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Horadam Polynomials Estimates for λ-Pseudo-Starlike Bi-Univalent Functions

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Abstract. In the present investigation, we use the Horadam Polynomials to establish upper bounds for the second and third coefficients of functions belongs to a new subclass of analytic and λ-pseudo-starlike bi-univalent functions defined in the open unit disk U. Also, we discuss Fekete-Szegö problem for functions belongs to this subclass.

Keywords: Bi-univalent functions, Coefficient bounds, Horadam polynomials, λ-Pseudo-starlike functions, Fekete-Szegö problem, Subordination.

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1. Introduction

Let $\mathcal{A}$ stand for the family of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Further, let S indicate the subclass of $\mathcal{A}$ consisting of the form (1.1) which are univalent in U. It is well known (see [3]) that every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ stands for the class of bi-univalent functions in U given by (1.1). In fact, Srivastava et al. [13] has apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [4], Goyal and Goswami [5], Caglar et al. [2] and others (see, for example [9, 10, 11, 12, 14]).

A function $f \in S$ is said to be λ-pseudo-starlike function of order β $0 \leq \beta < 1$ in U, if it satisfies (see [1]): This class of functions was denoted by $\mathcal{L}_\lambda(\beta)$. It is observed that for $\lambda = 1$, we have the class of starlike functions.

With a view to recalling the principal of subordination between analytic functions, let the functions f and g be analytic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [6]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}),$$

with

$$h_1(r) = a \quad and \quad h_2(r) = br, \tag{1.3}$$

for some real constant a, b, p and q.

The generating function of the Horadam polynomials $h_n(r)$ (see [7]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{1.4}$$

2. Main Results

We begin this section by defining the subclass $\mathcal{L}_\Sigma(\delta, \lambda, r)$ as follows:

Definition 2.1. For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$ if it satisfies the subordinations:

$$1 + \frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^\lambda}{f(z)} - 1\right) \prec \Pi(r, z) + 1 - a$$

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and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)\prec\Pi(r,w)+1-a,$$

where a is real constant and the function $g=f^{-1}$ is given by (1.2).

Theorem 2.2. *For $\delta\in\mathbb{C}\setminus\{0\}$, $\lambda\geq 1$ and $r\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathcal{L}_{\Sigma}(\delta,\lambda,r)$. Then*

$$|a_2|\leq\frac{\sqrt{2}\delta\left|br\right|\sqrt{\left|br\right|}}{\sqrt{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p\left(\lambda-1\right)^2\right]br^2-2qa\left(\lambda-1\right)^2\right|}}$$

and

$$|a_3|\leq\frac{\delta\left|br\right|}{3\lambda-1}+\frac{\delta^2b^2r^2}{\left(\lambda-1\right)^2}.$$

Proof. Let $f\in\mathcal{L}_{\Sigma}(\delta,\lambda,r)$. Then there are two analytic functions $u,v:U\longrightarrow U$ given by

$$u(z)=u_1z+u_2z^2+u_3z^3+\cdots\quad(z\in U)\tag{2.1}$$

and

$$v(w)=v_1w+v_2w^2+v_3w^3+\cdots\quad(w\in U),\tag{2.2}$$

with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$ such that

$$1+\frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right)=\Pi(r,u(z))+1-a$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)=\Pi(r,v(w))+1-a.$$

Or, equivalently

$$1+\frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right)=1+h_1(r)+h_2(r)u(z)+h_3(r)u^2(z)+\cdots\tag{2.3}$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)=1+h_1(r)+h_2(r)v(w)+h_3(r)v^2(w)+\cdots.\tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4) yields

$$1+\frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right)=1+h_2(r)u_1z+\left[h_2(r)u_2+h_3(r)u_1^2\right]z^2+\cdots\tag{2.5}$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)=1+h_2(r)v_1w+\left[h_2(r)v_2+h_3(r)v_1^2\right]w^2+\cdots.\tag{2.6}$$

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It is quite well-known that if $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in U$, then

$$|u_i| \leq 1 \quad and \quad |v_i| \leq 1 \; forall \; i \in \mathbb{N}. \tag{2.7}$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$\frac{2\lambda - 1}{\delta} a_2 = h_2(r) u_1, \tag{2.8}$$

$$\frac{3\lambda - 1}{\delta} a_3 + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r) u_2 + h_3(r) u_1^2, \tag{2.9}$$

$$-\frac{2\lambda - 1}{\delta} a_2 = h_2(r) v_1 \tag{2.10}$$

and

$$\frac{3\lambda - 1}{\delta}(2a_2^2 - a_3) + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r) v_2 + h_3(r) v_1^2. \tag{2.11}$$

