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Higher Order Degenerate Hermite-Bernoulli Polynomials Arising from p-Adic Integrals on $\mathbb{Z}_p$

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Abstract. Our principal interest in this paper is to study higher order degenerate Hermite-Bernoulli polynomials arising from multivariate p-adic invariant integrals on $\mathbb{Z}_p$. We give interesting identities and properties of these polynomials that are derived using the generating functions and p-adic integral equations. Several familiar and new results are shown to follow as special cases. Some symmetry identities are also established.

Keywords: Multivariate p-adic invariant integral on $\mathbb{Z}_p$, Degenerate Bernoulli polynomials, Hermite polynomials.

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1. Introduction

For a fixed number p (odd and prime), we consider the following notations:
$\mathbb{Z}_p \rightarrow$ ring of p-adic integers.
$\mathbb{Q}_p \rightarrow$ field of p-adic rational numbers.

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$\mathbb{C}_p \rightarrow$ completion of algebraic closure of $\mathbb{Q}_p$.
And let

$$\mathbb{H} = \left\{ \lambda, t \in \mathbb{C}_p \middle| |\lambda t|_p < p^{\frac{-1}{p-1}} \right\}. \tag{1.1}$$

Let υ_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|\, p \,|_p = p^{-\upsilon_p(p)} = \frac{1}{p}$. Suppose $\mathrm{U}D(\mathbb{Z}_p)$ be the space of all uniformly differentiable functions on $\mathbb{Z}_p$, then f is said to be uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$\mathfrak{F}_f(u, w) = \frac{f(u) - f(w)}{u - w},$$

has a limit $l = f^{'}(a)$ as $(u, w) \rightarrow (a, a)$, (see [13, 18]). We begin with the Riemann sums for f given by the expression

$$\frac{1}{p^N} \sum_{0 \leq i < p^N} f(i) = \sum_{0 \leq i < p^N} f(i) \mu(i + p^N \mathbb{Z}_p);\ (N \in \mathbb{N}), \tag{1.2}$$

which converges to a limit in $\mathbb{Z}_p$, denoted by

$$\int_{\mathbb{Z}_p} f(y) d\mu_0(y), \text{(see [2])}.$$

In (1.2), μ denotes the p-adic Haar measure defined as:

Definition 1.1. (see [24]) Let $B(\mathbb{Q}_p)$ be the σ-algebra generated by the family of open subsets of $(\mathbb{Q}_p)$. A *Haar measure* on $(\mathbb{Q}_p)$ is a measure defined by $B(\mathbb{Q}_p)$ with the following properties.
(i). $\mu \neq 0$,
(ii). $\mu(K) < \infty$ *for all compact subset* K *of* $(\mathbb{Q}_p)$,
(iii). $\mu(a + M) = \mu(M)$ *for all* $a \in \mathbb{Q}_p$ *and all* $M \in B(\mathbb{Q}_p)$.

For $f \in \bigcup D(\mathbb{Z}_p)$, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined as (see [6, 7, 9-21]):

$$I_0(f) = \int_{\mathbb{Z}_p} f(y) d\mu_0(y) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{j=0}^{p^N - 1} f(j). \tag{1.3}$$

If $f_n(y) = f(y + n)$, then for $n = 1$, from (1.3), we get:

$$I_0(f_1) = I_0(f) + f^{'}(0), \text{(see [13, 15])}. \tag{1.4}$$

Here, $f^{'}(0)$ refers to the differential coefficient $\frac{d}{dy} f(y)$ at $y = 0$ and $f_1 = f(y + 1)$.

Bernoulli polynomials, named after *Jacob Bernoulli*, occur as a fundamental study for several special functions in mathematics. The exponential generating function for *Bernoulli polynomials* [1-23] is given as

$$\left(\frac{t}{e^t - 1} \right) e^{xt} = \sum_{l \geq 0} B_l(x) \frac{t^l}{l!}. \tag{1.5}$$

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Using Taylor's formula for polynomials, we conclude that $B_l(x+y) = \sum_{k=0}^{l} \binom{l}{k} B_{l-k}(y)x^k$, which on substituting $y=0$ gives Bernoulli polynomials as a sequence of Bernoulli numbers:

$$B_l(x) = \sum_{k=0}^{l} \binom{l}{k} B_{l-k}x^k. \tag{1.6}$$

Several generalized members belonging to the family of Bernoulli polynomials are well defined by many authors. For Example see Table 1:

