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The Graded Classical Prime Spectrum with the Zariski Topology as a Notherian Topological Space

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Abstract. Let G be a group. Let R be a G-graded commutative ring with identity and let M be a graded R-module. The graded classical prime spectrum $Cl.Spec_g(M)$ is defined to be the set of all graded classical prime submodule of M. In this paper we establish necessary and sufficient conditions for $Cl.Spec_g(M)$ with the Zariski topology to be a Noetherian topological space.

Keywords: Graded classical prime submodule, Graded classical prime spectrum, Zariski topology.

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1. Introduction

In recent years, the study of modules whose spectra space have a Zariski topology has grown in various directions. Some researchers have investigated the interplay between algebraic properties of a module and the topological properties of its spectrum (see for example [1, 2, 6, 7, 8, 15, 17, 23, 24, 26, 30, 32]). Also the Zariski topology on the graded spectrum of graded rings in [33, 34, 35, 36, 37] was generalized in different ways to the graded spectrum of graded modules over graded commutative rings as in [3, 4, 13, 14, 33]. In the present work, we study graded modules whose graded classical spectrums equipped with the Zariski topology are Noetherian spaces. For this purpose, in

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section 2, we study and develop the basic properties of graded classical Zariski radicals of graded submodules that are essential for the later sections of this article. In section 3, one of the main purposes of this article is the presentation of conditions under which a graded module has Noetherian graded classical prime spectrum with the Zariski topology. In Theorem 3.2, we show that for a graded R-module M, $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space if and only if the ACC for the Z_g^{cl}-radical submodules of M holds, also we extend this result to graded g-Cl.Top R-modules in Theorem 3.3, and moreover, in Corollary 3.14, whenever $(Spec_g(R),\ \tau_R^g)$ is a Noetherian topological space (for example R is Noetherian), then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space if the natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$ is surjective. In Theorem 3.15 we show that if M is a Noetherian graded R-module, then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space. Also we characterize graded modules with Noetherian topological space, we show in Theorem 3.12 and Theorem 3.18, if M is a graded R-module with surjective natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space if and only if $(Spec_g(\overline{R}),\ \tau_{\overline{R}}^g)$ is a Noetherian topological space if and only if M has property (GFG), that is for every graded submodule K, there exists a finitely generated graded ideal I of R so that $Z\text{-}Gr_M^{cl}(K) = Z\text{-}Gr_M^{cl}(IK)$. In Section 4, we obtain theorems related to the irreducible components and the combinatorial dimension of the graded classical prime spectrum. Theorem 4.5, for example, states that a graded R-module M with Noetherian graded classical prime spectrum has the property that every closed subset of $Cl.Specg(M)$ has a finite number of irreducible components, property (GFC). We also show Theorem 4.13, for a graded module M with surjective natural map ψ, the combinatorial dimension of $(Cl.Spec_g(M),\ \varrho^g)$ is the same as the Krull dimension of $\overline{R} = R/Ann(M)$.
Throughout this paper all rings are commutative with identity and all modules are unitary. Before we state some results let us introduce some notation and terminology. We refer to [18], [28] and [29] for these basic properties and more information on graded rings and modules.

Let G be a group and R be a G-graded commutative ring. A proper graded ideal I of R is said to be a graded prime ideal if whenever $rs \in I$, we have $r \in I$ or $s \in I$, where $r, s \in h(R)$, (for more details see [36, 37, 38]). Recall that the spectrum $Spec_g(R)$ of a graded ring R consists of all graded prime ideals of R. For every graded ideal I of R, we set $V_R^g(I) = \{P \in Spec_g(R) | I \subseteq P\}$. Then the sets $V_R^g(I)$ satisfy the axioms for the closed sets of a topology on $Spec_g(R)$, called the Zariski topology; we denote the Zariski topology over the graded prime spectrum as a topological space by $(Spec_g(R), \tau_R^g)$, (for more details see [33, 34, 35, 36, 37]).

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Let R be a G-graded ring and M an R-module. The colon graded ideal of M into K is $(K :_M M) = \{r \in R | rM \subseteq K\} = Ann(M/K)$. Dually, the colon graded submodule of M into a graded ideal I of R is $(K :_M I) = \{m \in M | Im \subseteq K\}$. In the case that $I = Rr$, we write $(K :_R r)$ (see [9]).

A proper graded submodule K of M is called a graded classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in K$, then either $rm \in K$ or $sm \in K$, (for more details see [5, 14]). It is show in [5, Lemma 3.1] that if P is a graded classical prime submodule of M, then $p := (P :_R M)$ is a graded prime ideal of R and P is called a graded p-classical prime submodule.

In the literature, there are many different generalizations of the Zariski topology of graded rings to graded modules.

The quasi-Zariski topology on $Cl.Spec_g(M)$ is defined as follows: For each graded submodule K of M, put $\mathbb{V}^g_*(K) = \{P \in Cl.Spec_g(M) | K \subseteq P\}$ and $\eta^g_*(M) = \{\mathbb{V}^g_*(K) | K$ is a graded submodule of $M\}$. Then there exists a topology ϱ^g_* on $Cl.Spec_g(M)$ having $\eta^g_*(M)$ as the family of closed subsets of $Cl.Spec_g(M)$ if and only if it is closed under finite unions. When this is the case, ϱ^g_* is called the quasi-Zariski topology on $Cl.Spec_g(M)$ and M is called g-Cl.Top module, written as a topological space as $(Cl.Spec_g(M),\ \varrho^g_*)$, (see [3, 14]).

K. Al-Zoubi, M. Jaradat in [4], defined another variety for a graded submodule K of a graded R-module M. They defined the variety of K to be $\mathbb{V}^g(K) = \{P \in Cl.Spec_g(M) : (P :_R M) \supseteq (K :_R M)\}$. Then the set $\eta^g(M) = \{\mathbb{V}^g(K) | K$ is a graded submodule of $M\}$ satisfies the axioms for the closed sets of a topology on $Cl.Spec_g(M)$, called the Zariski topology on $Cl.Spec_g(M)$ and denoted by ϱ^g, written as a topological space as $(Cl.Spec_g(M),\ \varrho^g)$. And they studied some properties on this topology as T_1-space and spectral space.

In [20], M. Jaradat investigated more properties of the Zariski topology on $Cl.Spec_g(M)$ and some conditions under which the graded classical prime spectrum of M is a spectral space for its Zariski topology.

We note that the case that $Cl.Spec_g(M) = \phi$ is the trivial case and we will not discuss it, so throughout the paper we assume that $Cl.Spec_g(M) \neq \phi$.

2. Graded Classical Radical And Graded Zariski Classical Radical Of Graded Submodules

In this section, we obtain results about graded Zariski classical radicals of graded submodules that are needed in the later sections.

Let R be a G-graded ring and M a graded R-module. For a graded submodule K of M, the graded classical radical of K, denoted by $Gr^{cl}_M(K)$, is the intersection of all graded classical prime submodules of M containing K; that

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is, $Gr^{cl}_M(K) = \cap\{P \mid P \in \mathbb{V}^g_*(K)\} = \cap\{P \mid P \supseteq K\}$. If $\mathbb{V}^g_*(K) = \phi$, then $Gr^{cl}_M(K) = M$. If $K = 0$, then $Gr^{cl}_M(0)$ is called the graded classical nil-radical of M. (See [14]). In the next definition we will define a new radical for a graded submodule of a graded module.

Definition 2.1. Let R be a G-graded ring and M be a graded R-module. The graded Zariski classical radical of a graded submodule K of M, denoted by Z-$Gr^{cl}_M(K)$, is the intersection of all members of $\mathbb{V}^g(K)$ for the Zariski topology, that is, Z-$Gr^{cl}_M(K) = \cap\{P \mid P \in \mathbb{V}^g(K)\} = \cap\{P \in Cl.Spec_g(M) \mid (P :_R M) \supseteq (K :_R M)\}$. If $\mathbb{V}^g(K) = \phi$, then Z-$Gr^{cl}_M(K) = M$. We say a graded submodule K is a Z^{cl}_g-radical submodule (graded Zariski classical radical submodule) if $K = Z$-$Gr^{cl}_M(K)$.

