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Strong I^K-Convergence in Probabilistic Metric Spaces

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Abstract. In this paper we introduce strong I^K-convergence of functions which is common generalization of strong I^*-convergence of functions in probabilistic metric spaces. We also define and study strong I^K-limit points of functions in same space.

Keywords: Strong I^K-convergence, Strong $(I \vee K)^K$-convergence, Strong I^K-Cauchy, Strong I^K-limit points.

2000 Mathematics subject classification: 54E70, 40A05, 40A35

1. Introduction

The works of generalizations of convergence of sequences were taken into consideration in the early sixties of twentieth century. In the year 1951 the concept of usual convergence of a real sequences was extended to statistical convergence by H. Fast [11] and then H. Steinhaus [28] and later it was developed by many authors [1, 12, 26, 27]. Now we recall the definition of natural density of a set $A \subset \mathbb{N}$ where $\mathbb{N}$ denotes the set of natural numbers. If A_n denote the set $\{a \in A : a \leq n\}$ and $|A_n|$ stands for the cardinality of A_n, the natural density of A is then defined by

$$d(A) = \lim_n \frac{|A_n|}{n}$$

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if the limit exits. A real sequence $\{x_n\}$ is said to be statistically convergent to ξ if for every $\epsilon > 0$ the set $A(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}$ has natural density zero [11, 18]. Extensions of statistical convergence to more general spaces can be found in [20, 21, 23]. The another generalization of statistical convergence is the concept of ideal convergence (i.e., I and I^*-convergence) which depends on the construction of ideals of subsets of $\mathbb{N}$ introduced by P. Kostyrko et al. [17] in the beginning of twenty first century. I-convergence of a sequence of real numbers coincides with the ordinary convergence if I is the ideal of all finite subsets of natural numbers and with the statistical convergence if I is the ideal of $\mathbb{N}$ of natural density zero [14, 17].

The concept of I^*-convergence was introduced by P. Kostyrko et al. [17]. Subsequently the concept of I and I^*-convergence was extended from the real number space to the metric spaces and to the normed linear spaces by many authors and finally to topological spaces by B. K. Lahiri and P. Das [14] in the year 2005. They proved that some basic properties are preserved also in a topological space. Later many works on I-convergence were done in topological spaces [2, 3, 4, 6].

In the year 2010, M. Macaj and M. Sleziak [19] introduced the idea of I^K-convergence in a topological space where I and K are ideals of an arbitrary set S and showed that this type of convergence is a common generalization for all types of I and I^*-convergence in some restrictions. They also gave the idea of AP(I, K) condition which is generalization of AP condition given in [17].

The concept of I-Cauchy condition was studied first by K. Dems [10] in 2004 and then further investigation on I^*-Cauchyness was studied in [25] by A. Nabiev et al. in 2007. In the year 2014, P. Das et al. [8] studied on I^K-Cauchy functions.

The idea of probabilistic metric spaces was first introduce by Menger [22] as a generalization of ordinary metric spaces. The notion distance has a probabilistic nature which has led to do a remarkable development of the probabilistic metric space(in short PM Space). PM Spaces have nice topological properties and several topologies can be defined on this space and the topology that is found to be most useful is the strong topology. The theory was brought to its present form by Schweizer and Sklar [30] and Tardiff [34]. In the year 2009, the concept of statistical convergence and then strong ideal convergence in probabilistic metric spaces was studied in [31, 32] by C. Sencimen et al. In the year 2012, M. Mursaleen et al. studied ideal convergence in probabilistic normed spaces [24].

The recent works of generalizations of convergence via ideals in probabilistic metric spaces have been developed by many authors. It seems therefore reasonable to think if we extend the same in the same space using double ideals and in that case we intend to investigate how far several the basic properties (such as results on limit points, Cauchy sequences etc.) are affected. In our

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paper we study the idea of strong I^K-convergence of functions in a probabilistic metric space which also generalizes the strong I^*-convergence studied in [32]. Since the convergence in PM space is very significant to probabilistic analysis, we realize that the idea of convergence via double ideals in a PM space would give more general frame for analysis of PM spaces.

2. Preliminaries

We recall on some basic ideas related to theory of PM spaces which are already studied in depth in the book by Schweizer and Sklar [29]. A non-decreasing function $F : \mathbb{R} \to [0,1]$ with $F(-\infty) = 0$ and $F(\infty) = 1$ is called a distribution function. In particular if F is defined on $[0,\infty]$ and left continuous on $(0,\infty)$ is called a distance distribution function (d.d.f). The set of all distribution function and set of all distance distribution functions are respectively denoted by $\triangle$ and $\triangle^+$. For any $x \in (-\infty,\infty)$ the unit step function at x is denoted by ϵ_x and is defined to be a function in the family of distribution functions given by

$$\epsilon_x(s) = \begin{cases} 0 & \text{if } s \leq x \\ 1 & \text{if } s > x \end{cases}$$

Definition 2.1. The distance between F and G in $\triangle$ is defined by $d_L(F,G) = inf\{t \in (0,1] : \textit{ both } (F,G;t) \textit{ and } (G,F;t) \textit{ hold}\}$ where for $t \in (0,1]$, the condition $(F,G;t)$ holds if $F(s-t) - t \leq G(s) \leq F(s+t) + t$ for every $s \in (-\frac{1}{t}, \frac{1}{t})$.