S.No.	Members of Bernoulli polynomials	Generating function and series definition
1	Degenerate Bernoulli polynomials [23]	$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} B_l(x\|\lambda)\frac{t^l}{l!}$
2	Higher order Bernoulli polynomials[15, 17]	$(\frac{t}{e^t-1})^m e^{xt} = \sum_{l=0}^{\infty} B_l^{(m)}(x)\frac{t^l}{l!}$
3	Higher order Degenerate Bernoulli polynomials [14]	$\left(\frac{\log(1+\lambda t)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_n^{(m)}(x\|\lambda)\frac{t^n}{n!}$
4	Generalized Bernoulli polynomials [11]	$\frac{t\sum_{i=0}^{d-1}\chi(i)e^{it}}{e^{dt}-1}e^{xt} = \sum_{l=0}^{\infty} B_{l,\chi}(x)\frac{t^l}{l!}$
5	Generalized Degenerate Bernoulli polynomials [11]	$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{a}{\lambda}}-1}\sum_{i=0}^{a-1}\chi(i)(1+\lambda t)^{\frac{i+x}{\lambda}} = \sum_{l=0}^{\infty} B_{l,\lambda,\chi}(x)\frac{t^l}{l!}$
6	Carlitz's type Bernoulli polynomials [4, 19]	$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} \beta_l(x,\lambda)\frac{t^l}{l!}$
7	q-Bernoulli polynomials [3, 21]	$\frac{t+\log q}{qe^t-1}e^{xt} = \sum_{l=0}^{\infty} B_{l,q}(x)\frac{t^l}{l!}$

TABLE 1. Some generalized members of Bernoulli polynomials.

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Note that, for $x=0$ in the generating functions (1-7, cf. Table 1), the corresponding Bernoulli numbers are obtained, (see [1-23]).

Applying (1.4) with $f(x) = e^{(x+y)t}$, we get the p-adic integral representation of the generating function for the *Bernoulli polynomials* $B_n(x)$ defined by Kim [10-18]:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.7}$$

where $B_n = B_n(0)$ are the *Bernoulli numbers.*

The *degenerate Bernoulli polynomials* $B_n(x|\lambda)$ are defined in [10, 19] as:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_0(y) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} B_l(x|\lambda)\frac{t^l}{l!}, (t \in \mathbb{H}) \tag{1.8}$$

One must remember that these degenerate Bernoulli polynomials $B_n(x|\lambda)$ are different from the Carlitz's degenerate Bernoulli polynomials $\beta_n(x|\lambda)$. Indeed,

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{l=0}^{\infty}\beta_l(x|\lambda)\frac{t^l}{l!}, \text{(see[4])} \tag{1.9}$$

with $\beta_l(\lambda)=\beta_l(0|\lambda)$, However,

$$\lim_{\lambda\to 0}B_n(x|\lambda)=\lim_{\lambda\to 0}\beta_n(x|\lambda)=B_n(x),$$

where $B_n(x)$ are ordinary Bernoulli polynomials.
Now, comparing (1.8) and (1.9), we have

$$\sum_{l=0}^{\infty}B_l(x|\lambda)\frac{t^l}{l!}=\frac{\log(1+\lambda t)}{\lambda t}\sum_{l=0}^{\infty}\beta_l(x|\lambda)\frac{t^l}{l!}. \tag{1.10}$$

For $m\in\mathbb{N}$, we note that (see [12, 17]):

$$\underbrace{\int_{\mathbb{Z}_p}...\int_{\mathbb{Z}_p}}_{m-times}(1+\lambda t)^{\frac{x+x_1+...+x_m}{\lambda}}d\mu_0(x_1)d\mu_0(x_2)...d\mu_0(x_m)$$

$$=\left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{l=0}^{\infty}B_l^{(m)}(x|\lambda)\frac{t^l}{l!}, \tag{1.11}$$

where $B_l^{(m)}(x|\lambda)$ are the *degenerate Bernoulli polynomials of order* m and $B_l^{(m)}(\lambda)=B_l^{(m)}(0|\lambda)$ are the *degenerate Bernoulli numbers of order* m.

The *Stirling numbers of the first kind* are defined as (see [21]):

$$(x)_n=\prod_{j=0}^{n-1}(x-j)=\sum_{l=0}^{n}S_1(n,l)x^l,\ (n\geq 0). \tag{1.12}$$

In light of (1.12), the *degenerate Stirling numbers of the first kind* are defined by Kim [17]:

$$(x)_{n,\lambda}=\sum_{l=0}^{n}S_1(n,l|\lambda)(x)^l,\ (n\geq 0). \tag{1.13}$$

We observe that

$$(1+\lambda t)^{\frac{x+x_1+\cdots+x_m}{\lambda}}=\sum_{k=0}^{\infty}\binom{\frac{1}{\lambda}(x+x_1+\cdots+x_m)}{k}\lambda^k t^k$$

$$=\sum_{k=0}^{\infty}\lambda^k\left(\frac{x+x_1+...+x_m}{\lambda}\right)_k\frac{t^k}{k!} \tag{1.14}$$

$$=\sum_{k=0}^{\infty}(x+x_1+...+x_m)(x+x_1+...+x_m-\lambda)...(x+x_1+...+x_m-(k-1)\lambda)\frac{t^k}{k!}$$