Let X be a topological space, if Y is a nonempty subset of X, then we let $\Im(Y)$ denote the intersection of the members of Y. Thus, if Y_1 and Y_2 are subsets of $Cl.Spec_g(M)$, then $\Im(Y_1 \cup Y_2) = \Im(Y_1) \cap \Im(Y_2)$.

Theorem 2.2. [4, Theorem 4.4]. *Let R be a G-graded ring, M a graded R-module, K be a graded submodule of M and Y be a subset of $Cl.Spec_g(M)$. Then $\mathbb{V}^g(\Im(Y)) = Cl\{Y\}$, the closure of Y. Hence Y is closed if and only if $\mathbb{V}^g(\Im(\mathbb{V}^g(K))) = \mathbb{V}^g(Z\text{-}Gr^{cl}_M(K)) = \mathbb{V}^g(K)$.*

The assertions in the following proposition are followed easily from Definition 2.1 and [4, Lemma 3.3 and Lemma 3.6].

Proposition 2.3. *Let R be a G-graded ring, M be a graded R-module, K and L be graded submodules of M, $P \in Cl.Spec_g(M)$, and I be a graded ideal of R. Then we have the following statements:*
(i) If $Q \in \mathbb{V}^g_*(K)$, then $Gr^{cl}_M(K) \subseteq Q$.
(ii) If $P \in \mathbb{V}^g(K)$, then Z-$Gr^{cl}_M(K) \subseteq P$.
(iii) Z-$Gr^{cl}_M(K) \subseteq Gr^{cl}_M(K)$.
(iv) Z-$Gr^{cl}_M(IM) = Z$-$Gr^{cl}_M(Gr(I)M) = Gr^{cl}_M(IM) = Gr^{cl}_M(Gr(I)M)$.
(v) Z-$Gr^{cl}_M(K) = Z$-$Gr^{cl}_M((K :_R M)M) = Z$-$Gr^{cl}_M(Gr((K :_R M))M) = Gr^{cl}_M((K :_R M)M) = Gr^{cl}_M(Gr((K :_R M))M)$.
(vi) If $K \subseteq L$, then $\mathbb{V}^g_*(K) \supseteq \mathbb{V}^g_*(L)$ if and only if $Gr^{cl}_M(K) \subseteq Gr^{cl}_M(L)$. The converse is true if $K \subseteq Gr^{cl}_M(K)$.
(vii) If $(K :_R M) \subseteq (L :_R M)$, then $\mathbb{V}^g(K) \supseteq \mathbb{V}^g(L)$ if and only if Z-$Gr^{cl}_M(K) \subseteq Z$-$Gr^{cl}_M(L)$. The converse is true if $K \subseteq Z$-$Gr^{cl}_M(K)$.
(viii) If P is a graded classical prime submodule, then $(K :_R M) \subseteq (P :_R M)$ if and only if $\mathbb{V}^g(K) \supseteq \mathbb{V}^g(P)$; consequently, $(K :_R M) = (P :_R M)$ if and only if $\mathbb{V}^g(K) = \mathbb{V}^g(P)$ if and only if Z-$Gr^{cl}_M(K) = Z$-$Gr^{cl}_M(P)$.

In the next proposition, we list more properties of both $\mathbb{V}^g(K)$ and Z-$Gr^{cl}_M(K)$ for a graded submodule K of M, which are useful in next section.
Let M be a graded R-module. The map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ where

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$\overline{R} = R/Ann(M)$, defined by $\psi(P) = \overline{(P :_R M)}$ for every $P \in Cl.Spec_g(M)$ will be called the natural map of $Cl.Spec_g(M)$, (see [4]).

Proposition 2.4. *Let R be a G-graded ring, M be a graded R-module, K and L are graded submodules of M and ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, be the natural map. Then we have the following statements:*
(i) $Z\text{-}Gr^{cl}_M(Z\text{-}Gr^{cl}_M(K)) = Z\text{-}Gr^{cl}_M(K)$.
(ii) $Z\text{-}Gr^{cl}_M(K \cap L) = Z\text{-}Gr^{cl}_M(K) \cap Z\text{-}Gr^{cl}_M(L)$.
(iii) If ψ is surjective, then $K \neq M$ if and only if $\mathbb{V}^g_*(K) \neq \phi$ if and only if $Gr^{cl}_M(K) \neq M$.
(iv) If ψ is surjective, then $K \neq M$ if and only if $\mathbb{V}^g(K) \neq \phi$ if and only if $Z\text{-}Gr^{cl}_M(K) \neq M$.
(v) $Gr((K :_R M)) \subseteq (Z\text{-}Gr^{cl}_M(K) :_R M) \subseteq (Gr^{cl}_M(K) :_R M)$ and thus $Gr((K :_R M))M \subseteq Z\text{-}Gr^{cl}_M(K) \subseteq Gr^{cl}_M(K)$. If ψ is surjective, then $Gr((K :_R M)) = (Z\text{-}Gr^{cl}_M(K) :_R M)$.

Proof. (i) Follows directly from Theorem 2.2.
(ii) Holds by [4, Lemma 3.1(iii)]. Since $Z\text{-}Gr^{cl}_M(K \cap L) = \Im(\mathbb{V}^g(K \cap L)) = \Im(\mathbb{V}^g(K) \cup \mathbb{V}^g(L)) = \Im(\mathbb{V}^g(K)) \cap \Im(\mathbb{V}^g(L)) = Z\text{-}Gr^{cl}_M(K) \cap Z\text{-}Gr^{cl}_M(L)$.
(iii) $K = M$ if and only if $\mathbb{V}^g_*(K) = \phi$ if and only if $Gr^{cl}_M(K) = M$.
(iv) Suppose that $K \neq M$. Then $(K :_R M) \neq R$, and so there exists a $p \in Spec_g(R)$ such that $(K : M) \subseteq p$. Since $Ann(M) \subseteq p$ and ψ is surjective, there exists a $P \in Cl.Spec_g(M)$ with $p = (P :_R M) \supseteq (K :_R M)$ by [20, Proposition 4.3]. It follows that $P \in \mathbb{V}^g(K)$, and so $\mathbb{V}^g(K) \neq \phi$. Next suppose that $\mathbb{V}^g(K) \neq \phi$ and let $P \in \mathbb{V}^g(K)$. Then, by Proposition 2.3, $Z\text{-}Gr^{cl}_M(K) = \Im(\mathbb{V}^g(K)) \subseteq P \neq M$. Also $Z\text{-}Gr^{cl}_M(K) \neq M$ implies $K \neq M$ by Definition 2.1.
(v) We assume that $\mathbb{V}^g(K) \neq \phi$ (if not, $Z\text{-}Gr^{cl}_M(K) = M$). Set $H := \{(P :_R M) \in Spec_g(R) \mid P \in \mathbb{V}^g(K)\} = \{(P :_R M) \in Spec_g(R) \mid P \in Cl.Spec_g(M)$ and $(K :_R M) \subseteq (P :_R M)\}$. Then we have $H \subseteq V^g_R((K :_R M))$ and $\Im(H) = \cap_{P \in \mathbb{V}^g(K)}(P :_R M) = ((\cap_{P \in \mathbb{V}^g(K)} P) :_R M) = (Z\text{-}Gr^{cl}_M(K) :_R M)$. Thus $(Z\text{-}Gr^{cl}_M(K) :_R M) = \Im(H) \supseteq \Im(V^g_R((K :_R M))) = Gr((K :_R M))$, whence $Gr((K :_R M))M \subseteq (Z\text{-}Gr^{cl}_M(K) :_R M)M \subseteq Z\text{-}Gr^{cl}_M(K)$. Therefore, by Proposition 2.3 we are done for the first part. To prove the second part, assume that M is a graded module with surjective natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$ and $p \in \mathbb{V}^g((K :_R M))$. Then $Ann(M) \subseteq (K : M) \subseteq p$, and by [20, Proposition 4.3] there exists a graded classical prime submodule P with $p = (P :_R M) \supseteq (K :_R M)$; that is, $p \in H$. Thus $V^g_R((K :_R M)) \subseteq H$. We conclude that $H = \mathbb{V}^g((K :_R M))$. Therefore, $\Im(H) = Z\text{-}Gr^{cl}_M(K :_R M) = \Im(\mathbb{V}^g((K :_R M))) = Gr((K :_R M))$. □

In the next proposition we study the relationship between $Z\text{-}Gr^{cl}_M(K)$ and $Gr^{cl}_M(K)$ for a graded submodule K of M.