Definition 2.2. A sequence $\{F_n\}_{n\in\mathbb{N}}$ of d.d.f's is said to converge weakly to a d.d.f F and if $\{F_n(s)\}_{n\in\mathbb{N}}$ converges to $F(s)$ at each continuity point s of F and then we write $F_n \xrightarrow{w} F$.

In order to present the definition of a probabilistic metric space, we need the notion of triangle function introduced by Serstnev in [33].

Definition 2.3. A triangle function $\tau : \triangle^+ \times \triangle^+ \to \triangle^+$ is a binary operation on $\triangle^+$ which is non-decreasing, associative, commutative in each of its variables and has ϵ_0 as the identity.

Definition 2.4. A probabilistic metric space (briefly PM space) is a triplet $(P,\mathcal{F},\tau)$ where P is a non-empty set, $\mathcal{F} : P \times P \to \triangle^+$ is a function, τ is a triangle function satisfying the following condition for all $a,b,c \in P$
(i) $\mathcal{F}(a,a) = \epsilon_0$
(ii) $\mathcal{F}(a,b) \neq \epsilon_0$ if $a \neq b$
(iii) $\mathcal{F}(a,b) = \mathcal{F}(b,a)$
(iv) $\mathcal{F}(a,c) \geq \tau(\mathcal{F}(a,b),\mathcal{F}(b,c))$

Henceforth we shall denote $\mathcal{F}(a,b)$ by F_{ab} and its value at s by $F_{ab}(s)$.

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Theorem 2.5. *Let $G \in \triangle^+$ be given then for any $t > 0$, $G(t) > 1 - t$ if and only if $d_L(G, \epsilon_0) < t$*

Definition 2.6. Let $(P, \mathcal{F}, \tau)$ be a PM space. For $t > 0$ and $a \in P$, the strong t-neighborhood of $a \in P$ is defined by the set $\mathcal{N}_a(t) = \{b \in P : F_{ab}(t) > 1 - t\}$.

The collection $\aleph_a = \{\mathcal{N}_a(t) : t > 0\}$ is called strong neighborhood system at a and the union $\aleph = \cup_{a \in P} \aleph_a$ is said to be strong neighborhood system of S and the strong topology is introduced by a strong neighborhood system. Applying theorem 2.5 we can write strong t-neighborhood as $\mathcal{N}_a(t) = \{b \in P : d_L(F_{ab}, \epsilon_0) < t\}$.

Theorem 2.7. *Let $(P, \mathcal{F}, \tau)$be a PM space. If τ is continuous, then the strong neighborhood system $\aleph$ satisfies (i) and (ii).*
(i) If V is a strong neighborhood of $p \in P$ and $q \in V$, then there is a strong neighborhood W of q such that $W \subseteq V$.
(ii) If $p \neq q$, then there is a $V \in \mathcal{N}_p$ and a W in $\mathcal{N}_q$ such that $V \cap W = \phi$ and thus the strong neighborhood system $\mathcal{N}$ determines a Hausdorff topology for P.

Definition 2.8. Let $(P, \mathcal{F}, \tau)$ be PM space. Then for any $t > 0$, the subset $U(t)$ of $P \times P$ given by $U(t) = \{(a, b) : F_{ab}(t) > 1 - t\}$ is called strong-t-vicinity.

Theorem 2.9. *Let $(P, \mathcal{F}, \tau)$ be PM space and τ be continuous. Then for any $t > 0$, there is an $\eta > 0$ such that $U(\eta) \circ U(\eta) \subseteq U(t)$, where $U(\eta) \circ U(\eta) = \{(a, c) : for\ some\ b, (a, b)\ and\ (b, c) \in U(\eta)\}$*

Note 2.10. *Under the hypothesis of theorem 2.9 we can say that for any $t > 0$ there is an $\eta > 0$ such that $F_{ac}(t) > 1 - t$ whenever $F_{ab}(\eta) > 1 - \eta$ and $F_{bc}(\eta) > 1 - \eta$ i.e. from the theorem 2.5 we can say that $d_L(F_{ac}, \epsilon_0) < t$ whenever $d_L(F_{ab}, \epsilon_0) < \eta$ and $d_L(F_{bc}, \epsilon_0) < \eta$.*

Definition 2.11. Let S be a non-empty set then a family of sets $I \subset P(S)$ is called to be an ideal if
(i) $A, B \in I \Rightarrow A \cup B \in I$
(ii) $A \in I, B \subset A \Rightarrow B \in I$

I is said to be nontrivial ideal if $S \notin I$ and $I \neq \{\phi\}$. In view of condition (ii) $\phi \in I$. If $I \subsetneq P(S)$ we say that I is proper ideal on S. A nontrivial ideal I is said to be admissible if it contains all the singletons of S. A nontrivial ideal I is said to be non-admissible if it is not admissible. The ideal of all finite subsets of S which we shall denote by Fin(S). If $S = \mathbb{N}$, set of all natural number, then we denote Fin instead of Fin($\mathbb{N}$) for short.