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$$= \sum_{k=0}^{\infty}(x + x_1 + \cdots + x_m)_{k,\lambda}\frac{t^k}{k!}.$$

where the *degenerate falling factorial sequence* is defined as

$$(x)_{k,\lambda} = \begin{cases} x(x-\lambda)\cdots(x-(k-1)\lambda) \; ; & k \geq 1 \\ 1 \; ; & k = 0 \end{cases}$$

Thus, we indeed can write

$$(x + x_1 + \cdots + x_m)_{k,\lambda} = \sum_{l=0}^{k} S_1(k, l|\lambda)(x + x_1 + \cdots + x_m)^l, \text{ (see [13]).} \quad (1.15)$$

With the viewpoint of the deformed Bernoulli polynomials, the *Daehee polynomials $D_n(x)$ $(i \geq 0)$ of the first kind* are defined as

$$\int_{\mathbb{Z}_p}(1+t)^{x+x_1}d\mu_0(x_1) = \frac{log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{ (see [15])} \quad (1.16).$$

For $x = 0$, $D_n = D_n(0)$ are called the *Daehee numbers*.

For $r \in \mathbb{N}$ the *Bernoulli polynomials of the second kind of order* r are defined by Jang et. al. [7]:

$$\left(\frac{t}{log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x)\frac{t^n}{n!}, \quad (1.17)$$

where $b_n^{(r)} = b_n^{(r)}(0)$ are the *higher order Bernoulli numbers of the second kind.*

In 2016, Khan [8] introduced *the degenerate Hermite polynomials* defined by means of the following generating function

$$(1+\lambda t)^{x/\lambda}(1+\lambda t^2)^{y/\lambda} = \sum_{n=0}^{\infty} H_n(x, y; \lambda)\frac{t^n}{n!}. \quad (1.18)$$

where for $\lambda \to 0, H_n(x, y; \lambda)$ reduces to the 2-variable Kampe de Feriet Hermite polynomials $H_n(x, y)$ (read [1, 5]).

Very recently, Haroon and Khan [6] studied the p-adic integrals for the *degenerate Hermite Bernoulli polynomials* ${}_HB_n(x, y|\lambda)$:

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \sum_{l=0}^{\infty} {}_HB_l(x, y|\lambda)\frac{t^l}{l!}, \quad (1.19)$$

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where, $\lambda, t \in \mathbb{H}$.

The current note is prepared with an objective to study higher order versions of degenerate Hermite Bernoulli polynomials ${}_HB_n^{(m)}(x, y|\lambda)$ arising from multivariate p-adic invariant integrals on $\mathbb{Z}_p$. In the upcomimg sections we investigate the generating function and the p-adic integral expression for degenerate Hermite Bernoulli polynomials of order m to establish some combinatorial properties, explicit expressions and symmetry identities. The results are new and contribute to a number of special cases.

2. Hermite based degenerate Bernoulli polynomials of order m arising from p-adic integrals

The construction of p-adic measures is done mainly to integrate functions. In this section, we derive the generating function for Hermite based degenerate Bernoulli polynomials of order m by using the intimate connection of Bernoulli numbers and polynomials with p-adic measures. We assume that $\lambda \neq 0$ and $\lambda, t \in \mathbb{H}$.

We define the *Hermite based degenerate Bernoulli polynomials of order* m in terms of multivariate p-adic invariant integrals on $\mathbb{Z}_p$ as:

$$\underbrace{\int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p}}_{m-times} (1+\lambda t)^{\frac{x+x_1+...+x_m}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1)...d\mu_0(x_m)$$

$$= \left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^m (1+\lambda t)^{\frac{x}{\lambda}} (1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_HB_n^{(m)}(x, y|\lambda) \frac{t^n}{n!}. \quad (2.1)$$

For $x = y = 0$, ${}_HB_n^{(m)} = {}_HB_n^{(m)}(0, 0|\lambda)$ are called the *degenerate Hermite-Bernoulli numbers of order m.*

Theorem 2.1. *For $m \geq 1$ and $n \geq 0$, we state*

$$\begin{aligned} {}_HB_n^{(m)}(x, y|\lambda) &= \sum_{j=0}^{n-2k} S_1(n-2k, j|\lambda) \int_{\mathbb{Z}_p^m} (x + x_1 + ... + x_m)^j \prod_{i=1}^m d\mu_0(x_i) \\ &= \sum_{k=0}^{[\frac{n}{2}]} \left(\frac{y}{\lambda}\right)_k \lambda^k \sum_{l=0}^{n-2k} S_1(n-2k, l|\lambda) B_j^{(m)}(x) \frac{n!}{k!(n-2k)!} \end{aligned} \quad (2.2)$$

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where

$$\int_{\mathbb{Z}_p^m} f(x_1+\cdots+x_m)\prod_{i=1}^{m} d\mu_0(x_i) = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{m-times} f(x_1+\cdots+x_m)d\mu_0(x_1)\cdots d\mu_0(x_m).$$

