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Coupled Coincidence and Coupled Common Fixed Points of (ψ, ϕ) Contraction Type T-coupling in Metric Spaces

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Abstract. In this paper, we define (ψ, ϕ)-Contraction Type T-coupling, establish a theorem satisfying such contraction condition, and prove the existence and uniqueness of coupled coincidence and coupled common fixed points in metric space. Here ψ and ϕ are two altering distance functions and T is a SCC-Map for metric spaces. Our results extend and generalize several related results in the existing literature. We also provided two examples to verify our main results.

Keywords: Coupled coincidence point, Coupled common fixed Point, (ψ, ϕ)-Contraction Type T-coupling, SCC-Map.

2020 Mathematics subject classification: 54H25, 47H10.

1. Introduction

The theoretical framework of metric fixed point theory has been an active research field and the contraction principle is one the most important theorems in functional analysis. The contraction principle introduced by Banach [2] has wide range of applications in a fixed point theory. The family of contractive mappings in different spaces is a great interest and has already been studied extensively in the existing literatures(see [10, 11, 18, 22, 23]).

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The concept of coupled fixed point and the study of coupled fixed point problems appeared for the first time in some papers of [15, 16, 17]. Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point of mapping. Lakshmikantham and Ciric [14] also introduced the concept of coupled coincidence point. The concept of coupling was introduced by [8, 9]. The results on existence of coupled fixed point and coupled coincidence points appeared in many papers [1, 4, 5, 6, 8, 9, 10, 20, 21]. They proved the existence and uniqueness of strong coupled fixed point for couplings using Kannan type contractions for complete metric spaces.

Choudhury et al. [9] posed an open problem regarding the investigation of fixed point and related properties for couplings satisfying other type of inequalities. Aydi et al. [1] proved the existence and uniqueness of strong coupled fixed point for (ψ, ϕ)-contraction type coupling in complete partial metric spaces. Rashid and Khan [20] attempted to answer this open problem by introducing SCC-Map and ϕ-contraction type T-coupling and generalize ϕ-contraction type coupling given by Aydi et al. [1] to ϕ-contraction type T-coupling and proved the existence theorem of coupled coincidence point for metric spaces which are not complete.

In this paper, we generalize the works of Rashid and Khan [20] by defining a new contractive type namely (ψ, ϕ)-Contraction Type T-coupling and establish a theorem satisfying such contraction condition, and prove the existence and uniqueness of coupled coincidence and coupled common fixed points in metric space. Our results extend and generalize several related results in the existing literature. We also provided two examples to verify our main results.

2. Preliminaries

In this section, we need to recall some basic definitions, lemmas, and necessary results from existing literature.

Definition 2.1. A sequence $\{x_n\}$ in a metric space (X, d) is said to converge to a point $x \in X$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.2. A sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Furthermore, a metric space (X, d) is called complete if every Cauchy sequence $\{x_n\}$ in (X, d) converges to a point $x \in X$.

Lemma 2.3. *Let (X, d) be a metric space. we have*

(1) *If $d(x, y) = 0$, then $x = y$.*
(2) *If $x \neq y$, then $d(x, y) > 0$.*

Lemma 2.4. *Let $\{x_n\}$ and $\{y_n\}$ be sequences such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a metric space (X, d). If $d(x_n, y_n) \to 0$ as $n \to \infty$, then $x = y$.*

Definition 2.5. [9] Let (X, d) be a metric space A and B be two nonempty subsets of X. Then a function $F : X \times X \to X$ is said to be a coupling with respect to A and B if $F(x, y) \in B$ and $F(y, x) \in A$ where $x \in A$ and $y \in B$.

Definition 2.6. [3] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.7. [6] An element $(x, y) \in X \times X$ where X is any nonempty set, is called a strong coupled fixed point of the mapping $F : X \times X \to X$ if (x, y) is the coupled fixed point and $x = y$ that is, $F(x, x) = x$.

Definition 2.8. [9] Let A and B be two nonempty subsets of a complete metric space (X, d). A coupling $F : X \times X \to X$ is called a Banach type coupling with respect to A and B if it satisfies the following inequality:

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)]$$

where $x, v \in A, y, u \in B$, and $k \in [0, 1)$.

Theorem 2.9. [9] *Let A and B be two nonempty closed subsets of a complete metric space (X, d). Let $F : X \times X \to X$ be Banach type coupling with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.*

Definition 2.10. [13] Let A and B be two nonempty subsets of a given set X. Any function $T : X \to X$ is said to be cyclic (with respect to A and B) if $T(A) \subset B$ and $T(B) \subset A$.