It follows from (2.8) and (2.10) that

$$u_1 = -v_1 \tag{2.12}$$

and

$$\frac{2\left(\lambda - 1\right)^2}{\delta^2} a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \tag{2.13}$$

If we add (2.9) to (2.11), we find that

$$\frac{4\lambda^2 - 5\lambda + 1}{\delta} a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \tag{2.14}$$

Substituting the value of $u_1^2 + v_1^2$ from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2 = \frac{\delta^2 h_2^3(r)(u_2 + v_2)}{\delta\left(4\lambda^2 - 5\lambda + 1\right) h_2^2(r) - 2h_3(r)\left(\lambda - 1\right)^2}. \tag{2.15}$$

Further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2| \leq \frac{\sqrt{2}\delta\,|br|\sqrt{|br|}}{\sqrt{\left|\left[\delta\left(4\lambda^2 - 5\lambda + 1\right) b - 2p\left(\lambda - 1\right)^2\right] br^2 - 2qa\left(\lambda - 1\right)^2\right|}}.$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$\frac{2(3\lambda - 1)}{\delta}(a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \tag{2.16}$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3 = \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^2(r)(u_1^2 + v_1^2)}{2\left(\lambda - 1\right)^2}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{\delta\,|br|}{3\lambda - 1} + \frac{\delta^2 b^2 r^2}{\left(\lambda - 1\right)^2}.$$

This completes the proof of Theorem 2.2 □

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In the next theorem, we discuss the Fekete-Szegö problem for the subclass $\mathcal{L}_{\Sigma}(\delta,\lambda,r)$.

Theorem 2.3. *For $\delta \in \mathbb{C}\setminus\{0\}$, $\lambda \geq 1$ and $r,\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_{\Sigma}(\delta,\lambda,r)$. Then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\delta|br|}{3\lambda-1}; \\ \qquad for\ |\mu-1| \leq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}, \\ \frac{2\delta^2|br|^3|\mu-1|}{\left|\left[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}; \\ \qquad for\ |\mu-1| \geq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}. \end{cases}$$

Proof. It follows from (2.15) and (2.16) that

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{\delta h_2(r)(u_2-v_2)}{2(3\lambda-1)}+(1-\mu)a_2^2 \\ &= \frac{\delta h_2(r)(u_2-v_2)}{2(3\lambda-1)}+\frac{\delta^2 h_2^3(r)(u_2+v_2)(1-\mu)}{\delta\left(4\lambda^2-5\lambda+1\right)h_2^2(r)-2h_3(r)\left(\lambda-1\right)^2} \\ &= h_2(r)\left[\left(\psi(\mu,r)+\frac{\delta}{2(3\lambda-1)}\right)u_2+\left(\psi(\mu,r)-\frac{\delta}{2(3\lambda-1)}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu,r)=\frac{\delta^2 h_2^2(r)(1-\mu)}{\delta\left(4\lambda^2-5\lambda+1\right)h_2^2(r)-2h_3(r)\left(\lambda-1\right)^2}.$$

According to (1.3), we find that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\delta|br|}{3\lambda-1}, & 0 \leq |\psi(\mu,r)| \leq \frac{\delta}{2(3\lambda-1)}, \\ 2\left|br\right|\left|\psi(\mu,r)\right|, & |\psi(\mu,r)| \geq \frac{\delta}{2(3\lambda-1)}. \end{cases}$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\delta|br|}{3\lambda-1}; \\ \qquad for\ |\mu-1| \leq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}, \\ \frac{2\delta^2|br|^3|\mu-1|}{\left|\left[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}; \\ \qquad for\ |\mu-1| \geq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}. \end{cases}$$

□

Putting $\mu=1$ in Theorem 2.3, we obtain the following result:

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Corollary 2.4. *For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_{\Sigma}(\delta, \lambda, r)$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{\delta\,|br|}{3\lambda - 1}.$$

Remark 2.5. If we put $\lambda = 1$ in our Theorems, we have the result for well-known class $S_{\Sigma}^{*}(r)$ of bi-starlike functions which was considered recently by Srivastava et al. [8].

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