Note 2.12. *A filter on S is a non-empty collection of subsets of S, which is closed under finite intersection and super sets. If I is a non-trivial on a non-void set S then $F = F(I) = \{A \subset S : S \setminus A \in I\}$ is clearly a filter on S and conversely. $F(I)$ is called the associated filter with respect to ideal I.*

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Note 2.13. *If I is an ideal on S and $M \subseteq S$ then we denote by $I|_M$ the trace of the ideal I on the subset M i.e. $I|_M = \{A \cap M : A \in I\}$ and the dual filter is $F(I|_M) = \{G \cap M : G \in F(I)\}$.*

3. Strong I^K-Convergence of Functions

Throughout the paper P stands for a probabilistic metric space(briefly PM space) and we always assume that in a PM space P, the triangle function τ is continuous and P is endowed with strong topology and I, K are non-trivial ideals of a non empty set S unless otherwise stated. First we will give the definition of Fin-convergence of a function in a PM space

Definition 3.1. Let $(P, \mathcal{F}, \tau)$ be a PM space. A function $f : S \to P$ is said to be Fin-convergent to $p \in P$ if $f^{-1}(P \setminus \mathcal{N}_p(t)) = \{s \in S : f(s) \notin \mathcal{N}_p(t)\}$ is a finite set for every strong t-neighborhood $\mathcal{N}_p(t)$ of p.

We use the notation Fin(S)-$f = p$. Now we give the definition of strong I-convergence using functions instead of sequences in a probabilistic metric space.

Definition 3.2. (cf.[32]) Let I be an ideal on a non-empty set S and $(P, \mathcal{F}, \tau)$ be a PM space. A function $f : S \to P$ is said to be strong I-convergent to $p \in P$ if

$$f^{-1}(\mathcal{N}_p(t)) = \{s \in S : f(s) \in \mathcal{N}_p(t)\} \in F(I)$$

holds for every strong t-neighborhood $\mathcal{N}_p(t)$ of p.

That is $f^{-1}(P \setminus \mathcal{N}_p(t)) = \{s \in S : f(s) \notin \mathcal{N}_p(t)\} \in I$ for every strong t-neighborhood. We use the notation $f \xrightarrow{str-I} p$. If $S = \mathbb{N}$ we obtain the usual definition of strong I-convergence of sequences in a PM space. In this case the notation $p_n \xrightarrow{str-I} p$ is used for a real sequence $\{p_n\}$. Now we consider some primary results regarding strong I-convergence for future reference.

Note 3.3. *(i) If I is an ideal on an arbitrary set S and P is PM space then it can be easily verified that strong I-limit of a function is unique.*
(ii) If I_1, I_2 are ideals on an arbitrary set S such that $I_1 \subseteq I_2$ then for each function $f : S \to P$, we get $f \xrightarrow{str-I_1} p$ implies $f \xrightarrow{str-I_2} p$.
(iii) Again if P, Q are two PM spaces and $g : P \to Q$ is a continuous mapping and if $f : S \to P$ is strong I-convergent to p then $g \circ f$ is strong I-convergent to $g(p)$.

Since we are working with function, we modify the definition of strong I^*-convergence in PM space.

Definition 3.4. Let I be an ideal on an arbitrary set S and let $f : S \to P$ be a function to a PM space P. The function f is called strong I^*-convergent to

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$p \in P$ if there exists a set $M \in F(I)$ such that the function $g : S \to P$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ p & \text{if } s \notin M \end{cases}$$

is Fin(S)-convergent to p.

If f is strong I^*-convergent to p, then we write $f \xrightarrow{str-I^*} p$. The usual notion of strong I^*-convergence of sequence is a special case for $S = \mathbb{N}$. We write $p_n \xrightarrow{str-I^*} p$ for a real sequence $\{p_n\}$. In the definition of strong I^K-convergence we simply replace the Fin by an ideal on the set S. Strong I^K-convergence as a common generalization of all types of strong I^*-convergence of sequences and functions from S to P. Here we shall work with functions instead of sequences. One of the reasons is that using functions sometimes helps to simplify notation.

Definition 3.5. Let K and I be an ideal on an arbitrary set S, P be a PM space and let p be an element of P. The function $f : S \to P$ is called strong I^K-convergent to $p \in P$ if there exists a set $M \in F(I)$ such that the function $g : S \to P$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ p & \text{if } s \notin M \end{cases}$$

is strong K-convergent to p.

Remark 3.6. We can reformulate the definition of strong I^K-convergence in the following way: if there exists an $M \in F(I)$ such that the function $f|_M$ is strong $K|_M$-convergent to p where $K|_M = \{A \cap M : A \in K\}$.

If f is strong I^K-convergent to p, then we write $f \xrightarrow{str-I^K} p$. As usual, notion of I^K-convergence of sequence is a special case for $S = \mathbb{N}$.

Lemma 3.7. *If I and K are ideals on an arbitrary set S and $f : S \to P$ is a function such that $f \xrightarrow{str-K} p$, then $f \xrightarrow{str-I^K} p$.*

Proof. The proof is simple. Choose $M = S \in F(I)$ in Definition 3.5. □

Proposition 3.8. *Let I, J, K and L be ideals on a set S such that $I \subseteq J$ and $K \subseteq L$ and let P be a PM space. Then for any function $f : S \to P$, we have*
(i)$f \xrightarrow{str-I^K} p \;\Rightarrow\; f \xrightarrow{str-J^K} p$ and
(ii)$f \xrightarrow{str-I^K} p \;\Rightarrow\; f \xrightarrow{str-I^L} p$

Proof. (i) Now as $f \xrightarrow{str-I^K} p$ so there exist a set $M \in F(I)$ such that the function $g : S \to P$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ p & \text{if } s \notin M \end{cases}$$

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is strong K-convergent to p. Here $M \in F(I) \subseteq F(J)$ as $I \subseteq J$. So obviously $f \xrightarrow{str-J^K} p$. (ii) The proof directly follows from the fact that $K \subset L$ and the note 3.3(ii). □

Theorem 3.9. *Let I,K be ideals on an arbitrary set S, P be a PM space and let f be a function from S to P then*
(i) $f \xrightarrow{str-I^K} p \;\Rightarrow\; f \xrightarrow{str-I} p$ if $K \subseteq I$. (ii) $f \xrightarrow{str-I} p \;\Rightarrow\; f \xrightarrow{str-I^K} p$ if $I \subseteq K$.