Definition 2.11. [20] Let A and B be two nonempty subsets of a given set X. Any function $T : X \to X$ is said to be self-cyclic (with respect to A and B) if $T(A) \subseteq A$ and $T(B) \subseteq B$.

Definition 2.12. [14] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 2.13. [14] An element $(x, y) \in X \times X$, where X is any nonempty set, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and and $g : X \to X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.

Definition 2.14. [6] An element $(x, y) \in X \times X$ where X is any nonempty set, is called a strong coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = y$. That is, $F(x, x) = g(x) = x$.

Definition 2.15. [20] An element $(x, y) \in X \times X$ is called a strong coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = y$. That is, $F(x, x) = g(x)$.

Definition 2.16. [12] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied:

(i) ψ is monotonically non-deceasing and continuous.
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.17. [20] Let A and B be any two nonempty subsets of a metric space (X,d) and $T : X \to X$ be a self-map on X. Then T is said to be SCC-Map with respect to A and B), if

(i) $T(A) \subseteq A$ and $T(B) \subseteq B$,
(ii) $T(A)$ and $T(B)$ are closed in X.

Definition 2.18. [20] Let A and B be two nonempty subsets of a metric space (X,d) and ψ, ϕ are two altering distance functions. Then a coupling $F : X \times X \to X$ is said to be (ψ,ϕ)-contraction type coupling with respect to A and B if it satisfies the following inequality:

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\})$$

for any $x,v \in A$ and $y,u \in B$.

Theorem 2.19. [20] *Let A and B be two nonempty closed subsets of a complete metric space (X,d) and $F : X \times X \to X$ is a (ψ,ϕ)-contraction type coupling (with respect to A and B). That is, there exist altering distance functions ψ, ϕ such that*

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\})$$

for any $x,v \in A$ and $y,u \in B$. Then

(i) $A \cap B \neq \emptyset$.
(ii) F has a unique strong coupled fixed point in $A \cap B$.

Definition 2.20. [14] Let X be nonempty. The mappings $F : X \times X \to X$ and $g : X \to X$ are called weakly compatible if $g(F(x,y)) = F(gx,gy)$ and $g(F(y,x)) = F(gy,gx)$ whenever $gx = F(x,y)$ and $gy = F(y,x)$.

3. Main Results

Before presenting and proving the main theorem, we introduce the following definition.

Definition 3.1. Let A and B be two nonempty subsets of a metric space (X,d) and $T : X \to X$ is a SCC-Map on X (with respect to A and B). Then a coupling $F : X \times X \to X$ is said to be (ψ,ϕ)-contraction type T-coupling (with respect to A and B) if there exist altering distance functions ψ, ϕ such that

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(Tx,Tu),d(Ty,Tv)\}) - \phi(max\{d(Tx,Tu),d(Ty,Tv)\}) \tag{3.1}$$

for any $x, v \in A$ and $y, u \in B$.

Theorem 3.2. *Let A and B be any two nonempty closed subsets of a complete metric space (X, d), $T : X \to X$ is a SCC-Map on X (with respect to A and B), and a coupling F: $X \times X \to X$ be (ψ, ϕ)-contraction type T-coupling (with respect to A and B), then*

(i) $T(A) \cap T(B) \neq \emptyset$.
(ii) *F and T have a coupled coincidence point in $A \times B$.*
(iii) *If F and T are weakly compatible, then F and T have a unique coupled common fixed point in $A \times B$.*

Proof. Since A and B are non-empty subsets of X and F is (ψ, ϕ)-contraction type-T coupling with respect to A and B, then for $x_0 \in A$ and $y_0 \in B$, we define the sequence $\{x_n\}$ and $\{y_n\}$ in A and B respectively such that

$$Tx_{n+1} = F(y_n, x_n) \text{ and } Ty_{n+1} = F(x_n, y_n). \tag{3.2}$$

If for some $n, Tx_{n+1} = Ty_n$ and $Ty_{n+1} = Tx_n,$ then using (3.2), we have $Tx_n = Ty_{n+1} = F(x_n, y_n)$ and $Ty_n = Tx_{n+1} = F(y_n, x_n)$. This show that (x_n, y_n) is a coupled coincidence point of F and T. So, we are done in this case. Thus we assume that $Tx_n \neq Ty_{n+1}$ or $Ty_n \neq Tx_{n+1}$ for all $n \geq 0$.
Let us define a sequence $\{D_n\}$ by