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Proposition 2.5. *Let R be a G-graded ring, M be a graded R-module and K be a graded submodule of M. Then the following statements are equivalent:*
(i) $K \subseteq Z\text{-}Gr^{cl}_M(K)$,
(ii) $\mathbb{V}^g(K) = \mathbb{V}^g_*(K)$,
(iii) $Z\text{-}Gr^{cl}_M(K) = Gr^{cl}_M(K)$.
Hence a Z^{cl}_g-radical submodule is a graded classical radical submodule.

Proof. (i) $\Rightarrow$ (ii) Clearly, $\mathbb{V}^g(K) \supseteq \mathbb{V}^g_*(K)$. Let $P \in \mathbb{V}^g(K)$. Then $Z\text{-}Gr^{cl}_M(K) \subseteq P$ so that $K \subseteq P$ by (i). Thus $\mathbb{V}^g(K) \subseteq \mathbb{V}^g_*(K)$ and, therefore, $\mathbb{V}^g(K) = \mathbb{V}^g_*(K)$.
(ii) $\Rightarrow$ (iii) By Definition 2.1.
(iii) $\Rightarrow$ (i) Follows as $K \subseteq Gr^{cl}_M(K)$.
For the last statement, let L be a Z^{cl}_g-radical submodule. Then $L = Z\text{-}Gr^{cl}_M(L)$ and (i) $\Rightarrow$ (iii) imply that $L = Z\text{-}Gr^{cl}_M(L) = Gr^{cl}_M(L)$. □

Recall that a graded R-module M is called a graded multiplication if for each graded submodule N of M, $N = IM$ for some graded ideal I of R. One can easily show that if N is graded submodule of a graded multiplication module M, then $N = (N :_R M)M$, (see [31]).

Proposition 2.6. *Let R be a G-graded ring and M be a graded R-module. If M is a graded multiplication module, then the following hold:*
(i) $Z\text{-}Gr^{cl}_M(K) = Gr^{cl}_M(K)$.
(ii) $Z\text{-}Gr^{cl}_M(Gr^{cl}_M(K)) = Gr^{cl}_M(Z\text{-}Gr^{cl}_M(K))$.

Proof. (i) Since M is a graded multiplication module, then $N = IM$ for some graded ideal I of R. Then by [4, Lemma 3.6(ii)], we have $\mathbb{V}^g(K) = \mathbb{V}^g(IM) = \mathbb{V}^g_*(IM) = \mathbb{V}^g_*(K)$, thus $Z\text{-}Gr^{cl}_M(K) = Gr^{cl}_M(K)$.
(ii) It is clear by (i) and Proposition 2.4(i). □

3. Noetherian Graded Classical Prime Spectrum

In this section, we examine the graded classical prime spectrum for a certain type of graded modules, and we give necessary and sufficient conditions for it to form a Noetherian topological space, with respect to the Zariski topology for graded classical prime submodules. We also investigate other aspects of this topology.

Remark 3.1. A topological space X is said to be Noetherian if the open subsets of X satisfy the ascending chain condition (if the closed subsets of X satisfy the descending chain condition). (See [10, p. 79, Exercises 5-12]).

Theorem 3.2. *Let R be a G-graded ring and M be a graded R-module. Then $(Cl.Spec_g(M), \varrho^g)$ is a Noetherian topological space if and only if the ACC for the Z^{cl}_g-radical submodules of M holds.*

Proof. Suppose the ACC holds for Z^{cl}_g-radical submodules of M. Let $\mathbb{V}^g(K_1) \supseteq \mathbb{V}^g(K_2) \supseteq ...$ be a descending chain of closed sets $\mathbb{V}^g(K_i)$ of $Cl.Spec_g(M)$, where K_i is a graded submodule of M. Then $\Im(\mathbb{V}^g(K_1)) = Z\text{-}Gr^{cl}_M(K_1) \subseteq \Im(\mathbb{V}^g(K_2)) = Z\text{-}Gr^{cl}_M(K_2) \subseteq ...$ is an ascending chain of Z^{cl}_g-radical submodules of M. So, by assumption there exists $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $Z\text{-}Gr^{cl}_M(K_n) = Z\text{-}Gr^{cl}_M(K_{n+i})$. Now, by Theorem 2.2, $\mathbb{V}^g(K_n) = \mathbb{V}^g(Z\text{-}Gr^{cl}_M(K_n)) = \mathbb{V}^g(Z\text{-}Gr^{cl}_M(K_{n+i})) = \mathbb{V}^g(K_{n+i})$. Thus $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space. Conversely, suppose that $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space. Let $K_1 \subseteq K_2 \subseteq ...$ be an ascending chain of Z^{cl}_g-radical submodules of M. Thus $\mathbb{V}^g(K_1) \supseteq \mathbb{V}^g(K_2) \supseteq ...$ be a descending chain of closed sets $\mathbb{V}^g(K_i)$ of $Cl.Spec_g(M)$. By assumption there is $n \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $\mathbb{V}^g(K_n) = \mathbb{V}^g(K_{n+i})$. Therefore, $K_n = Z\text{-}Gr^{cl}_M(K_n) = \Im(\mathbb{V}^g(K_n)) = \Im(\mathbb{V}^g(K_{n+i})) = Z\text{-}Gr^{cl}_M(K_{n+i}) = K_{n+i}$. Therefore the ACC for the Z^{cl}_g-radical submodules of M holds. □

Theorem 3.3. *Let R be a G-graded ring and M be a g-$Cl.Top$ R-module. Then $(Cl.Spec_g(M),\ \varrho^g_*)$ is a Noetherian topological space if and only if the ACC for the graded classical radical submodules of M holds.*

Proof. The proof is similar to that of Theorem 3.2. □

A graded submodule K of M will be called a graded classical semiprime if K is an intersection of graded classical prime submodules of M, (see [14, p. 162]). Thus $Gr^{cl}_M(K)$, $Z\text{-}Gr^{cl}_M(K)$ are graded classical semiprime submodules, and hence we have the following corollary.

Corollary 3.4. *Let R be a G-graded ring and M be a graded R-module. If M satisfies the ACC on graded classical semiprime submodules, then $(Cl.Spec_g(M),\ \varrho^g)$ and $(Cl.Spec_g(M),\ \varrho^g_*)$ are Noetherian topological spaces.*

Proof. By Theorem 3.2 and Theorem 3.3. □

In the next theorem we give different conditions under which a graded R-module M has an injective natural map.

Theorem 3.5. *Let R be a G-graded ring and M be a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then ψ is injective if and only if one of the following cases satisfied:*
(i) $(Cl.Spec_g(M),\ \varrho^g)$ is a T_0-space.
(ii)$\mid Cl.Spec^p_g(M) \mid \leq 1$ for every $p \in Spec_g(R)$.
(iii) For any graded submodules N_1, N_2 of M, if $\mathbb{V}^g(N_1) = \mathbb{V}^g(N_2)$, then $N_1 = N_2$.
(iv) M is a graded classical weak multiplication R-module.
(v) M is a fully graded classical semiprime submodule R-module.

Proof. By [4, Theorem 4.10], [20, Theorem 4.7], [20, Theorem 4.8]. □

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In the following, we give a formal definition of $Cl.Spec_g(M)$-inj graded R-module M.

Definition 3.6. Let R be a G-graded ring and M be a graded R-module. Then M is said to be $Cl.Spec_g(M)$-inj if $Cl.Spec_g(M) = \phi$ or $Cl.Spec_g(M) \neq \phi$ and M satisfies one of the statements in Theorem 3.5.