Proof. From (2.1), we observe

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_m}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1)\cdots d\mu_0(x_m) \tag{2.3}$$

$$= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}...\int_{\mathbb{Z}_p}(x+x_1+...+x_m)_{n,\lambda}d\mu_0(x_1)...d\mu_0(x_m)\frac{t^n}{n!},$$

which on using (1.15), looks like

$$= (1+\lambda t^2)^{\frac{y}{\lambda}}\sum_{n=0}^{\infty}\sum_{j=0}^{n}S_1(n,j|\lambda)\int_{\mathbb{Z}_p}...\int_{\mathbb{Z}_p}(x+x_1+...+x_m)^j d\mu_0(x_1)...d\mu_0(x_m)\frac{t^n}{n!}. \tag{2.4}$$

$$= \left[\sum_{k=0}^{\infty}\left(\frac{y}{\lambda}\right)_k \lambda^k\frac{t^{2k}}{k!}\right]\left[\sum_{n=0}^{\infty}\sum_{j=0}^{n}S_1(n,j|\lambda)\int_{\mathbb{Z}_p^m}(x+x_1+...+x_m)^j\prod_{i=1}^{m}d\mu_0(x_i)\frac{t^n}{n!}\right] \tag{2.5}$$

From [15, 16], we know that

$$\int_{\mathbb{Z}_p}...\int_{\mathbb{Z}_p}(x+x_1+...+x_m)^n d\mu_0(x_1)...d\mu_0(x_m) = B_n^{(m)}(x), \tag{2.6}$$

where $B_n^{(m)}(x)$ are the *Bernoulli polynomials of order m.* Therefore, from (2.5) and (2.6), we get

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \left[\sum_{i=0}^{\infty}\left(\frac{y}{\lambda}\right)_i\lambda^i\frac{t^{2i}}{i!}\right]\left[\sum_{n=0}^{\infty}\sum_{j=0}^{n}S_1(n,j|\lambda)B_j^{(m)}(x)\frac{t^n}{n!}\right]. \tag{2.7}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (2.1), (2.5) and (2.7), we get the expected result of the Theorem 2.1. □

Remark 2.2. Putting $m = 1$ in (2.2), we get the following result for degenerate Hermite Bernoulli polynomials of order 1 [cf. 6(Theorem 1)].

Corollary 2.3. *For $n \geq 0$, we have*

$$ {}_HB_n(x,y|\lambda) = \sum_{k=0}^{[\frac{n}{2}]}\left(\frac{y}{\lambda}\right)_k\lambda^{n-k-j}\sum_{j=0}^{n-2k}S_1(n-2k,j)B_j(x)\frac{n!}{k!(n-2k)!}.$$

Theorem 2.4. *For $m \geq 1$ and $n \geq 0$, we state*

$$ {}_HB_n^{(m)}(x,y|\lambda) = n!\sum_{r=0}^{[\frac{n}{2}]}\int_{\mathbb{Z}_p^m}\lambda^r\left(\frac{y}{\lambda}\right)_r\frac{(x+x_1+\cdots+x_m)_{n-2r,\lambda}}{(n-2r)!r!}\prod_{i=0}^{m}d\mu_0(x_i)$$

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$$= \sum_{j=0}^{n} \binom{n}{j} \frac{S_1(j+m,m)}{\binom{j+m}{m}} \lambda^j {}_H\beta_{n-j}^{(m)}(x,y|\lambda), \tag{2.8}$$

where, ${}_H\beta_n^{(m)}(x,y|\lambda)$ *are called the Carlitz's degenerate Hermite-Bernoulli polynomials of order m.*

Proof. Using (1.14) in the integral (2.1), we get

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^m} \lambda^n \left(\frac{x+x_1+\cdots+x_m}{\lambda}\right)_n (1+\lambda t^2)^{\frac{y}{\lambda}} \prod_{i=0}^{m} d\mu_0(x_i)\frac{t^n}{n!}.$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^m} (x+x_1+\cdots+x_m)_{n,\lambda} (1+\lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1)\cdots d\mu_0(x_m)\frac{t^n}{n!}.$$

$$= n! \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{2}]} \int_{\mathbb{Z}_p^m} \lambda^r \left(\frac{y}{\lambda}\right)_r \frac{(x+x_1+\cdots+x_m)_{n-2r,\lambda}}{(n-2r)!r!} d\mu_0(x_1)\cdots d\mu_0(x_m)\frac{t^n}{n!}. \tag{2.9}$$

Again from (2.1), we have

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m \left(\frac{log(1+\lambda t)}{\lambda t}\right)^m (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}. \tag{2.10}$$

We can write

$$\left(\frac{log(1+\lambda t)}{\lambda t}\right)^m = m! \sum_{j=0}^{\infty} \frac{S_1(j+m,m)j!\lambda^j}{(j+m)!}\frac{t^j}{j!},$$

$$= \sum_{j=0}^{\infty} \frac{S_1(j+m,m)}{\binom{j+m}{j}} \lambda^j \frac{t^j}{j!}. \tag{2.11}$$

By (1.10) and (2.11) we get

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \left[\sum_{n=0}^{\infty} {}_H\beta_n^{(m)}(x,y|\lambda)\frac{t^n}{n!}\right]\left[\sum_{j=0}^{\infty} \frac{S_1(j+m,m)}{\binom{j+m}{j}} \lambda^j \frac{t^j}{j!}\right]. \tag{2.12}$$

Finally, we conclude the proof of our assertion by comparing the coefficients of t^n in (2.9) and (2.12).