$$D_n = max\{d(Tx_{n+1}, Ty_n), d(Ty_{n+1}, Tx_n)\}. \tag{3.3}$$

Then by lemma 2.3, we have $\{D_n\} \subseteq [0, \infty)$ for all $n \in \mathbb{N}$. Now using (3.1) and (3.2) and the fact that $x_n \in A$ and $y_n \in B$ for all n, we have

$$\begin{aligned} \psi(d(Tx_n, Ty_{n+1})) &= \psi[d(F(y_{n-1}, x_{n-1}), F(x_n, y_n))] \\ &= \psi[d(F(x_n, y_n), F(y_{n-1}, x_{n-1}))] \\ &\leq \psi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}] - \\ &\quad \phi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}]. \end{aligned} \tag{3.4}$$

Using the properties of ϕ, we have

$$\psi(d(Tx_n, Ty_{n+1})) \leq \psi(max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}).$$

Again using the properties of ψ, we get

$$d(Tx_n, Ty_{n+1}) \leq max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}. \tag{3.5}$$

Now, using (3.1) and (3.2) and the fact that $x_n \in A$ and $y_n \in B$ for all n, we have

$$\begin{aligned} \psi(d(Ty_n, Tx_{n+1})) &= \psi[d(F(x_{n-1}, y_{n-1}), F(y_n, x_n))] \\ &\leq \psi[max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}] - \\ &\quad \phi[max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}]. \end{aligned} \tag{3.6}$$

Now, using the properties of ψ and ϕ, we get

$$d(Ty_n, Tx_{n+1}) \leq max\{d(Tx_{n-1}, Ty_n), d(Ty_{n-1}, Tx_n)\}. \tag{3.7}$$

By using (3.5) and (3.7), we get
$max\{d(Ty_n, Tx_{n+1}), d(Ty_{n+1}, Tx_n)\} \leq max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}$.
That is,

$$max\{d(Tx_{n+1}, Ty_n), d(Ty_{n+1}, Tx_n)\} \leq max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}. \tag{3.8}$$

From (3.3) and (3.8), we have $D_n \leq D_{n-1}$ for all $n \geq 1$.
Therefore, $\{D_n\}$ is monotonically decreasing sequence of non-negative real numbers.
There exists $r \geq 0$ such that $\lim_{n\to\infty} D_n = r$. That is,

$$\lim_{n\to\infty}\{d(Tx_{n+1}, Ty_n), d(Ty_{n+1}, Tx_n)\} = r. \tag{3.9}$$

Suppose $r > 0$.
Since $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing, then for all $a, b \in [0, \infty)$, we have

$$max\{\psi(a), \psi(b)\} = \psi(max\{a, b\}). \tag{3.10}$$

Now, using (3.4), (3.8), and (3.10), we get

$$\begin{aligned} \psi[max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\}] &= max[\psi\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\}] \\ &\leq \psi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}] - \\ &\quad \phi[max\{d(Tx_n, Ty_{n-1}), d(Ty_n, Tx_{n-1})\}]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (3.9) and continuities of ψ and ϕ, we have $\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$ which is a contradiction. Hence $\phi(r) = 0$ since ϕ is an altering distance function. So $r = 0$. Hence,

$$\lim_{n\to\infty} D_n = 0.$$

That is,

$$\lim_{n\to\infty} max\{d(Tx_n, Ty_{n+1}), d(Ty_n, Tx_{n+1})\} = 0.$$

Thus

$$\lim_{n\to\infty} d(Tx_n, Ty_{n+1}) = 0 \text{ and } \lim_{n\to\infty} d(Ty_n, Tx_{n+1}) = 0. \tag{3.11}$$

Now, we define a sequence $\{R_n\}$ by $R_n = d(Tx_n, Ty_n)$ and show that $R_n \to 0$ as $n \to \infty$. By using (3.1) and (3.2), we get

$$\begin{aligned} \psi(R_n) &= \psi(d(Tx_n, Ty_n)) \\ &= \psi(d(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \psi(max(\{d(Ty_{n-1}, Tx_{n-1})\})) - \phi(max(\{d(Ty_{n-1}, Tx_{n-1})\})). \end{aligned} \tag{3.12}$$

By properties of ψ and ϕ, we have $R_n \leq d(Tx_{n-1}, Ty_{n-1}) = R_{n-1}$. That is, $R_n \leq R_{n-1}$ for all $n \geq 1$. Thus, $\{R_n\}$ is monotone decreasing sequence of non-negative real numbers which implies that there exists $s \geq 0$ such that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(Tx_n, Ty_n) = s. \tag{3.13}$$