Theorem 3.7. *Let R be a G-graded ring and M be a graded R-module.*
(i) Suppose that M is $Cl.Spec_g(M)$-inj. If R satisfies the ACC on graded prime ideals, then M satisfies the ACC on graded classical prime submodules.
(ii) Suppose that M is $Cl.Spec_g(M)$-inj. If $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space, then M satisfies the ACC on graded classical radical submodules.
(iii) For each graded submodule K of M, if K satisfies one of the statements in Proposition 2.5, then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space if and only if M satisfies the ACC on graded classical radical submodules.

Proof. (i) Let $P_1 \subseteq P_2 \subseteq ...$ be an ascending chain of graded classical prime submodules of M. This induces the following chain of graded prime ideals, $\psi(P_1) \subseteq \psi(P_2) \subseteq ....$ Since R satisfies the ACC on graded prime ideals, there exists a positive integer k such that for each $i \in \mathbb{N}$, $\psi(P_k) = \psi(P_{k+i})$. Now since ψ is injective, by Theorem 3.5, we have $P_k = P_{k+i}$ as required.
(ii) Let $K_1 \subseteq K_2 \subseteq ...$ be an ascending chain of graded classical prime submodules of M. Then $\mathbb{V}^g(K_1) \supseteq \mathbb{V}^g(K_2) \supseteq ...$ is a descending chain of closed subsets of $(Cl.Spec_g(M),\ \varrho^g)$, which is stationary by assumption. There exists an integer $n \in \mathbb{N}$ such that $\mathbb{V}^g(K_n) = \mathbb{V}^g(K_{n+i})$ for each $i \in \mathbb{N}$. By Theorem 3.5, we have $K_n = K_{n+i}$ for each $i \in \mathbb{N}$.
(iii) By Theorem 3.2. □

In the next corollary we give different cases under which the graded classical prime spectrum is a Noetherian topological space with the quasi-Zariski topology.

Corollary 3.8. *Let R be a G-graded ring and M be a g-Cl.Top R-module. Then $(Cl.Spec_g(M),\ \varrho^g_*)$ is a Noetherian topological space in the following cases:*
(i) R satisfies the ACC on graded prime ideals.
(ii) $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space.
(iii) For each graded submodule K of M, if K satisfies one of the statements in Proposition 2.5 and M satisfies the ACC on Z^{cl}_g-radical submodules.
Moreover, in part (iii) $(Cl.Spec_g(M),\ \varrho^g_*)$ is Noetherian topological space if and only if $(Cl.Spec_g(M),\ \varrho^g)$ is Noetherian topological space.

Proof. It follows from [20, Theorem 6.10], Theorem 3.2, Theorem 3.3 and Theorem 3.7. □

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Proposition 3.9. *Let R be a G-graded ring and M be a graded multiplication R-module. Then $(Cl.Spec_g(M),\ \varrho^g_*)$ is a Noetherian topological space if and only if $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space.*

Proof. It follows from Proposition 2.6. □

In the sequel, we present more conditions under which $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space.
A topological space X is a cofinite topological space when its open sets are the emptyset, X and all subsets with a finite complement. This topology is denoted by τ^{fc}. (See [25]).

Theorem 3.10. *Let R be a G-graded ring such that the intersection of every infinite collection of graded prime ideals of R is zero and let M be a graded R-module. Then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space.*

Proof. Let $\mathbb{V}^g(K)$ be a closed subset of $(Cl.Spec_g(M),\ \varrho^g)$ for some graded submodule N of M. If $\mathbb{V}^g(N)$ is infinite, then $(K :_R M)$ is contained in an infinite number of graded prime ideals of R. Since the intersection of every infinite collection of graded prime ideals of R is zero, $(N :_R M) = (0)$ so that $\mathbb{V}^g(N) = Cl.Spec_g(M)$. It follows that $\varrho^g \subseteq \tau^{fc}$ and hence $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space because every cofinite topological space is Noetherian. □

Theorem 3.11. *Let R be a G-graded ring and M be a graded R-module. Then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space in the following cases:*
(i) If for every graded submodule N of M there exists a finitely generated graded submodule L of K such that $Gr^{cl}_M(K) = Gr^{cl}_M(L)$.
(ii) M satisfies the ACC on extended graded submodules, IM, where I is a graded ideal of R.

Proof. (i) Let $K_1 \subseteq K_2 \subseteq ...$ be an ascending chain of graded classical semiprime submodules of M, and let $K = \cup_i K_i$. By assumption, there exists a finitely generated graded submodule L of K such that $Gr^{cl}_M(K) = Gr^{cl}_M(L)$. Hence there exists a positive integer n such that $L \subseteq K_n$. Then $Gr^{cl}_M(K) = Gr^{cl}_M(L) \subseteq K_n \subseteq K \subseteq Gr^{cl}_M(K)$, so that $K_n = K_{n+1} = K_{n+2} = ...$ Hence, M satisfies the ACC on graded classical semiprime submodules. By Corollary 3.4, $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space.
(ii) Let $\mathbb{V}^g(K_1) \supseteq \mathbb{V}^g(K_2) \supseteq ...$, be a descending chain of closed subsets of $(Cl.Spec_g(M),\ \varrho^g)$. Then we have an ascending chain of graded submodules of M, $\Im(\mathbb{V}^g(K_1)) \subseteq \Im(\mathbb{V}^g(K_2)) \subseteq ...$, and the ascending chain of ideals, $(\Im(\mathbb{V}^g(K_1)) :_R M) \subseteq (\Im(\mathbb{V}^g(K_2)) :_R M) \subseteq ....$ Thus, there is a positive integer n such that $(\Im(\mathbb{V}^g(K_n)) :_R M)M = (\Im(\mathbb{V}^g(K_{n+i})) :_R M)M$ for each $i = 1, 2, ...$ By [4, Lemma 3.6(i)], $\mathbb{V}^g(\Im(\mathbb{V}^g(K_n))) = \mathbb{V}^g(\Im(\mathbb{V}^g(K_{n+i})))$. So, by Theorem 2.2, $\mathbb{V}^g(K_n) = \mathbb{V}^g(K_{n+i})$, and so $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space. □

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The surjectivity of the natural map of $(Cl.Spec_g(M),\, \varrho^g)$ is particularly important in studying properties of $(Cl.Spec_g(M),\, \varrho^g)$. A graded modules M with surjective natural map ψ, plays important roles in the following theorem. In Theorem 3.12, the surjectivity of ψ yields the characterization that $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space exactly if $(Spec_g(\overline{R}),\, \tau^g_{\overline{R}})$ is.

Theorem 3.12. *Let R be a G-graded ring and M be a graded R-module with surjective natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space if and only if $(Spec_g(\overline{R}),\, \tau^g_{\overline{R}})$ is a Noetherian topological space.*