□

Theorem 2.5. *For* $m \geq 1$ *and* $n \geq 0$*, we state*

$${}_HB_n^{(m)}(x,y|\lambda) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} D_{n-j}^{(m)} {}_H\beta_j^{(m)}(x,y|\lambda). \tag{2.13}$$

where $D_j^{(m)}$ *are the higher order Daehee numbers (see [20]).*

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Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_m}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\prod_{i=0}^{m} d\mu_i(x_m)$$
$$= \left(\frac{log(1+\lambda t)}{\lambda t}\right)^m \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}.$$

By (1.16), we define

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{x_1+\ldots+x_m} d\mu_0(x_1)\cdots d\mu_0(x_m)$$
$$= \left(\frac{\log(1+\lambda t)}{\lambda t}\right)^m = \sum_{k=0}^{\infty} D_k^{(m)}\frac{\lambda^k t^k}{k!}, \qquad (m\in\mathbb{N}). \tag{2.14}$$

Now by (2.14), we obtain

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \left[\sum_{n=0}^{\infty} D_k^{(m)}\frac{(\lambda t)^n}{n!}\right]\left[\sum_{j=0}^{\infty} {}_H\beta_j^{(m)}(x,y|\lambda)\frac{t^j}{j!}\right],$$

which on using Cauchy product gives

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}D_{n-j}^{(m)}{}_H\beta_j^{(m)}(x,y|\lambda)\right]\frac{t^n}{n!}.$$

Comparing the coefficients of t^n yields the asserted result in (2.13). □

Remark 2.6. With an immediate substitution $y=0$ in Theorems 2.1, 2.4 and 2.5, we get the formally proved result of Kim [15, (Theorem 1)].

Corollary 2.7. *For $n\geq 0$, we have*

$$\int_{\mathbb{Z}_p^m}(x+x_1+\ldots+x_m)_{n,\lambda}d\mu_0(x_1)\ldots d\mu_0(x_m) = \sum_{j=0}^{n}\binom{n}{j}\lambda^{n-j}\beta_j^{(m)}(x|\lambda)D_{n-j}^{(m)}$$
$$= \sum_{j=0}^{n}\binom{n}{j}\frac{S_1(j+m,m)}{\binom{j+m}{m}}\lambda^j\beta_{n-j}^{(m)}(x|\lambda).$$

Theorem 2.8. *For $m\geq 1$ and $n\geq 0$, we state*

$${}_H\beta_n^{(m)}(x,y|\lambda) = \sum_{k=0}^{n}\binom{n}{k}b_{n-k}^{(m)}\lambda^n\int_{\mathbb{Z}_p^m}\left(\frac{x+x_1+\ldots+x_m}{\lambda}\right)_k(1+\lambda t^2)^{\frac{y}{\lambda}}\prod_{i=0}^{m}d\mu_0(x_i)$$
$$= \sum_{k=0}^{n}\binom{n}{k}b_k^{(m)}\lambda^k{}_HB_{n-k}^{(m)}(x,y|\lambda). \tag{2.15}$$

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Proof. From (2.1), we consider the expression

$$\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^m \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\ldots+x_m}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1)\ldots d\mu_0(x_m) \tag{2.16}$$

$$=\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^m\left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$=\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^m(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_H\beta_n^{(m)}(x,y|\lambda)\frac{t^n}{n!}. \tag{2.17}$$

By (1.17), we define

$$\left(\frac{\lambda t}{\ln(1+\lambda t)}\right)^m=\sum_{n=0}^{\infty}b_n^{(m)}(\lambda t)^n, \tag{2.18}$$

which is the λ-analogue of higher order Bernoulli numbers $b_n^{(m)}$ of the second kind.

By (2.16) and (2.18), we get

$$=\left[\sum_{k=0}^{\infty}b_k^{(m)}\lambda^k\frac{t^k}{k!}\right]\left[\sum_{n=0}^{\infty}\lambda^n\int_{\mathbb{Z}_p^m}\left(\frac{x+x_1+\ldots+x_m}{\lambda}\right)_n(1+\lambda t^2)^{\frac{y}{\lambda}}\prod_{i=0}^{m}d\mu_0(x_i)\frac{t^n}{n!}\right] \tag{2.19}$$

$$=\left[\sum_{k=0}^{\infty}b_k^{(m)}\lambda^k\frac{t^k}{k!}\right]\left[\sum_{n=0}^{\infty}{}_HB_n^{(m)}(x,y|\lambda)\frac{t^n}{n!}\right]. \tag{2.20}$$

Now, comparing the coefficients of $\frac{t^n}{n!}$ in (2.17), (2.19) and (2.20), we obtain the required identities of the Theorem 2.8.