Taking $n \to \infty$ in (3.12) and using continuities of ψ and ϕ, we have $\psi(s) \leq \psi(s) - \phi(s) < \psi(s)$. Since ϕ is an altering distance function, it follows that $\phi(s) = 0$ which in turn implies that $s = 0$. That is,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} d(Tx_n, Ty_n) = 0. \tag{3.14}$$

Now, applying the triangle inequality and using (3.11) and (3.14), we have

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) \leq \lim_{n\to\infty} d(Tx_n, Ty_n) + \lim_{n\to\infty} d(Ty_n, Tx_{n+1}) = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty} d(Ty_n, Ty_{n+1}) \leq \lim_{n\to\infty} d(Ty_n, Tx_n) + \lim_{n\to\infty} d(Tx_n, Ty_{n+1}) = 0. \tag{3.16}$$

Now, we will prove that the sequences $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in $T(A)$ and $T(B)$ respectively. If possible, let $\{Tx_n\}$ or $\{Ty_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and a sequence of positive integer $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, with $n(k) > m(k) > k$, we have

$$g_k = max\{d(Tx_{m(k)}, Tx_{n(k)}), d(Ty_{m(k)}, Ty_{n(k)})\} \geq \epsilon. \tag{3.17}$$

Furthermore, corresponding to m_k, we can choose n_k such that k is the smallest positive integer with $n(k) \geq m(k) > k$ and satisfying (3.17), then

$$max\{d(Tx_{m(k)}, Tx_{n(k)-1}), d(Ty_{m(k)}, Ty_{n(k)-1})\} < \epsilon. \tag{3.18}$$

Now, we show that:
$d(Ty_{n(k)}, Tx_{m(k)+1}) \leq max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}$.
By using (3.1) and (3.2), we get

$$\begin{aligned} \psi[d(Ty_{n(k)}, Tx_{m(k)+1})] &= \psi[d(F(x_{n(k)-1}, y_{n(k)-1}), F(y_{m(k)}, x_{m(k)}))] \\ &\leq \psi[max\{d(Tx_{n(k)-1}, Ty_{m(k)}), d(Ty_{n(k)-1}, Tx_{m(k)})\}] - \\ &\quad \phi[max\{d(Tx_{n(k)-1}, Ty_{m(k)}), d(Ty_{n(k)-1}, Tx_{m(k)})\}]. \end{aligned}$$

Using properties of ψ and ϕ, we have

$$d(Ty_{n(k)}, Tx_{m(k)+1}) \leq max\{d(Tx_{n(k)-1}, Ty_{m(k)}), d(Ty_{n(k)-1}, Tx_{m(k)})\}. \tag{3.19}$$

Similarly, we can show by the same steps that

$$d(Tx_{n(k)}, Ty_{m(k)+1}) \leq max\{d(Ty_{n(k)-1}, Tx_{m(k)}), d(Tx_{n(k)-1}, Ty_{m(k)})\}. \tag{3.20}$$

From (3.19) and (3.20), we have

$$max\{d(Ty_{n(k)}, Tx_{m(k)+1}), d(Tx_{n(k)}, Ty_{m(k)+1})\} \leq \lambda, \tag{3.21}$$

where $\lambda = max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\}$.
It is a fact that for nonnegative real numbers a, b, c,

$$max\{a + c, b + c\} = c + max\{a, b\}.$$

Therefore, by the triangle inequality on (3.18) and the above fact, we have

$$\begin{aligned} \lambda &= max\{d(Tx_{m(k)}, Ty_{n(k)-1}), d(Ty_{m(k)}, Tx_{n(k)-1})\} \\ &\leq max\{d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Ty_{n(k)-1}), d(Ty_{m(k)}, Ty_{n(k)-1}) + \\ &\quad d(Ty_{n(k)-1}, Tx_{n(k)-1})\} \\ &= d(Tx_{n(k)-1}, Ty_{n(k)-1}) + max\{d(Tx_{m(k)}, Tx_{n(k)-1}), d(Ty_{m(k)}, Ty_{n(k)-1})\} \\ &< d(Tx_{n(k)-1}, Ty_{n(k)-1}) + \epsilon. \end{aligned} \tag{3.22}$$

Thus from (3.21) and (3.22), we get

$$max\{d(Ty_{n(k)}, Tx_{m(k)+1}), d(Tx_{n(k)}, Ty_{m(k)+1})\} < d(Tx_{n(k)-1}, Ty_{n(k)-1}) + \epsilon. \tag{3.23}$$