Proof. (i) Now $f \xrightarrow{str-I^K} p$, then by the definition of strong I^K-convergence there exist a set $M \in F(I)$ such that the function $g : S \to P$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ p & \text{if } s \notin M \end{cases}$$

is strong K-convergent to p. i.e. $g^{-1}(P \setminus \mathcal{N}_p(t)) = f^{-1}(P \setminus \mathcal{N}_p(t)) \cap M \in K \subseteq I$, for every strong t-neighborhood of p. Consequently, $f^{-1}(P \setminus \mathcal{N}_p(t)) \subseteq (S \setminus M) \cup g^{-1}(P \setminus \mathcal{N}_p(t)) \in I$ [as $S \setminus M \in I$]. Thus $f \xrightarrow{str-I} p$.
(ii) Proof follows from the note 3.3(ii) and lemma 3.7. □

Now we give an example which is strong I^K-convergence but not strong I-convergence.

Example 3.10. Let K and I be two ideals on non-empty set S such that $K \not\subset I$ and $I \not\subset K$, but $K \cap I \neq \phi$. Consider a set $B \in K \setminus I$. Let $\mathcal{N}_p(t)$ be strong t-neighborhood of $p \in P$ and $q \in P \setminus \mathcal{N}_p(t)$. Let us define the function $f : S \to P$ by

$$f(s) = \begin{cases} x & \text{if } s \in S \setminus B \\ y & \text{if } s \in B \end{cases}$$

Clearly, $f^{-1}(P \setminus \mathcal{N}_p(t)) = B \in K$ so $f \xrightarrow{str-K} p$ and by the lemma 3.7, $f \xrightarrow{str-I^K} p$. But $f^{-1}(P \setminus \mathcal{N}_p(t)) = B \notin I$ i.e. $f \overset{str-I}{\nrightarrow} p$.

Note 3.11. *Consider the two sets $M_1 = \{2n : n \in \mathbb{N}\}$ and $M_2 = \{3n : n \in \mathbb{N}\}$ then 2^{M_1} and 2^{M_2} are two ideals such that $2^{M_1} \not\subset 2^{M_2}$ and $2^{M_2} \not\subset 2^{M_1}$ but $2^{M_1} \cap 2^{M_2} \neq \phi$.*

3.1. **Strong I^I and $(I \vee K)^K$-Convergence.** In this part, for any two ideals I, K on a non-empty set S, we discuss strong I^K-convergence when $I = K$ and strong $(I \vee K)^K$-convergence where $I \vee K = \{G \cup H : G \in I, H \in K\}$ is the new ideal containing both I and K. Then it is clear that $I, K \subseteq I \vee K$. It is noted that if $I \vee K$ is non-trivial ideal and $I, K \subsetneq I \vee K$ then both I and K are non-trivial. But following examples shows that converse part may or may not be true always.

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Example 3.12. Consider the two sets $M_1 = \{3n : n \in \mathbb{N}\}$ and $M_2 = \{3n-1 : n \in \mathbb{N}\}$ now it is clear that 2^{M_1}, 2^{M_2} and $2^{M_1} \vee 2^{M_2}$ all ideals are non-trivial on $\mathbb{N}$.

Example 3.13. Now let M_1 be set of all odd integers and M_2 be set of all even integers. Then $I = 2^{M_1}$, $K = 2^{M_2}$ both are non-trivial on the whole set $\mathbb{N}$ but $I \vee K$ is not a non-trivial ideal on $\mathbb{N}$.

If $I \vee K$ is a non-trivial ideal on a non-empty set S then the dual filter of $I \vee K$ is $F(I \vee K) = \{A \cap B : A \in F(I), B \in F(K)\}$.

Theorem 3.14. *Let $f : S \to P$ be a map, I, K be ideals on the set S and P be a PM space. Then*
(i)$f \xrightarrow{str-I} p \Leftrightarrow f \xrightarrow{str-I^I} p$ *and*
(ii)$f \xrightarrow{str-I^K} p \Leftrightarrow f \xrightarrow{str-(I\vee K)^K} p$

Proof. (i) Proof of one implication follows from lemma 3.7 taking $K = I$.
Conversely, let f be strong I^I-convergent to p then there is a set $M \in F(I)$ such that $f|_M$ is strong $I|_M$-convergent. So for any strong t-neighborhood $\mathcal{N}_p(t)$ of p there exists $G \in F(I)$ such that

$$f^{-1}(\mathcal{N}_p(t)) \cap M = G \cap M$$

Clearly $G \cap M \in F(I)$ and $G \cap M \subseteq f^{-1}(\mathcal{N}_p(t))$ i.e. $f^{-1}(\mathcal{N}_p(t)) \in F(I)$ i.e. f is strong convergence to p.
(ii) Suppose that f is strong I^K-convergent to p. Then there is a set $M \in F(I)$ such that $f|_M$ is strong $K|_M$-convergent. Clearly $M \in F(I \vee K)$, since $M \in F(I)$. Therefore f is also strong $(I \vee K)^K$-convergent to p.
Conversely, let f is strong $(I \vee K)^K$-convergent to p i.e. there is a set $M \in F(I \vee K)$ such that $f|_M$ is strong $K|_M$-convergent. Then for any strong t-neighborhood $\mathcal{N}_p(t)$ of p there exists $G \in F(K)$ such that $f^{-1}(\mathcal{N}_p(t)) \cap M = G \cap M$. Since $M \in F(I \vee K)$, then $M = M_1 \cap M_2$ for some $M_1 \in F(I)$ and $M_2 \in F(K)$. Now we have

$$f^{-1}(\mathcal{N}_p(t)) \cap M_1 \supseteq f^{-1}(\mathcal{N}_p(t)) \cap M = (G \cap M_2) \cap M_1$$