□

Remark 2.9. Replacing $m=1$ in (2.15), we obtain the following corollary [6(Theorem 2.8)].

Corollary 2.10. *For $n\geq 0$, we have*

$${}_H\beta_n(x,y|\lambda)=\sum_{k=0}^{n}\binom{n}{k}b_k\lambda^k{}_HB_{n-k}(x,y|\lambda).$$

Remark 2.11. Further replacement $y=0$ in (2.15), gives the following identity [16(Theorem 6)].

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Corollary 2.12. *Let $m \geq 1$ and $n \geq 0$, then*

$$\beta_n^{(m)}(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}^{(m)} \lambda^{n-k} \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (x_1 + ...x_m + x|\lambda)_k d\mu_0(x_1)...d\mu_0(x_m).$$

Theorem 2.13. *For $m \geq 1$ and $n \geq 0$, we state*

$$_HB_n^{(m)}(x + w, y|\lambda) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} \beta_{n-k}^{(m)}(x|\lambda) H_{k-j}(w, y; \lambda)$$
$$\times \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (x_1 + x_2 + ...x_m)_{j,\lambda} d\mu_0(x_1)...d\mu_0(x_m), \quad (2.21)$$

where $H_k(x, y; \lambda)$ are degenerate Hermite polynomials (cf. (1.18)).

Proof. In the integral identity (2.1), we replace x by $x + w$,

$$\sum_{n=0}^{\infty} {}_HB_n^{(m)}(x + w, y|\lambda)\frac{t^n}{n!}$$
$$= \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x_1+...+x_m+x+w}{\lambda}} (1 + \lambda t^2)^{\frac{y}{\lambda}} d\mu_0(x_1)...d\mu_0(x_m) \quad (2.22)$$
$$= \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}\right)^m (1 + \lambda t)^{\frac{x}{\lambda}} \left(\frac{\log(1 + \lambda t)}{\lambda t}\right)^m (1 + \lambda t)^{\frac{w}{\lambda}} (1 + \lambda t^2)^{\frac{y}{\lambda}}$$
$$= \left[\sum_{n=0}^{\infty} \beta_n^{(m)}(x|\lambda)\frac{t^n}{n!}\right] \int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x_1+...+x_m}{\lambda}} d\mu_0(x_1)...d\mu_0(x_m)$$
$$\times \left[\sum_{k=0}^{\infty} H_k(w, y; \lambda)\frac{t^k}{k!}\right],$$
$$= \left[\sum_{n=0}^{\infty} \beta_n^{(m)}(x|\lambda)\frac{t^n}{n!}\right] \left[\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} H_{k-j}(x, y; \lambda)\right.$$
$$\left.\times \int_{\mathbb{Z}_p^m} (x_1 + ... + x_m)_{j,\lambda} \, d\mu_0(x_1)...d\mu_0(x_m)\frac{t^k}{k!}\right]. \quad (2.23)$$

An easy comparison of coefficients in (2.22) and (2.23) completes the proof.

□

Remark 2.14. Giving appropriate values to the parameters and the variables in (2.1), we obtain certain known polynomials as special cases of degenerate Hermite Bernoulli polynomials of order m. A systematic account of these special cases are mentioned in Table 2.

S.No.	Values of the parameters and variables	Name of the resultant polynomials	p-adic integral and series representation of the special cases
i	m=1	Degenerate Hermite-Bernoulli polynomials [6]	$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}d\mu_0(x_1) = \sum\limits_{n=0}^{\infty} {}_HB_n(x,y\vert\lambda)\frac{t^n}{n!}$
ii	m=1, $y=0$	Degenerate Bernoulli polynomials [15, 16]	$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1}{\lambda}}d\mu_0(x_1) = \sum\limits_{n=0}^{\infty} B_n(x\vert\lambda)\frac{t^n}{n!}$
iii	$y=0$	Higher order Degenerate Bernoulli polynomials [14, 19]	$\int_{\mathbb{Z}_p{}^m}(1+\lambda t)^{\frac{x+x_1+...+x_m}{\lambda}}d\mu_0(x_1)...d\mu_0(x_m) = \sum\limits_{n=0}^{\infty} B_n^{(m)}(x\vert\lambda)\frac{t^n}{n!}$
iv	m=1, $y=0$, $\lambda\to 0$	General Bernoulli polynomials[9]	$\int_{\mathbb{Z}_p} e^{(x+x_1)t}d\mu_0(x_1) = \sum\limits_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$
v	$\lambda\to 0$, $y=0$	Higher order Bernoulli polynomials [18, 20]	$\int_{\mathbb{Z}_p{}^m} e^{(x+x_1+...+x_m)t}d\mu_0(x_1)...d\mu_0(x_m) = \sum\limits_{n=0}^{\infty} B_n^{(m)}(x)\frac{t^n}{n!}$
vi	m=1, $\lambda=1$, $y=0$	Daehee polynomials [17]	$\int_{\mathbb{Z}_p}(1+t)^{x+x_1}d\mu_0(x_1) = \sum\limits_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}$
vii	$\lambda=1$, $y=0$	Higher order Daehee polynomials [20]	$\int_{\mathbb{Z}_p{}^m}(1+t)^{(x+x_1+...+x_m)}d\mu_0(x_1)...d\mu_0(x_m) = \sum\limits_{n=0}^{\infty} D_n^{(m)}(x)\frac{t^n}{n!}$