Now again by the triangle inequality, we have

$$\begin{aligned} d(Tx_{n(k)}, Tx_{m(k)}) &\leq d(Tx_{n(k)}, Ty_{n(k)}) + d(Ty_{n(k)}, Tx_{m(k)+1}) + \\ &\quad d(Tx_{m(k)+1}, Tx_{m(k)}) \end{aligned} \tag{3.24}$$

$$\begin{aligned} d(Ty_{n(k)}, Ty_{m(k)}) &\leq d(Ty_{n(k)}, Tx_{n(k)}) + d(Tx_{n(k)}, Ty_{m(k)+1}) + \\ &\quad d(Ty_{m(k)+1}, Ty_{m(k)}). \end{aligned} \tag{3.25}$$

From (3.17), (3.23), (3.24), and (3.25), we get

$$\begin{aligned} \epsilon \leq g_k &= max\{d(Tx_{n(k)}, Tx_{m(k)}), d(Ty_{n(k)}, Ty_{m(k)})\} \\ &\leq d(Tx_{n(k)}, Ty_{n(k)}) + max\{d(Tx_{m(k)}, Tx_{m(k)+1}), d(Ty_{m(k)}, Ty_{m(k)+1})\} + \\ &\quad max\{d(Ty_{n(k)}, Tx_{m(k)+1}), d(Tx_{n(k)}, Ty_{m(k)+1})\} \\ &< d(Tx_{n(k)}, Ty_{n(k)}) + max\{d(Tx_{m(k)}, Tx_{m(k)+1}), d(Ty_{m(k)}, Ty_{m(k)+1})\} + \\ &\quad d(Tx_{n(k)-1}, Ty_{n(k)-1}) + \epsilon. \end{aligned} \tag{3.26}$$

Taking $k \to \infty$ in (3.26) and using (3.14), (3.15), (3.16), and (3.17), we have $\epsilon < \epsilon$, which is a contradiction.
Hence $\{Tx_n\}$ and $\{Ty_n\}$ are Cauchy sequences in $T(A)$ and $T(B)$ respectively. Since $T(A)$ and $T(B)$ are closed subset of a complete metric space X, $\{Tx_n\}$ and $\{Ty_n\}$ are convergent in $T(A)$ and $T(B)$ respectively.
Thus, there exist $r \in T(A)$ and $s \in T(B)$ such that

$$Tx_n \to r \text{ and } Ty_n \to s \text{ as } n \to \infty. \tag{3.27}$$

From (3.14), we have

$$d(Tx_n, Ty_n) \to 0 \text{ as } n \to \infty. \tag{3.28}$$

Therefore, from (3.27) and (3.28), we have

$$s = r. \tag{3.29}$$

As $r \in T(A)$ and $s \in T(B)$, it follows that $s = r \in T(A) \cap T(B)$.
This proves part (i) i.e., $T(A) \cap T(B) \neq \emptyset$.
Now, since $r \in T(A)$ and $s \in T(B)$, there exist $a \in A$ and $b \in B$ such that $r = T(a)$ and $s = T(b)$.
From (3.27) and (3.29), we have

$$Tx_n \to T(a), Ty_n \to T(b) \tag{3.30}$$

$$T(a) = T(b). \tag{3.31}$$

Now, by (3.1), (3.2), (3.30), and (3.31) and the triangle inequality, we have

$$d(r, F(a, b)) \leq d(r, Ty_{n+1}) + d(Ty_{n+1}, F(a, b)).$$

Letting $n \to \infty$, we get

$$d(r, F(a, b)) \leq \lim_{n\to\infty} d(Ty_{n+1}, F(a, b)).$$

It follows that

$$\begin{aligned} \psi(d(r, F(a, b))) &\leq \lim_{n\to\infty} \psi(d(F(x_n, y_n), F(a, b))) \\ &\leq \lim_{n\to\infty} \psi(max\{d(Tx_n, Ta), d(Ty_n, T(b))\}) - \\ &\quad \lim_{n\to\infty} \phi(max\{d(Tx_n, T(a)), d(Ty_n, T(b))\}) \\ &= \psi(max\{d(r, T(a)), d(s, T(b))\}) - \\ &\quad \phi(max\{d(r, T(a)), d(s, T(b))\}) \\ &\leq \psi(max\{d(r, T(a)), d(s, T(b))\}). \end{aligned}$$