Proof. If $M = 0$, then trivially $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space. Hence we assume that M is a nonzero graded R-module. Let $\Psi : R \to \overline{R}$ be the natural graded epimorphism. For every $\overline{p} \in Spec_g(\overline{R})$, we identify $\overline{p}$ with $\Psi(p)$, where $p \in Spec_g(R)$; we also write $\overline{(K :_R M)}$ for $\Psi((K :_R M))$. Thus $p \in V^g_R((K :_R M))$ if and only if $\overline{p} \in V^g_{\overline{R}}(\overline{(K :_R M)})$. Since ψ is surjective, ψ is a closed mapping by [4, Proposition 3.9], whence $\psi(\mathbb{V}^g(K)) = V^g_{\overline{R}}(\overline{(K :_R M)}) \subseteq Spec_g(\overline{R})$ for every graded submodule K of M. Suppose that $(Spec_g(\overline{R}),\, \tau^g_{\overline{R}})$ is a Noetherian topological space, and let $\mathbb{V}^g(K_1) \supseteq \mathbb{V}^g(K_2) \supseteq ...$ be a descending chain of closed sets in $Cl.Spec_g(M)$, where K_i is a graded submodule of M. Then $\psi(\mathbb{V}^g(K_1)) \supseteq \psi(\mathbb{V}^g(K_2)) \supseteq ...$ is a descending chain of closed sets in $Spec_g(\overline{R})$. Hence there exists a j such that $\psi(\mathbb{V}^g(K_j)) = \psi(\mathbb{V}^g(K_{j+1}))$, that is, $V^g_{\overline{R}}(\overline{(K_j :_R M)}) = V^g_{\overline{R}}(\overline{(K_{j+1} :_R M)})$ in $Spec_g(\overline{R})$. Namely, $V^g_R((K_j :_R M)) = V^g_R((K_{j+1} :_R M))$ in $Spec_g(R)$. Thus, $\mathbb{V}^g(K_j) = \{P \in Cl.Spec_g(M) \mid (P :_R M) \supseteq (K_j :_R M)\} = \{P \in Cl.Spec_g(M) \mid (P :_R M) \in V^g_R((K_j :_R M))\} = \{P \in Cl.Spec_g(M) \mid (P :_R M) \in V^g_R((K_{j+1} :_R M))\}{=}\mathbb{V}^g((K_{j+1}))$. By Remark 3.1, $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space.
Conversely suppose that $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space, and let $\mathbb{V}^g(\overline{I_1}) \supseteq \mathbb{V}^g(\overline{I_2}) \supseteq ...$ be a descending chain of closed sets in $Spec_g(\overline{R})$, where each $\overline{I_i}$ is a graded ideal of $\overline{R}$. Since ψ is continuous by [4, Proposition 3.9], $\psi^{-1}(\mathbb{V}^g(\overline{I}_i)) \supseteq \psi^{-1}(\mathbb{V}^g(\overline{I}_2)) \supseteq ...$ is a descending chain of closed sets in $(Cl.Spec_g(M),\, \varrho^g)$. By hypothesis, there exists an i such that $\psi^{-1}(\mathbb{V}^g(\overline{I}_i)) = \psi^{-1}(\mathbb{V}^g(\overline{I}_{i+1}))$, whence $\psi \circ \psi^{-1}(\mathbb{V}^g(\overline{I}_i)) = \psi \circ \psi^{-1}(\mathbb{V}^g(\overline{I}_{i+1}))$. We have that $\mathbb{V}^g(\overline{I}_i) = \mathbb{V}^g(\overline{I}_{i+1})$ because ψ is surjective. Therefore, $(Spec_g(\overline{R}),\, \tau^g_{\overline{R}})$ is a Noetherian topological space. □

Recall that $(Spec_g(R),\, \tau^g_R)$ is a Noetherian topological space if and only if the ascending chain condition (ACC) for graded radical ideals holds, and R is said to be a graded Noetherian ring if it satisfies the ascending chain condition (ACC) on graded ideals of R. Equivalently, R is graded Noetherian if and only if every graded ideal of R is finitely generated (see [29]).

Lemma 3.13. *Let R be a G-graded ring. If R is a graded Noetherian ring then $(Spec_g(R),\, \tau_R^g)$ is a Noetherian topological space.*

Proof. Let $Y_1 \supset Y_2 \supset ...$ be any decreasing chain of closed subsets of $(Spec_g(R),\, \tau_R^g)$. Then we can write $Y_i = V_R^g(I_i)$ for every i. Define $J_i = \cap_{p\in Y_i} p$. Then clearly $V_R^g(J_i) = V_R^g(I_i)$ and $\{J_i\}_i$ is an increasing chain of graded ideals of R. Since R is a graded Noetherian ring, this sequence $J_1 \supset J_2 \supset ...$ becomes stationary and therefore the chain $Y_1 \supset Y_2 \supset ...$ becomes stationary. Hence, $(Spec_g(R),\, \tau_R^g)$ is a Noetherian topological space. □

Corollary 3.14. *Let R be a G-graded ring and M be a graded R-module with surjective natural map ψ: $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then we have the following:*
(i) If $(Spec_g(R),\, \tau_R^g)$ is a Noetherian topological space, then $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space.
(ii) If R is a graded Noetherian G-ring, then $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space.

Proof. (i) Let M be a graded R-module with surjective natural map. It is well-known that the mapping $\mu : Spec_g(\overline{R}) \to Spec_g(R)$ given by $J/Ann(M) \to J$ is a graded R-homeomorphism, and hence $(Spec_g(\overline{R}),\, \tau_{\overline{R}}^g)$ is homeomorphic to a closed subspace of $(Spec_g(R),\, \tau_R^g)$, and the corresponding topological properties are inherited by closed. Thus if $(Spec_g(R),\, \tau_R^g)$ is a Noetherian topological space, so is $(Spec_g(\overline{R}),\, \tau_{\overline{R}}^g)$. Thus, by Theorem 3.12, $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space.
(ii) Follows by Lemma 3.13 and part (i). □

Recall that a graded R-module M is called a graded Noetherian module if it satisfies the ascending chain condition on its graded submodules.
We know from Lemma 3.13 that for a graded Noetherian ring R, $(Spec_g(R),\, \tau_R^g)$ is a Noetherian topological space. In the following theorem we will generalize this fact.

Theorem 3.15. *Let R be a G-graded ring. If M is a graded Noetherian R-module, then $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space.*

Proof. Let $Y_1 \supset Y_2 \supset ...$, be any decreasing sequence of closed subset of $(Cl.Spec_g(M),\, \varrho^g)$. Then we can write, $Y_i = \mathbb{V}^g(K_i)$, where K_i is a graded submodule of M for every i. Define $K_i^{'} = \cap_{N\in Y_i} N = Z\text{-}Gr_M^{cl}(K_i^{'})$. Then clearly $\mathbb{V}^g(K_i) = \mathbb{V}^g(K_i^{'})$ and $\{K_i^{'}\}_i$ is an increasing sequence of graded submodule of M. Since M is a graded Noetherian R-module, this sequence becomes stationary and therefore the sequence $Y_1 \supset Y_2 \supset ...$, becomes stationary. Hence. $(Cl.Spec_g(M),\, \varrho^g)$ is a Noetherian topological space. □

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We continue to study some conditions under which the graded classical prime spectrum of a graded R-module M is a Noetherian topological space, here we will use the property (GFG), defined below.

Definition 3.16. Let R be a G-graded ring and M be a graded R-module.
(i) A graded submodule K of M is called a graded FG-submodule if $Z\text{-}Gr^{cl}_M(K) = Z\text{-}Gr^{cl}_M(IM)$ for some finitely generated graded ideal I of R.
(ii) M is said to have property (GFG) if every graded submodule of M is a graded FG-submodule.
(iii) A graded ideal of R is called a graded FG-ideal if and only if it is a graded FG-submodule of R. i.e., for any graded ideal I of R we have $Gr(I) = Gr(J)$ for finitely generated graded ideal J of R.

Proposition 3.17. *Let R be a G-graded ring, M be a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and K be a graded submodule of M. Then we have the following statements:*
(i) Suppose that $Gr((K :_R M)) = Gr(I)$ for some finitely generated graded ideal I of R. Then K is a graded FG-submodule of M.
(ii) If ψ is surjective, then K is a graded FG-submodule of M if and only if $(K :_R M)$ is a graded FG-ideal of R.