Since $G \cap M_2 \in F(K)$, this shows that $f^{-1}(\mathcal{N}_p(t)) \cap M_1 \in F(K|_{M_1})$ i.e. f is strong I^K-convergent to p. □

4. Basic Properties of strong I^K-Convergence in PM Spaces

Theorem 4.1. *Let $I \vee K$ be a nontrivial ideal on an arbitrary non empty set S and let P be a PM-space. Then a strong I^K-convergent function $f : S \to P$ has unique strong I^K-limit.*

Proof. If possible suppose that the strong I^K-convergent function f has two distinct strong I^K-limits say p and q. Since every PM-space is Hausdorff,

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there exists strong t-neighborhood $\mathcal{N}_p(t))$ and $\mathcal{N}_q(t))$ for $(t > 0)$ such that $\mathcal{N}_p(t) \cap \mathcal{N}_q(t) = \phi$.
Now f has strong I^K-limit p, so there exists a set $M_1 \in F(I)$ such that the function $g : S \to P$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M_1 \\ p & \text{if } s \notin M_1 \end{cases}$$

is strong K-convergent to p. So, $g^{-1}(\mathcal{N}_p(t) = \{s \in M_1 : g(s) \in \mathcal{N}_p(t)\} \cup \{s \in S \setminus M_1 : g(s) \in \mathcal{N}_p(t)\} = (S \setminus M_1) \cup f^{-1}(\mathcal{N}_p(t)) = S \setminus (M_1 \setminus f^{-1}(\mathcal{N}_p(t))) \in F(K)$ i.e. $M_1 \setminus f^{-1}(\mathcal{N}_p(t)) \in K$ or $M_1 \setminus N_1 \in K$ where $N_1 = f^{-1}(\mathcal{N}_p(t))$.
Similarly, f has strong I^K-limit q so there exists a set $M_2 \in F(I)$ such that $M_2 \setminus f^{-1}(\mathcal{N}_q(t)) \in K$ or $M_2 \setminus N_2 \in K$ where $N_2 = f^{-1}(\mathcal{N}_q(t))$. So $(M_1 \setminus N_1) \cup (M_2 \setminus N_2) \in K$. Then $(M_1 \cap M_2) \cap (N_1 \cap N_2)^c \subset (M_1 \cap N_1^c) \cup (M_2 \cap N_2^c) \in K$. Thus $(M_1 \cap M_2) \cap (N_1 \cap N_2)^c \in K$ i.e. $(M_1 \cap M_2) \setminus (f^{-1}(\mathcal{N}_p(t)) \cap f^{-1}(\mathcal{N}_q(t))) \in K$ i.e. Since $f^{-1}(\mathcal{N}_p(t) \cap \mathcal{N}_q(t)) = \phi$ then $M_1 \cap M_2 \in K$ i.e.

$$S \setminus (M_1 \cap M_2) \in F(K) \tag{4.1}$$

As $M_1, M_2 \in F(I)$,

$$M_1 \cap M_2 \in F(I) \tag{4.2}$$

As $I \vee K$ is a non-trivial ideal so the dual filter $F(I \vee K) = \{A \cap B : A \in F(I), B \in F(K)\}$ exists. Now from 4.1 and 4.2 we get $\phi \in F(I \vee K)$ which is a contradiction. Hence the strong I^K-limit is unique. □

Theorem 4.2. *If I and K both are admissible ideals and if $f : S \to X \subset P$ is an injective function which is strong I^K-convergent to $p_0 \in P$ then p_0 is a accumulation point of X.*

Proof. The function f has strong I^K-limit p_0, so there exists a set $A \in F(I)$ such that the function $g : S \to P$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ p_0 & \text{if } s \notin A \end{cases}$$

is strong K-convergent to p_0. Let $\mathcal{N}_{p_0}(t)$ be an arbitrary strong t-neighborhood. Then the set C (say)= $g^{-1}(\mathcal{N}_{p_0}(t)) = \{s : g(s) \in \mathcal{N}_{p_0}(t)\} \in F(K)$. So $C \notin K$ i.e. the set C is an infinite set, since K is admissible ideal. Choose $k_0 \in \{s : g(s) \in \mathcal{N}_{p_0}(t)\}$ such that $g(k_0) \neq p_0$ then $g(k_0) \in \mathcal{N}_{p_0}(t) \cap (X \setminus \{p_0\})$. Thus p_0 is a accumulation point of X. □

Theorem 4.3. *A Continuous function $h : P \to P$ always preserves strong I^K-convergence.*