Similarly, $\psi(d(s, F(b, a))) \leq \psi(max\{d(s, T(b)), d(r, T(a))\})$.
Since

$$\begin{aligned} \psi(max\{d(r, F(a, b)), d(s, F(b, a))\} &= max\{\psi(d(r, F(a, b))), \psi(d(s, F(b, a)))\} \\ &\leq \psi(max\{d(s, T(b)), d(r, T(a))\}) = 0, \end{aligned}$$

we have

$$max\{d(r, F(a, b)), d(s, F(b, a))\} = 0.$$

So, $F(a, b) = r$ and $F(b, a) = s$.
Hence, $F(a, b) = T(a) = r$ and $F(b, a) = T(b) = s$.
Therefore, $(a, b) \in A \times B$ is the coupled coincidence point, and $(T(a), T(b))$ is the coupled point of coincidence of F and T.
Now, we will show that the coupled point of coincidence of F and T is unique.
Let (a', b') be another coupled coincidence point of F and T.
So, we will prove that $T(a) = T(a')$ and $T(b) = T(b')$. The proof is as follows.

Suppose $T(a) \neq T(a')$. Using (3.1)

$$\begin{aligned}
\psi(d(T(a),T(a'))) &= \psi(d(F(a,b),F(a',b'))) \\
&\leq \psi(max\{d(T(a),T(a')),d(T(b),T(b'))\}) - \\
&\quad \phi(max\{d(T(a),T(a')),d(T(b),T(b'))\}) \\
&= \psi(max\{d(T(a),T(a')),d(T(a),T(a'))\}) - \\
&\quad \phi(max\{d(T(a),T(a')),d(T(a),T(a'))\}) \\
&= \psi(d(T(a),T(a'))) - \phi(d(T(a),T(a'))) \\
&< \psi(d(T(a),T(a'))).
\end{aligned}$$

So that $\phi(d(T(a),T(a'))) = 0$ (since ϕ is an altering distance function) which in turn implies that $d(T(a),T(a')) = 0$. Hence $T(a) = T(a')$. Similarly, we can show that $T(b) = T(b')$. Hence, the coupled point of coincidence of F and T is unique.

Using (3.31), we have $T(a) = T(b)$.

Thus, $(T(a),T(a))$ is the unique coupled point of coincidence of the mapping F and T with respect to A and B. Now, we show that F and T have unique coupled common fixed point. For this let $T(a) = z$, then, we have $z = T(a) = F(a,a)$ by the weakly compatibility of F and T, we have

$$Tz = T(T(a)) = T(F(a,a)) = F(T(a),T(a)) = F(z,z).$$

Thus, $(T(z),T(z))$ is coupled point of coincidence of F and T.

By the uniqueness of coupled point of coincidence of F and T, we have $T(z) = T(a)$.

Thus, we obtain $z = T(z) = F(z,z)$.

Therefore, (z,z) is the unique coupled common fixed point of F and T. □

Remark 3.3. If we take $T = I$ (the identity map) and A and B be any two non-empty closed subsets of a complete metric space, then Theorem 3.2 will reduce to Theorem 2.19 of Rashid and Khan [20].

The following are examples which support our main result.

EXAMPLE 3.4. Let $X = [0,5]$ with a metric d defined on X by

$$d(x,y) = |x-y|.$$

Let $A = \{1\}$ and $B = \{1,2\}$. Then A and B are closed subsets of X.

We define $F : X \times X \to X$ by $F(x,y) = min\{x,y\}$, for all $x,y \in X$.

Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} 1 & \text{if } 0 \leq x < 2 \\ 2 & \text{if } 2 \leq x \leq 5 \end{cases}.$$

Also, we define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t^2$ and $\psi(t) = t^3$.

Then clearly ψ and ϕ are altering distances functions.

First, we show that T is a SCC-map.
$T(A) = \{1\}$ and $T(B) = \{1, 2\}$.
So, $T(A)$ and $T(B)$ are closed subsets of a complete metric space $X = [0, 5]$.
Hence $T : X \to X$ is a SCC-Map.
Second, we show that F is T-coupling with respect to A and B. For all $x \in A$ and $y \in B$, we have $F(x, y) = 1 \in B$ and $F(y, x) = 1 \in A$ which show that F is T-coupling with respect to A and B.
Third, we prove that F is (ψ, ϕ)-Contraction type T-Coupling w.r.t. A and B. Let $x, v \in A$ and $y, u \in B$ i.e., $x = 1$ and $y = 1, 2$. Four cases will arise for y and u.
Case (i): $x = v = 1$ and $y = u = 1$.
Case (ii): $x = v = 1$ and $y = 1, u = 2$.
Case (iii): $x = v = 1$ and $y = 2, u = 1$.
Case (iv): $x = v = 1$ and $y = u = 2$.
For **case (i)**, i.e., $x = v = 1$ and $y = u = 1$, we have $F(x, y) = F(1, 1) = 1$, $F(u, v) = F(1, 1) = 1$, $T(x) = T(y) = T(u) = T(v) = T(1) = 1$, $d(1, 1) = 0$, and