TABLE 2

3. Some Symmetry Identities involving p-adic integrals for the degenerate Hermite-Bernoulli polynomials of order m

In this section, we consider the *quotient of integrals* $T(x)$ which on applying (1.4) can be expressed as the generating function for the integer power sums (cf. (3.4)). Further we establish symmetry identities involving these quotient of integrals and the p-adic integral expression of the generating function for the degenerate Hermite-Bernoulli polynomials of order m.

The *sum of integer powers* [22] $S_k(n), k\in\mathbb{N}_0$ is defined by

$$S_k(n)=\sum_{j=0}^{n}j^k=0^k+1^k+...+n^k, \tag{3.1}$$

where,

$$S_k(0)=1.$$

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We define

$$S_k(n|\lambda) = \sum_{j=0}^{n} (j|\lambda)_k, \text{(see [16])} \tag{3.2}$$

where $(x|\lambda)_n = x(x-\lambda)...(x-(n-1)\lambda)$ is the generalized falling factorial sequence for non-negative integer n.

We consider the quotient of integrals

$$T(x) = \frac{l \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{lx}{\lambda}} d\mu_0(x)} = \frac{\lambda}{\log(1+\lambda t)}$$

$$\times \left[\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+l}{\lambda}} d\mu_0(x) - \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x) \right]$$

$$= \sum_{k=0}^{l-1} (1+\lambda t)^{\frac{k}{\lambda}} = \sum_{s=0}^{\infty} \left[\sum_{k=0}^{l-1} \left(\frac{k}{\lambda}\right)_s \lambda^s \right] \frac{t^s}{s!}. \tag{3.3}$$

Thus, for $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbb{N}$, from (3.2) and (3.3) we obtain the identity

$$\frac{\mathfrak{h}_1 \mathfrak{h}_2 \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x}{\lambda}} d\mu_0(x)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{\mathfrak{h}_1 \mathfrak{h}_2}{\lambda} x} d\mu_0(x)} = \frac{(1+\lambda t)^{\frac{\mathfrak{h}_1 \mathfrak{h}_2}{\lambda}} - 1}{(1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}} - 1}$$

$$= \sum_{k=0}^{\infty} S_k \left(\mathfrak{h}_1 - 1 | \frac{\lambda}{\mathfrak{h}_2} \right) \frac{(\mathfrak{h}_2 t)^k}{k!}, \text{ (see [16])} \tag{3.4}$$

where $\lim_{\lambda \to 0} S_k(l|\lambda) = S_k(l)$,
and

$$\int_{\mathbb{Z}_p} ... \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}(x_1+...+x_r+\mathfrak{h}_2 x)} (1+\lambda t^2)^{\frac{\mathfrak{h}_1{}^2 \mathfrak{h}_2{}^2}{\lambda} y} d\mu_0(x_1)...d\mu_0(x_m)$$

$$= \left(\frac{\mathfrak{h}_1 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}} - 1} \right)^m (1+\lambda t)^{\frac{\mathfrak{h}_1 \mathfrak{h}_2}{\lambda} x} (1+\lambda t^2)^{\frac{\mathfrak{h}_1{}^2 \mathfrak{h}_2{}^2}{\lambda} y}$$

$$= \sum_{l=0}^{\infty} {}_H B^{(m)}_{l, \frac{\lambda}{\mathfrak{h}_1}} (\mathfrak{h}_2 x, \mathfrak{h}_2{}^2 y) \frac{(\mathfrak{h}_1 t)^l}{l!}, \tag{3.5}$$

where, $\lambda, t \in \mathbb{H}$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$.