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Proof. Let f has strong I^K-limit p, then $\exists\ M \in F(I)$ such that $g : S \to P$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ p & \text{if } s \notin M \end{cases}$$

is strong K-convergent to p. Let $\mathcal{N}_p(t)$ be a strong t-neighborhood of the point p. Then $g^{-1}(\mathcal{N}_p(t)) = (S \setminus M) \cup f^{-1}(\mathcal{N}_p(t)) = S \setminus (M \setminus f^{-1}(\mathcal{N}_p(t))) \in F(K)$ i.e. $M \setminus f^{-1}(\mathcal{N}_p(t)) \in K$. Now we shall show $h(f(p)) \xrightarrow{str-I^K} h(p)$. So it suffices to show that the function $g_1 : S \to P$ given by

$$g_1(s) = \begin{cases} (h \circ f)(s) & \text{if } s \in M \\ h(p) & \text{if } s \notin M \end{cases}$$

is strong K-convergent to $h(p)$. Let $\mathcal{N}_{h(p)}(t)$ be a strong t-neighborhood containing $h(p)$. Since h is continuous so there exists a strong t-neighborhood $\mathcal{N}_p(t)$ containing p such that $h(\mathcal{N}_p(t)) \subset \mathcal{N}_{h(p)}(t)$. Clearly $\{s : h(f(s)) \notin \mathcal{N}_{h(p)}(t)\} \subset \{s : f(s) \notin \mathcal{N}_p(t)\}$ which implies that $\{s : f(s) \in \mathcal{N}_p(t)\} \subset \{s : h \circ f(s) \in \mathcal{N}_{h(p)}(t)\}$ i.e. $f^{-1}(\mathcal{N}_p(t)) \subset (h \circ f)^{-1}(\mathcal{N}_{h(p)}(t))$. So $M \setminus (h \circ f)^{-1}(\mathcal{N}_{h(p)}(t)) \subset M \setminus f^{-1}(\mathcal{N}_p(t))$. Therefore $M \setminus (h \circ f)^{-1}(\mathcal{N}_{h(x)}(t)) \in K$ as $M \setminus f^{-1}(\mathcal{N}_p(t)) \in K$. So its complement $g_1^{-1}(\mathcal{N}_{h(p)}(t)) \in F(K)$, as required. Hence $h(f(p)) \xrightarrow{str-I^K} h(p)$. □

Theorem 4.4. *If P has no limit point then strong I-convergence implies strong I^K-convergence, where I and K both are admissible ideals.*

Proof. Let $f : S \to P$ be a function such that $f \xrightarrow{str-I} p$. Since P has no limit point so $\mathcal{N}_p(t) = \{p\}$ is open where $\mathcal{N}_p(t)$ is strong t-neighborhood. Now we have $f^{-1}(P \setminus \mathcal{N}_p(t)) = \{s \in S : f(s) \notin \mathcal{N}_p(t)\} \in I$. Then $M = f^{-1}(\mathcal{N}_p(t)) = \{s \in S : f(s) \in \mathcal{N}_p(t)\} \in F(I)$. Then there exists $M \in F(I)$ such that the function $g : S \to P$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ p & \text{if } s \notin M \end{cases}$$

is strong K-convergent to p. (Since for any strong t-neighborhood $\mathcal{N}_p(t)$ containing p, $\{s \in S : g(s) \notin \mathcal{N}_p(t)\} = \phi \in K$). So $f \xrightarrow{str-I^K} p$. □

Note 4.5. *Converse of the theorem4.4 may not be true. Let I and K be two ideals on an arbitrary non-void set S. Consider a set $B \in K \setminus I$. Let $q \in P \setminus \{p\}$ be a fixed point and consider a function $f : S \to P$ by*

$$f(s) = \begin{cases} p & \text{if } s \in S \setminus A \\ q & \text{otherwise} \end{cases}$$

Now if $\mathcal{N}_p(t)$ is any strong t-neighborhood containing p then $f^{-1}(\mathcal{N}_p(t)) = S \setminus B$ if $q \notin \mathcal{N}_p(t)$ and $f^{-1}(\mathcal{N}_p(t)) = S$ if $q \in \mathcal{N}_p(t)$ i.e. in both case $f^{-1}(\mathcal{N}_p(t)) \in F(K)$. Hence strong K-$\lim f = p$ then by lemma(3.7) we get strong I^K-$\lim f =$

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p. But $\mathcal{N}_p(t_0) = \{p\}$ is also a strong t_0-neighborhood containing p, since P has no limit point and $f^{-1}(P \backslash \mathcal{N}_p(t_0)) = B \notin I$. Hence f is not strong I-convergent to p.

4.1. **Additive Property with strong I and I^K-Convergence.** When we are trying to find the relationship between strong I and I^K-convergence, the following condition is important. Before giving the definition of AP(I, K)-condition which is defined in [19], we need to state the definition of K-pseudo intersection.

Definition 4.6. [19] Let K be an ideal on a set S. We write $A \subset_K B$ whenever $A \setminus B \in K$. If $A \subset_K B$ and $B \subset_K A$ then we write $A \sim_K B$. Clearly $A \sim_K B \Leftrightarrow A \triangle B \in K$.
We say that a set A is K-pseudo intersection of a system $\{A_n : n \in \mathbb{N}\}$ if $A \subset_K A_n$ holds for each $n \in \mathbb{N}$.

Definition 4.7. [19] Let I, K be ideals on the set S. We say that I has additive property with respect to K or that the condition AP(I, K) holds if any one of the following equivalent conditions holds:

(i) For every sequence $(A_n)_{n\in\mathbb{N}}$ of sets from I there is $A \in I$ such that $A_n \subset_K A$ for all n's.

(ii) Any sequence $(F_n)_{n\in\mathbb{N}}$ of sets from $F(I)$ has K-pseudo intersection in $F(I)$.

(iii) For every sequence $(A_n)_{n\in\mathbb{N}}$ of sets from I there exists a sequence $(B_n)_{n\in\mathbb{N}} \in I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j\in\mathbb{N}} B_j \in I$.