$$\begin{aligned} 0 = \psi(d(F(x,y), F(u,v))) &\leq \psi(max\{d(T(x),T(u)), d(T(y),T(v))\}) - \\ &\quad \phi(max\{d(T(x),T(u)), d(T(y),T(v))\}) \\ &= \psi(max\{0,0\}) - \phi(max\{0,0\}) \\ &= \psi(0) - \phi(0) = 0, \end{aligned}$$

which proves case (i).
For **case (ii)**, i.e., $x = v = 1$ and $y = 1, u = 2$, we have $F(x, y) = F(1, 1) = 1$, $F(u, v) = F(2, 1) = 1, T(x) = T(y) = T(v) = T(1) = 1$,
$T(u) = T(2) = 2, d(1, 1) = 0$, $d(1, 2) = 1$, and

$$\begin{aligned} 0 = \psi(d(F(x,y), F(u,v))) &\leq \psi(max\{d(T(x),T(u)), d(T(y),T(v))\}) - \\ &\quad \phi(max\{d(T(x),T(u)), d(T(y),T(v))\}) \\ &= \psi(max\{1,0\}) - \phi(max\{1,0\}) \\ &= \psi(1) - \phi(1) = 0, \end{aligned}$$

which proves case (ii).
For **case (iii)**, i.e., $x = v = 1$ and $y = 2, u = 1$, we have $F(x, y) = F(1, 2) = 1$, $F(u, v) = F(1, 1) = 1, T(x) = T(u) = T(v) = 1, T(y) = T(2) = 2, d(1, 1) = 0$, $d(2, 1) = 1$, and

$$\begin{aligned} 0 = \psi(d(F(x,y), F(u,v))) &\leq \psi(max\{d(T(x),T(u)), d(T(y),T(v))\}) - \\ &\quad \phi(max\{d(T(x),T(u)), d(T(y),T(v))\}) \\ &= \psi(max\{0,1\}) - \phi(max\{0,1\}) \\ &= \psi(1) - \phi(1) = 0, \end{aligned}$$

which proves case (iii).
For **case (iv)**, i.e., $x = v = 1$ and $y = u = 2$, we have
$F(x, y) = F(1, 2) = 1, F(u, v) = F(2, 1) = 1, T(x) = T(v) = 1,$
$T(y) = T(u) = T(2) = 2, d(1, 1) = 0, d(1, 2) = d(2, 1) = 1$, and

$$\begin{aligned} \psi(d(F(x,y), F(u,v))) &\leq \psi(max\{d(T(x), T(u)), d(T(y), T(v))\}) - \\ &\quad \phi(max\{d(T(x), T(u)), d(T(y), T(v))\}) \\ \psi(0) &\leq \psi(max\{1,1\}) - \phi(max\{1,1\}) \\ 0 &\leq \psi(1) - \phi(1) = 0, \end{aligned}$$

which proves case (iv).
From the cases (i) to (iv) F and T satisfy all the conditions of Theorem 3.2.
Thus F and T have a strong coupled fixed points in $A \cap B$.
Clearly $T(A) \cap T(B) = \{1\} \neq \emptyset$.
1 is the unique strong coupled coincidence point and (1, 1) is the unique coupled common fixed point of F and T in $A \cap B$ as $T(1) = F(1, 1) = min\{1, 1\} = 1$.

Example 3.5. Let $X = [-1, 1]$ with a metric d given by

$$d(x, y) = |x - y|$$

for all $x, y \in X$. Then (X, d) is a complete metric space.
Let $A = [-1, 0]$ and $B = [0, 1]$. Then $A \cap B = \{0\}$.
We define $F : X \times X \to X$ by $F(x, y) = \frac{y-x}{8}$, for all $x, y \in X$.
Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \, . \\ 1 & \text{if } x > 0 \end{cases}$$

Since $T(A) = \{-1, 0\} \subset A, T(B) = \{0, 1\} \subset B$, $T(A)$ and $T(B)$ are closed sets in A and B respectively. Thus $T : X \to X$ is SCC-map with respect to A and B.
Now, we show that $F : X \times X \to X$ is a coupling with respect to A and B.