Proof. (i) By Proposition 2.3(v), $Z\text{-}Gr^{cl}_M(K) = Z\text{-}Gr^{cl}_M((K :_R M)M) = Z\text{-}Gr^{cl}_M(Gr((K :_R M))M)$. Hence $Z\text{-}Gr^{cl}_M(N) = Z\text{-}Gr^{cl}_M(Gr(I)M) = Z\text{-}Gr^{cl}_M(IM)$. Thus, K is a graded FG-submodule of M.
(ii) Suppose that $(K :_R M)$ is a graded FG-ideal. Then by part (i), K is a graded FG-submodule. Conversely, let K be a graded FG-submodule and I a finitely generated graded ideal of R such that $Z\text{-}Gr^{cl}_M(K) = Z\text{-}Gr^{cl}_M(IM)$. Since ψ is surjective by Proposition 2.4(v) and Proposition 2.3(iii), we have $Gr((K :_R M)) = (Z\text{-}Gr^{cl}_M(K) :_R M) = (Z\text{-}Gr^{cl}_M(IM) :_R M) = (Z\text{-}Gr^{cl}_M(Gr(I)M) :_R M) = Gr((Gr(I)M :_R M))$. Now by [20, Proposition 4.4(i)], we get $Gr((K :_R M)) = Gr((Gr(I)M :_R M)) = Gr(I)$. Thus $(K :_R M)$ is a graded FG-ideal of R. □

Recall that if M is a graded R-module with surjective natural map ψ, then the open sets $Cl.Spec_g(M) - \mathbb{V}^g(rM)$ for every $r \in h(R)$ are quasi-compact and form a base for $(Cl.Spec_g(M), \varrho^g)$, (see [4, Theorem 4.2]).

Theorem 3.18. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then M has property (GFG) if and only if $(Cl.Spec_g(M), \varrho^g)$ is a Noetherian topological space.*

Proof. Let K be a graded submodule of M. Then the following statements are equivalent:
(i) There exist a finitely generated graded ideal $I = \sum_{i=1}^{k} r_i R$ such that $Z\text{-}Gr^{cl}_M(K) = Z\text{-}Gr^{cl}_M(IM)$, where $r_i \in h(R)$ and $k \in \mathbb{Z}^+$ M.

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(ii) $\mathbb{V}^g(K) = \mathbb{V}^g(IM) = \mathbb{V}^g(\sum_{i=1}^n r_iM) = \mathbb{V}^g(\sum_{i=1}^n (r_iM :_R M)M) = \cap_{i=1}^n \mathbb{V}^g(r_iM)$, where $r_i \in h(R)$ and $n \in \mathbb{Z}^+$, (see [4, Lemma 3.6(ii) and Theorem 3.1(ii)] and Proposition 2.4).
(iii) The open set $\mathbb{U}^g(K) = Cl.Spec_g(M) - \mathbb{V}^g(K) = Cl.Spec_g(M) - (\cap_{i=1}^n \mathbb{V}^g(r_iM))$ $= \cup_{i=1}^n (Cl.Spec_g(M) - \mathbb{V}^g(r_iM))$, where $r_i \in h(R)$ and $n \in \mathbb{Z}^+$.
(iv) The open set $\mathbb{U}^g(K) = Cl.Spec_g(M) - \mathbb{V}^g(K)$ is quasi-compact as $\mathbb{U}^g(K)$ is a finite union of quasi-compact subsets $Cl.Spec_g(M) - \mathbb{V}^g(r_iM)$ with $r_i \in h(R)$, (see [4, Theorem 4.3]).
By using the equivalences above, we deduce that M has property (GFG) if and only if every open subset $\mathbb{U}^g(K)$ of form $Cl.Spec_g(M) - \mathbb{V}^g(K)$, where K is a graded submodule of M is quasi-compact if and only if $(Cl.Spec_g(M), \varrho^g)$ is a Noetherian topological space, (see [11, Chap. 1, 2. p. 123, Proposition 9]). □

Corollary 3.19. *Let R be a G-graded ring. Then R has property (GFG) if and only if $(Spec_g(R), \tau_R^g)$ is a Noetherian topological space.*

Proof. It is clear by Theorem 3.18. □

By [14, Proposition 3.3], the graded classical prime submodules of M/K are just the graded submodules N/K where N is a graded classical prime submodule of M with $K \subseteq N$. So we have the following lemma which is needed for Theorem 3.21.

Lemma 3.20. *Let R be a G-graded ring, M and $M^{'}$ be graded R-modules, and let $v : M \to M^{'}$ be a graded R-module homomorphism.*
(i) If $K^{'}$ is a graded submodule of M' and $K = v^{-1}(K^{'})$, then $(K^{'} :_R M^{'}) \subseteq (K :_R M)$. If v is surjective, then equality holds.
(ii) If P is a graded classical prime submodule of M containing $ker(v)$, then $v(P)$ is a graded classical prime submodule of $M^{'}$. If $P^{'}$ is graded classical prime in M', then $P = v^{-1}(P^{'})$ is graded classical prime in M. Thus, if v is surjective, there is a one-to-one correspondence between graded classical prime submodules P of M containing $ker(v)$ and graded classical prime submodules $\overline{P}$ of $M/ker(v) \cong M^{'}$.
(iii) Assume that v is surjective. Let $v^* : Cl.Spec(M) \to Cl.Spec(M')$ be a mapping such that $v^*(P) = v^{-1}(P^{'})$ for every $\acute{P} \in Cl.Spec(M')$. Then v^* is continuous.

Proof. By [14, Proposition 4.1] and [20, Theorem 4.10]. □

Theorem 3.21. *Let M be a graded R-module, K be a graded submodule of M, and $\varkappa : M \to M/K$ be the natural R-epimorphism. Then we have the following statements:*
(i) $\varkappa^* : Cl.Spec_g(M/K) \to Cl.Spec_g(M)$ induces a graded R-homeomorphism of $Cl.Spec_g(M/K)$ onto $\mathbb{V}_*^g(K)$.
(ii) If $(Cl.Spec_g(M), \varrho^g)$ is a Noetherian topological space. Then $(Cl.Spec_g(M/K), \varrho^g)$ is a Noetherian topological space.

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Proof. (i) By Lemma 3.20(ii), we can see that $\varkappa^*$ is a bijection of $Cl.Spec(M/K) = \{\overline{P} = P/K \mid P \in \mathbb{V}^g_*(K)\}$ onto $\mathbb{V}^g_*(K)$, where $\varkappa^*(P/K) = P$ for every $P/K \in Cl.Spec_g(M/K)$ and $(\varkappa^*)^{-1}(P) = P/K$ for every $P \in \mathbb{V}^g_*(K)$. Furthermore, $\varkappa^* : Cl.Spec_g(M/K) \to Cl.Spec_g(M)$ is continuous by to Lemma 3.20(iii). Consequently, $\varkappa^* : Cl.Spec_g(M/K) \to \mathbb{V}^g_*(K)$ is also continuous because $\varkappa^*(Cl.Spec_g(M/K)) = \mathbb{V}^g_*(K)$. To prove that $\varkappa^*$ is a homeomorphism, we only need to show that it is a closed mapping. Let $\mathbb{V}^g(\overline{L})$ be a closed set in $Cl.Spec_g(M/K)$, where $\overline{L} = L/K$ for some graded submodule L of M which contains K, and let $P \in Cl.Spec_g(M)$. Then $P \in \varkappa^*(\mathbb{V}^g(\overline{L}))$ if and only if $P \in \mathbb{V}^g_*(K)$ and $(\varkappa^*)^{-1}(P) = P/K \in \mathbb{V}^g(\overline{L})$ if and only if $P \in \mathbb{V}^g_*(N)$ and $P/N = P \in \mathbb{V}^g(\overline{L})$ if and only if $P \in \mathbb{V}^g_*(N)$ and $(\overline{P} : \overline{M}) \supseteq (\overline{L} : \overline{M})$ where $\overline{M} = M/N$ if and only if $P \in \mathbb{V}^g_*(K)$ and $(P :_R M) \supseteq (L :_R M)$ by Lemma 3.20(i) if and only if $P \in \mathbb{V}^g_*(K) \cap \mathbb{V}^g(L)$. We have that $\varkappa^*(\mathbb{V}^g(\overline{L})) = \mathbb{V}^g_*(K) \cap \mathbb{V}^g(L)$, a closed set in the subspace $\mathbb{V}^g_*(K)$ of $Cl.Spec_g(M)$. Thus $\varkappa^*$ is a closed mapping of $Cl.Spec_g(M/K)$ to $\mathbb{V}^g_*(K)$. Therefore, $\varkappa^*$ is a homeomorphism of $Cl.Spec(M/K)$ onto $\mathbb{V}^g_*(K)$.
(ii) If $(Cl.Spec_g(M), \varrho^g)$ is a Noetherian topological space, then the subspace $\mathbb{V}^g_*(K)$ of $Cl.Spec(M)$ is also a Noetherian topological space by [11, Chap. 1, 2. p. 123, Proposition 8]. It follows that $(Cl.Spec_g(M/K), \varrho^g)$ is a Noetherian topological space by (i). □

4. Noetherian Graded Classical Prime Spectrum And Its Irreducible Components

In this section, we investigate the relationship between the graded minimal classical prime of a graded R-module M and the irreducible components closed subsets of $(Cl.Spec_g(M), \varrho^g)$.
Recall that a topological space X is irreducible if the intersection of two non-empty open subsets of X is non-empty, (see [11, Ch. II, p. 119]). Every subset of a topological space consisting of a single point is irreducible and a subset Y of a topological space X is irreducible if and only if its closure is irreducible [11, Chap. 1, 2. p. 123, Proposition 8(i)]. A maximal irreducible subset Y of X is called an irreducible component of X and it is always closed, (see [11, Ch. II, p. 119]).

