\ i = 1, 2, \qquad m \leq \left(\frac{\Phi_i\left(\tilde{f}_s\right)}{\Phi_i\left(\tilde{f}_q\right)}\right)^{\frac{1}{s-q}} \leq M,\ i = 3, 4,$$

which together with the fact that $\mu_{s,q}(\Phi_i, \Omega)$ and $\tilde{\mu}_{s,q}\left(\Phi_i, \tilde{\Omega}\right)$ are continuous, symmetric and monotonous, shows that $\mu_{s,q}(\Phi_i, \Omega)$ and $\tilde{\mu}_{s,q}\left(\Phi_i, \tilde{\Omega}\right)$ are means.

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