(i) For every $x \in A$ and $y \in B, 0 \leq y-x \leq 2$, we have $0 \leq F(x, y) = \frac{y-x}{8} \leq \frac{1}{4}$. Thus, $F(x, y) \in [0, \frac{1}{4}] \subset B$.
(ii) Also, for every $y \in B$ and $x \in A$, $-2 \leq x - y \leq 0$, hence we have $-\frac{1}{4} \leq F(y, x) = \frac{x-y}{8} \leq 0$. Thus $F(y, x) \in [-\frac{1}{4}, 0] \subset A$.

From (i) and (ii), F is a coupling with respect to A and B.
It remains to show that F is (ψ, ϕ)-contraction type T-coupling with respect to A and B.
We define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2, \phi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. Clearly ψ and ϕ are altering distance functions.
We consider the following cases:

Case (i): $x = v = 0 \in A, y = u = 0 \in B$.
In this case,

$$\begin{aligned} 0 &= \psi(d(F(x,y),F(u,v)) \\ &\leq \psi(max\{d(Tx,Tu),d(Ty,Tv)\}) - \phi(max\{d(Tx,Tu),d(Ty,Tv)\}) = 0 \end{aligned}$$

Case (ii): $x = v = -1 \in A, y = u = 1 \in B$.
In this case, $F(x,y) = F(-1,1) = \frac{1}{4}, F(u,v) = F(1,-1) = -\frac{1}{4}$,
$d(F(x,y),F(u,v)) = d(\frac{1}{4},-\frac{1}{4}) = \frac{1}{2}$,
$\psi\left(\frac{1}{2}\right) = \frac{1}{4}$,
$T(x) = T(-1) = -1, T(u) = T(1) = 1, T(y) = T(1) = 1, T(v) = T(-1) = -1$,
$max\Big\{d(Tx,Tu),d(Ty,Tv)\Big\} = max\{d(-1,1),d(1,-1)\} = max\{2,2\} = 2$.
So, we have

$$\begin{aligned} \frac{1}{4} = \psi(d(F(x,y),F(u,v)) &\leq \psi\Big(maxd(Tx,Tu),d(Ty,Tv)\Big) \\ &\quad -\phi\Big(max\,\{d(Tx,Tu),d(Ty,Tv)\}\Big) = \frac{7}{2}. \end{aligned}$$

Case (iii): For $-1 \leq x, v < 0$ and $0 < u, y \leq 1$,
$|y-x| \leq 2, |v-u| \leq 2, max\Big\{d(Tx,Tu),d(Ty,Tv)\Big\} \in \{1,2\}$, but

$$\begin{aligned} d(F(x,y),F(u,v)) &= d(F(x,y),F(u,v)) = |F(x,y) - F(u,v)| \\ &= \left|\frac{y-x}{8} - \frac{v-u}{8}\right| \\ &\leq \left|\frac{y-x}{8}\right| + \left|\frac{v-u}{8}\right| \\ &\leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

Thus, since ψ is non-decreasing,

$$\begin{aligned} &\psi(d(F(x,y),F(u,v)) \\ &\qquad \leq \psi\left(\left|\frac{y-x}{8}\right| + \left|\frac{v-u}{8}\right|\right) \leq \psi(1/2) = 1/4 \\ &\qquad \leq \psi\Big(max\,\{d(Tx,Tu),d(Ty,Tv)\}\Big) - \phi\Big(max\,\{d(Tx,Tu),d(Ty,Tv)\}\Big) \\ &\qquad = \psi(1) - \phi(1) = \frac{3}{4} \text{ or } \psi(2) - \phi(2) = \frac{7}{2}. \end{aligned}$$

From cases (i)-(iii), F is (ψ,ϕ)-contraction type T-coupling with respect to A and B and satisfies all the conditions of Theorem 3.2. Thus F and T have a strong coupled common fixed point in $A \cap B = \{0\} \neq \emptyset$, $F(0,0) = T0 = 0$, and 0 the unique coupled common fixed point of F and T.

4. Conclusion

Rashid et al. [20] established and proved a theorem of coupled coincidence Point of (ψ,ϕ)-contraction type coupling in metric spaces. In this paper, we establish and prove existence of coupled coincidence point and existence and uniqueness of coupled common fixed point theorem for (ψ, ϕ)-contraction type T-coupling in metric spaces. Where ψ and ϕ are two altering distance functions and T is a SCC-Map. We also provide two examples in support of our main result. Our work extend coupled coincidence point of (ψ, ϕ)-contraction type coupling in metric spaces to coupled coincidence and coupled common fixed points of (ψ, ϕ)-contraction type T-coupling in metric spaces. Our result extend and generalize comparable results in the existing literature.

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