(iv) For every sequence of mutually disjoint sets $(A_n)_{n\in\mathbb{N}} \in I$ there exists a sequence $(B_n)_{n\in\mathbb{N}} \in I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j\in\mathbb{N}} B_j \in I$.

(v) For every non-decreasing sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \cdots$ of sets from I $\exists$ a sequence $(B_n)_{n\in\mathbb{N}} \in I$ such that $A_j \sim_K B_j$ for $j \in \mathbb{N}$ and $B = \cup_{j\in\mathbb{N}} B_j \in I$.

(vi) In the Boolean algebra $2^S/K$ the ideal I corresponds to a σ-directed subset,i.e. every countable subset has an upper bound.

The AP(I, K)-condition is more generalization of condition AP from [7][17].

Theorem 4.8. *Let I and K be two ideals on an arbitrary non-empty set S and P be a PM space. If the condition $AP(I, K)$ holds then strong I-convergence implies strong I^K-convergence.*

Proof. Let $f : S \to P$ be a function such that $f \xrightarrow{str-I} p$. Let $\mathcal{B} = \{\mathcal{N}_p(t_n) : n \in \mathbb{N}\}$ be a countable base for P at the point p. Now we have $f^{-1}(\mathcal{N}_p(t_n)) \in F(I)$ for each n, so there exists a set $A \in F(I)$ such that $A \subset_K f^{-1}(\mathcal{N}_p(t_n))$ i.e. $A \setminus f^{-1}(\mathcal{N}_p(t_n)) \in K$. Now we shall show that the function $g : S \to P$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ p & \text{if } s \notin A \end{cases}$$

is strong K-convergent to p. Now for $\mathcal{N}_p(t_n) \in B$,we have $g^{-1}(\mathcal{N}_p(t_n)) = (S\backslash A) \cup f^{-1}(\mathcal{N}_p(t_n)) = S\backslash(A\backslash f^{-1}(\mathcal{N}_p(t_n)))$. Since the set $A\backslash f^{-1}(\mathcal{N}_p(t_n)) \in K$,

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so $S\backslash(A\backslash f^{-1}(\mathcal{N}_p(t_n))) \in F(K)$ i.e. $g^{-1}(\mathcal{N}_p(t_n)) \in F(K)$. Therefore g is strong K-convergent to p i.e. f is strong I^K-convergent to p. □

5. Strong I^K-Cauchy Functions

Now we can define in a full generality the notion of Cauchy function and make some basic observations.

Definition 5.1. (cf [32]) Let $(P, \mathcal{F}, \tau)$ be a PM space. A function $f : S \to P$ is called strong I-Cauchy if for any $t > 0$ there exists an $m \in S$ such that

$$\{s \in S : f(s) \notin \mathcal{N}_{f(m)}(t)\} \in I$$

Lemma 5.2. *Let $(P, \mathcal{F}, \tau)$ be a PM space and I be an ideal on a set S. For a function $f : S \to P$ following are equivalent.*
(i) f is strong I-Cauchy.
(ii) For any $t > 0$ there is $m \in S$ such that $\{s \in S : f(s) \in \mathcal{N}_{f(m)}(t)\} \in F(I)$.
(iii) For every $t > 0$ there exists a set $A \in I$ such that $s, m \notin A$ implies $f(s) \in \mathcal{N}_{f(m)}(t)$

Proof. The proof is straightforward and so it is omitted. □

Note 5.3. *(i) Note that in a PM space $(P, \mathcal{F}, \tau)$ every strong I-convergent function is strong I-Cauchy.*
(ii) Clearly if I_1, I_2 are ideals on a set S such that $I_1 \subseteq I_2$ and if $f : S \to P$ is I_1-Cauchy then it is also I_2-Cauchy.

Definition 5.4. Let I, K be ideals on an arbitrary set S and $(P, \mathcal{F}, \tau)$ be a PM space. A function $f : S \to P$ is said to be strong I^K-Cauchy if there is $M \in F(I)$ such that the function $f|_M$ is strong $K|_M$-Cauchy.

If K =Fin we obtain the notion of strong I^*-Cauchy functions. It is relatively easy to see directly from definition and note 5.3(ii) that every strong I^K-convergent function is strong I^K-Cauchy.

Lemma 5.5. *If I and K are ideals on an arbitrary set S and P be a PM space and a function $f : S \to P$ is strong K-Cauchy then it is also strong I^K-Cauchy.*

Proof. If we take $M = S$ then $M \in F(I)$. In this case $K|_M = K$, hence f is strong $K|_M$-Cauchy. This shows that f is strong I^K-Cauchy. □

Lemma 5.6. *Let I, J, K and L be ideals on a set S such that $I \subseteq J$ and $K \subseteq L$ and let P be a PM space. Then for any function $f : S \to P$, we have*
(i)strong I^K-Cauchy ⇒ strong J^K-Cauchy and (ii)strong I^K-Cauchy ⇒ strong I^L-Cauchy.

Proof. (i)If $f : S \to P$ is strong I^K-Cauchy then there is a subset $M \in F(I)$ such that $f|_M$ is strong $K|_M$-Cauchy. Since $F(I) \subseteq F(J)$, we have $M \in F(J)$. This means that f is also strong J^K-Cauchy.