Remark 4.1. For a topological space X, we recall:
(i) If X is a Noetherian topological space, then every subspace of X is a Noetherian topological space, and X is a quasi-compact topological space (see [10, Chap. 6, Exc. 5]).
(ii) Every Noetherian topological space has only finitely many irreducible components (see [11, P. 124, Proposition 10]).

Remark 4.2. Let X be a topological space. We consider strictly decreasing (or strictly increasing) chain $Z_0, Z_1, ..., Z_r$ of length r of irreducible closed subsets

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Z_i of X. The supremum of the lengths, taken over all such chains, is called the combinatorial dimension of X and denoted by $dim(X)$. For the empty set, ϕ, the combinatorial dimension of ϕ is defined to be -1.

Definition 4.3. Let M be a graded R-module and K be a proper graded submodule of M. $P \in \mathbb{V}^g_*(K)$ is called a graded minimal classical prime submodule over K if there does not exist $Q \in \mathbb{V}^g_*(K)$ such that $Q \subset P$. If $\mathbb{V}^g_*(K) \neq \phi$, then the existence of graded minimal classical prime submodules over K can be verified easily by Zorn's lemma. We say that P is a graded classical prime divisor (resp. graded minimal classical prime divisor) of K if $P \in \mathbb{V}^g_*(K)$ (resp. $P \in Min^{cl}_g(\mathbb{V}^g_*(K))$, where $Min^{cl}_g(\mathbb{V}^g_*(K))$ is the set of all graded minimal classical prime submodule of M over K).

Definition 4.4. Let R be a G-graded ring and M be a graded R-module.
(i) R is said to have property (GFC) if every closed subset of $(Spec_g(R),\ \tau^g_R)$ has a finite number of irreducible components.
(ii) M is said to have property (GFP) if every graded submodule of M has a finite number of graded minimal classical prime divisors.
(iii) M is said to have property (GFC) if every closed subset of $(Cl.Spec_g(M),\ \varrho^g)$ has a finite number of irreducible components.
(iv) M is said to have property (GFD) if the $(Cl.Spec_g(M),\ \varrho^g)$ has a finite combinatorial dimension.

We next consider irreducible components of closed subsets of $Cl.Spec_g(M)$ for graded R-modules M that has a surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$.

Theorem 4.5. *Let R be a G-graded ring and M be a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$.*
(i) If ψ is surjective, then M has property (GFC) if and only if for every graded submodule K of M, the graded ideal $\overline{(K :_R M)}$ has a finite number of graded minimal prime divisors in $\overline{R}$.
(ii) If ψ is bijective and M has property (GFC), then $Cl.Spec_g(M)$ has finite number of minimal elements with respect to inclusion.
(iii) If ψ is bijective, then M has property (GFC) if and only if M has property (GFP).
(iv) If $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space, then it has property (GFC).

Proof. (i) Is a direct result of [20, Corollary 5.12(ii)].
(ii) By [20, Theorem 5.13].
(iii) Is a direct result of [20, Corollary 5.14].
(iv) By Remark 4.1(ii). □

[20, Theorem 3.7], the irreducible components of $(Spec_g(R),\ \tau^g_R)$ are the closed subsets $V^g_R(p)$, where p is a graded minimal prime ideal of R.

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Corollary 4.6. *Let R be a G-graded ring and M be a graded R-module with natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then we have the following statements:*
(i) If ψ is surjective, then R has property (GFC) if and only if every graded ideal I of R, $Gr(I)$ is contained in a finite number of minimal prime ideals.
(ii) If $(Spec_g(R),\ \tau_R^g)$ is a Noetherian topological space, then R has property (GFC).

Proof. (i) By Theorem 4.5(i).
(ii) By Remark 4.1(ii). □

Corollary 4.7. *Let R be a G-graded ring and M be a graded R-module with bijective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. If $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space, then every classical radical submodule of M is the intersection of a finite number of graded classical prime submodules.*

Proof. By Theorem 4.5(iv) and Theorem 4.5(ii), every graded submodule K of M has a finite number of graded minimal classical prime divisors. □

Corollary 4.8. *Let M be a R-graded module with bijective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. If $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space, then the following statements are true:*
(i) If K is a proper graded submodule of M, then $\mathbb{V}^g(K)$ has only finitely many minimal elements.
(ii) The set of graded minimal classical prime submodules of M is finite. In particular $Cl.Spec_g(M) = \cup_{i=1}^n \mathbb{V}^g(P_i)$, where P_i are all graded minimal classical prime submodules of M.

Proof. (i) We know that $\mathbb{V}^g(K)$ is homeomorphic to $Cl.Spec(M/K)$ by Theorem 3.21(i). Since $(Cl.Spec_g(M),\ \varrho^g)$ is Noetherian, $(Cl.Spec_g(M),\ \varrho^g_{M/K})$ has finitely many irreducible components by Theorem 3.21(ii) and Remark 4.1(ii). Hence by [20, Theorem 5.13], there is one-to-one correspondence between irreducible components of $(Cl.Spec_g(M),\ \varrho^g_{M/K})$ and graded minimal classical prime submodules of M/N. Also for $P \in Cl.Spec(M)$, P/K is a graded minimal classical prime submodule of M/K if and only if P is a graded minimal classical prime submodule of K.
(ii) Since $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space, the number of irreducible components of $(Cl.Spec_g(M),\ \varrho^g)$ is finite by Remark 4.1(ii). So the result follows from [20, Theorem 5.13]. □

Theorem 4.9. *Let R be a G-graded ring and M be a graded R-module. If M has property (GFD) then $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space in the following cases:*
(i) M has a surjective map ψ and for every graded submodule K of M, the

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graded ideal $\overline{(K :_R M)}$ *has a finite number of graded minimal prime divisors in* $\overline{R}$.
(ii) M *has bijective map* ψ *and* M *has property (GFP).*

Proof. It follows by Theorem 4.5(i), (iii) and [30, Proposition 1.1]. □

The next result is obtained by combining Remark 4.1, Theorem 3.2, Corollary 3.4, Theorem 3.7(iii), Theorem 3.10, Theorem 3.11, Theorem 3.12, Corollary 3.14(ii), Theorem 3.15, and Theorem 3.18.