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(ii) As $K \subseteq L$ implies $K|_M \subseteq L|_M$. From note 5.3(ii) we get that if $f|_M$ is strong $K|_M$-Cauchy then it is also strong $L|_M$-Cauchy i.e. f is strong I^L-Cauchy. □

Theorem 5.7. *Let P be a PM space and $f : S \to P$ be a map and let I, K be ideal on the arbitrary set S . then*
(i) f is strong I-Cauchy if and only if it is strong I^I-Cauchy. and
(ii) f is strong I^K-Cauchy if and only if it is strong $(I \vee K)^K$-Cauchy.

Proof. (i) Suppose that f is strong I-Cauchy. Then by lemma 5.5 it is strong I^I-Cauchy by taking $K = I$.
Conversely, let f be strong I^I-Cauchy. So there is a set $M \in F(I)$ such that $f|_M$ is strong $I|_M$-Cauchy. Then for any strong t-neighborhood $\mathcal{N}_f(q)(t)$ of $f(q)$, $q \in S$ the set C(say)$=\{p \in S : f(p) \in \mathcal{N}_f(q)(t)\} \cap M \in F(I|_M)$. So there exists $G \in F(I)$ such that $C = G \cap M$. Clearly $G \cap M \in F(I)$ and $G \cap M \subseteq f^{-1}(\mathcal{N}_{f(q)}(t))$ and so $f^{-1}(\mathcal{N}_{f(q)}(t)) \in F(I)$.
(ii) Suppose that f is strong I^K-Cauchy. Then there is a set $M \in F(I)$ such that $f|_M$ is strong $K|_M$-Cauchy. Clearly if $M \in F(I)$ then $M \in F(I \vee K)$. Therefore f is also strong $(I \vee K)^K$-Cauchy.
Conversely, let f be strong $(I \vee K)^K$-Cauchy. So there is a set $M \in F(I \vee K)$ such that $f|_M$ is strong $K|_M$-Cauchy. Then for any strong t-neighborhood $\mathcal{N}_{f(q)}(t)$, $q \in S$ there exists $G \in F(K)$ such that $f^{-1}(\mathcal{N}_{f(q)}(t)) \cap M = G \cap M$. Since $M \in F(I \vee K)$, then $M = M_1 \cap M_2$ for some $M_1 \in F(I)$ and $M_2 \in F(K)$. Now we have

$$f^{-1}(\mathcal{N}_{f(q)}(t)) \cap M_1 \supseteq f^{-1}(\mathcal{N}_{f(q)}(t)) \cap M = (G \cap M_2) \cap M_1$$

Since $G \cap M_2 \in F(K)$, this shows that $f^{-1}(\mathcal{N}_{f(q)}(t)) \cap M_1 \in F(K|_{M_1})$ i.e. f is strong I^K-Cauchy. □

6. Strong I^K-Limit Points

In this section, following the line of Fridy [13] and Leonetti et al. [15], we modify the definition of strong I-limit points given in [32].

Definition 6.1. Let $f : S \to P$ be a function and I be a non-trivial ideal of S. Then an element $q \in P$ is said to be a strong I-limit point of f if there exists a set M such that $M \notin I$ and the function $g : S \to P$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ q & \text{if } s \notin M \end{cases}$$

is Fin(S)-convergent to q.

In the definition of strong I^K-limit point we simply replace the finite ideal by an arbitrary ideal on the set S.

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Definition 6.2. Let $f : S \to P$ be a function and I, K be two non-trivial ideals of S. Then an element $q \in P$ is said to be a strong I^K-limit point of f if there exists a set M such that $M \notin I, K$ and the function $g : S \to P$ given by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ q & \text{if } s \notin M \end{cases}$$

is strong K-convergent to q.

We denote respectively by $\Lambda_f(I)$ and $\Lambda_f(I^K)$ the collection of all strong I and strong I^K-limit points of f.

Theorem 6.3. *If K is an admissible ideal then $\Lambda_f(I) \subseteq \Lambda_f(I^K)$ when $K \subseteq I$.*

Proof. Proof is obvious. So it is omitted. □

Theorem 6.4. *If every function $f : S \to P$ has a strong I^K-limit point then every infinite set Q in P has an ω-accumulation point when $|S| \leq |Q|$, where $|S|$ denotes the cardinality of the set S.*

Proof. Consider an injective function $f : S \to Q \subset P$ where Q is an infinite set. Then f has a strong I^K-limit point say q. So there exists a set M such that $M \notin I, K$ and the function $g : S \to P$ defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ q & \text{if } s \notin M \end{cases}$$

is strong K-convergent to q. Let $\mathcal{N}_q(t)$ be a strong t-neighborhood then $g^{-1}(\mathcal{N}_q(t)) = (S \setminus M) \cup f^{-1}(\mathcal{N}_q(t)) = S \setminus (M \setminus f^{-1}(\mathcal{N}_q(t))) \in F(K)$ i.e. $M \setminus f^{-1}(\mathcal{N}_q(t)) \in K$. So $f^{-1}(\mathcal{N}_q(t)) \notin K$.[For if $f^{-1}(\mathcal{N}_q(t)) \in K$ then we get $M \in K$, which is a contradiction.] So $\{s : f(s) \in \mathcal{N}_q(t)\}$ is an infinite set. Consequently, $\mathcal{N}_q(t)$ contains infinitely many elements of f in P. So $\mathcal{N}_q(t)$ contains infinitely many points of Q. Thus q becomes ω-accumulation point of Q. □

7. Open Question

An open question is whether I^K-convergence is equivalent to J-convergence, for some ideal $J = J(I, K)$. This would possibly open another line of research or provide a way of re-proving the same results with old techniques.

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