Theorem 4.10. *Let* R *be a* G*-graded ring and* M *be a graded* R*-module. Then* $(Cl.Spec_g(M),\ \varrho^g)$ *is Noetherian topological space and a quasi-compact space with property (GFC) in each of the following cases:*
(i) The ACC for Z_g^{cl}*-radical submodules of* M *holds.*
(ii) M *satisfies the ACC on graded classical semiprime submodules.*
(iii) For each graded submodule K *of* M*,* K *satisfies one of the statements in Proposition 2.5, and* M *satisfies the ACC on graded classical radical submodules.*
(iv) The intersection of every infinite collection of graded prime ideals of R *is zero.*
(v) For every graded submodule K *of* M *there exists a finitely generated graded submodule* L *of* K *such that* $Gr_M^{cl}(K) = Gr_M^{cl}(L)$.
(vi) M *satisfies the ACC on extended graded submodules,* IM*, where* I *is a graded ideal in* R.
(vii) M *has a surjective natural map* ψ*:* $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$*, where* $\overline{R} = R/Ann(M)$*, and* $(Spec_g(\overline{R}),\ \tau_{\overline{R}}^g)$ *is a Noetherian topological space.*
(viii) $(Spec_g(\overline{R}),\ \tau_{\overline{R}}^g)$ *is a Noetherian topological space and* M *has a surjective natural map* ψ*:* $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$*, where* $\overline{R} = R/Ann(M)$.
(ix) M *has a surjective natural map* ψ*:* $Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$*, where* $\overline{R} = R/Ann(M)$*, and* R *is a Noetherian* G*-graded ring.*
(x) M *is a Noetherian graded* R*-module.*
(xi) M *has a surjective natural map* $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$*, where* $\overline{R} = R/Ann(M)$*, and* M *has property (GFG).*

Definition 4.11. Let R be a G-graded ring. The Krull dimension of R, $K.dim_g(R)$, equals the combinatorial dimension of $Spec_g(R)$ equipped with the Zariski topology. For a graded R-module M**,** the classical Krull dimension of M is denoted by $K.dim_g^{cl}(M)$ and is defined by $K.\dim_g^{cl}(M) = K.\dim_g(\overline{R})$, where $\overline{R} = R/Ann(M)$.

Theorem 4.12. *Let* R *be a* G*-graded ring and* M *be a graded* R*-module with surjective natural map* $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$*, where* $\overline{R} = R/Ann(M)$*. Then* $(Cl.Spec_g(M),\ \varrho^g)$ *has a chain of irreducible closed subsets of* $Cl.Spec_g(M)$ *of length* r *if and only if* R *has a chain of graded prime ideals of length* r.

Proof. By [20, Theorem 5.9(i)], since $dim_g(Cl.Spec_g(M)) = dim_g(Spec_g(\overline{R}))$. □

Corollary 4.13. *Let R be a G-graded ring and M be a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$, and equip $Cl.Spec_g(M)$ and $Spec_g(R)$, respectively, with their Zariski topologies. Then, $dim_g(Cl.Spec_g(M)) = K.dim_g(\overline{R}) = dim_g(Spec_g(\overline{R})) = K.dim_g(V^g_R(Ann(M)))$, where $\mathbb{V}^g(Ann(M))$ is a closed subspace of $(Spec_g(R),\, \tau^g_R)$.*

Proof. By Theorem 4.12. □

Let R be a G-graded ring, the graded dimension, $dim_g(R)$ of R was defined in [9] as the supremum of all numbers n for which there exists a chain of graded prime ideals $p_0 \subseteq p_1 \subseteq ... \subseteq p_n$ in R, where $dim_g(R) = -1$ if $Spec_g(R) = \phi$ and $dim_g(R) = 0$ if every graded prime ideal is a maximal. A proper graded ideal I of R is said to be a graded maximum prime ideal if whenever $I \subseteq J$, we have $I = J$, where $J \in Spec_g(R)$, (see [36]), we will denote the set of graded maximum prime ideals of R by $Max^p_g(R)$. A proper graded ideal J of R is said to be a graded minimal prime ideal if whenever $I \subseteq J$, we have $I = J$, where $I \in Spec_g(R)$, (see [36]).

In the next theorem we study the relation between the Noetherian property and the graded dimension of a graded ring R. Let I a graded ideal of R, the minimal graded prime divisors of I correspond bijectively to the irreducible components of the subset $V^g_R(I) \subset (Spec_g(R),\, \tau^g_R)$, (see [20, Theorem 3.7]). In particular, any graded ideal in a graded ring R has minimal graded prime divisors, and any $p \in Spec_g(R)$ contains a minimal graded prime ideal of R.

Lemma 4.14. *Let R be a G-graded ring. Then we have the following statements:*
(i) If $(Spec_g(R),\, \tau^g_R)$ is Noetherian topological space (for example R is a graded Noetherian ring), then R has only finitely many graded minimal prime ideals. Moreover any graded ideal I of R, has only finitely many graded minimal prime divisors.
(ii) $(Spec_g(R),\, \tau^g_R)$ is a T_1-space if and only if $dim_g(R) = 0$ if and only if $Spec_g(R) = Max^p_g(R)$.
(iii) If $(Spec_g(R),\, \tau^g_R)$ is Noetherian topological space (for example R is a graded Noetherian ring) with $dim_g(R) = 0$, then $Spec_g(R)$ has only finitely many elements, which are both graded maximal and minimal prime ideals of R.

Proof. (i) By Remark 4.1(ii), $(Spec_g(R),\, \tau^g_R)$ has a finite number of irreducible components, which are exactly the graded minimal prime ideals of R by [20, Theorem 3.7]. The minimal graded prime divisors of I are just the generic points of the irreducible components of the set $V^g_R(I)$ by [20, Theorem 3.5].
(ii) [20, Theorem 3.8].
(iii) Since $(Spec_g(R),\, \tau^g_R)$ is Noetherian topological space, then by part (i) R

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has only finitely many graded minimal prime ideals. Also since $dim_g(R) = 0$, then by part (ii) we have $Spec_g(R) = Max^p_g(R)$. Therefore $(Spec_g(R),\ \tau^g_R)$ has only finitely many elements which are both graded maximal and minimal prime ideals of R. □

Theorem 4.15. *Let R be a G-graded ring and M be a graded R-module such that $(Cl.Spec_g(M),\ \varrho^g)$ has combinatorial dimension zero and $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space with surjective natural map $\psi : Cl.Spec_g(M) \to Spec_g(\overline{R})$, where $\overline{R} = R/Ann(M)$. Then the set of irreducible components of $(Cl.Spec_g(M),\ \varrho^g)$ is $\{\mathbb{V}^g(p_1M),\ \mathbb{V}^g(p_2M), \ldots,\ \mathbb{V}^g(p_kM)\}$ for some $k \in \mathbb{Z}^+$, where the p_i for $i = 1,\ 2, \ldots,\ k$ are all the graded minimal prime divisors of $Ann(M)$.*

Proof. Since $(Cl.Spec_g(M),\ \varrho^g)$ is a Noetherian topological space with $dim(Cl.Spec_g(M)) = 0$, $(Spec_g(\overline{R}),\ \tau^g_{\overline{R}})$ is a Noetherian topological space and $dim(\overline{R}) = 0$ by Theorem 3.12 and Corollary 4.13. Now $Spec_g(\overline{R}) = Max_g(\overline{R})$ and $(Spec_g(\overline{R}),\ \tau^g_{\overline{R}})$ has only finitely many elements $\overline{p}_1,\ \overline{p}_2, \ldots,\ \overline{p}_k$ which are both maximal and minimal graded prime ideals of $\overline{R}$ by Lemma 4.14(iii). Put $\overline{p}_i = p_i/Ann(M)$, where p_i is a graded ideal of R. Then p_i is a graded minimal prime divisor of $Ann(M)$, which is also a graded maximal ideal of R for every i. Thus $\{p_1,\ p_2, ..\ \ p_k\}$ is the set of all graded minimal prime divisors of $Ann(M)$. Since M is a graded R-module with surjective natural map $\psi : Cl.Spec_g(M) \longrightarrow Spec(\overline{R})$ and $p_iM \neq M$, so by [20, Proposition 4.4(i)] we have $(p_iM :_R M) = p_i$, a graded maximal ideal of R. Thus by [20, Proposition 4.4(ii)] p_iM is a graded p_i-classical prime submodule and $\mathbb{V}^g(p_iM)$ is an irreducible component of $(Cl.Spec_g(M),\ \varrho^g)$ for every i by [20, Corollary 5.10(i)]. Applying [20, Corollary 5.12(ii)], we can conclude that $\{\mathbb{V}^g(p_1M), \mathbb{V}^g(p_2M), \ldots, \mathbb{V}^g(p_kM)\}$ is the set of all irreducible components of $(Cl.Spec_g(M),\ \varrho^g)$. □

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