Theorem 3.1. *For $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbb{N}$ and $m \geq 1$, the following symmetry identity holds:*

$$\sum_{s=0}^{l} \binom{l}{s} \mathfrak{h}_1{}^{l-s-1} \mathfrak{h}_2{}^s \sum_{j=0}^{s} \binom{s}{j} {}_H B^{(m)}_{l-s, \frac{\lambda}{\mathfrak{h}_1}} (\mathfrak{h}_2 x, \mathfrak{h}_2{}^2 \mathcal{Z})$$

$$\times S_j \left(\mathfrak{h}_1 - 1 | \frac{\lambda}{\mathfrak{h}_2} \right) B^{(m-1)}_{s-j, \frac{\lambda}{\mathfrak{h}_2}} (\mathfrak{h}_1 y)$$

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$$= \sum_{s=0}^{l} \binom{l}{s} \mathfrak{h}_1 \mathfrak{h}_2{}^{l-s-1} \mathfrak{h}_1{}^{s} \sum_{j=0}^{s} \binom{s}{j} {}_H B^{(m)}_{l-s,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 x, \mathfrak{h}_1{}^2 \mathcal{Z})$$
$$\times S_j\left(\mathfrak{h}_2 - 1 | \frac{\lambda}{\mathfrak{h}_1}\right) B^{(m-1)}_{s-j,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 y). \qquad (3.6)$$

Proof. We start with the integral identity

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2) = \frac{\int_{\mathbb{Z}_p^{(m)}} (1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}(x_1+...+x_m+\mathfrak{h}_2 x)} (1+\lambda t^2)^{\frac{\mathfrak{h}_1{}^2 \mathfrak{h}_2{}^2}{\lambda} \mathcal{Z}} d\mu_0(x_1)...d\mu_0(x_m)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{\mathfrak{h}_1 \mathfrak{h}_2}{\lambda} x} d\mu_0(x)}$$
$$\times \int_{\mathbb{Z}_p^{(m)}} (1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}(x_1+...+x_m+\mathfrak{h}_1 y)} d\mu_0(x_1)...d\mu_0(x_m), \qquad (3.7)$$

which upon using (1.4), gives

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda) = \left(\frac{\mathfrak{h}_1 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}} - 1}\right)^m \frac{(1+\lambda t)^{\frac{\mathfrak{h}_1 \mathfrak{h}_2}{\lambda}(x+y)} (1+\lambda t^2)^{\frac{\mathfrak{h}_1{}^2 \mathfrak{h}_2{}^2}{\lambda} \mathcal{Z}}}{\mathfrak{h}_1 \mathfrak{h}_2 \log(1+\lambda t)^{\frac{1}{\lambda}}}$$
$$\times \left(\frac{\mathfrak{h}_2 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}} - 1}\right)^m \left((1+\lambda t)^{\frac{\mathfrak{h}_1 \mathfrak{h}_2}{\lambda}} - 1\right).$$

It is clear that $I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda)$ is symmetric in $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

Now, using (3.4) and (3.5) together with the symmetry of $I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda)$, we get

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda) = \left[\sum_{l=0}^{\infty} {}_H B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 x, \mathfrak{h}_2{}^2 \mathcal{Z}) \frac{\mathfrak{h}_1{}^l t^l}{l!}\right] \left[\sum_{j=0}^{\infty} S_j\left(\mathfrak{h}_1 - 1 | \frac{\lambda}{\mathfrak{h}_2}\right) \frac{\mathfrak{h}_2{}^j t^j}{j!}\right]$$
$$\times \left[\sum_{s=0}^{\infty} B^{(m-1)}_{s,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 y) \frac{\mathfrak{h}_2{}^s t^s}{s!}\right] . \frac{1}{\mathfrak{h}_1}$$

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda) = \sum_{l=0}^{\infty} \left[\sum_{s=0}^{l} \binom{l}{s} \mathfrak{h}_1{}^{l-s-1} \mathfrak{h}_2{}^{s} \sum_{j=0}^{s} \binom{s}{j} {}_H B^{(m)}_{l-s,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 x, \mathfrak{h}_2{}^2 \mathcal{Z})\right.$$
$$\left. \times S_j\left(\mathfrak{h}_1 - 1 | \frac{\lambda}{\mathfrak{h}_2}\right) B^{(m-1)}_{s-j,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 y)\right] \frac{t^l}{l!}. \qquad (3.8)$$

Similarly, (3.7) can also be written as

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda) = \sum_{l=0}^{\infty} \left[\sum_{s=0}^{l} \binom{l}{s} \mathfrak{h}_2{}^{l-s-1} \mathfrak{h}_1{}^{s} \sum_{j=0}^{s} \binom{s}{j} {}_H B^{(m)}_{l-i,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 x, \mathfrak{h}_1{}^2 \mathcal{Z})\right.$$
$$\left. \times S_j\left(\mathfrak{h}_2 - 1 | \frac{\lambda}{\mathfrak{h}_1}\right) B^{(m-1)}_{s-j,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 y)\right] \frac{t^l}{l!}. \qquad (3.9)$$

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Comparing the coefficients of $\frac{t^l}{l!}$ in (3.8) and (3.9), we obtain the required symmetry identity.

□

Theorem 3.2. *For $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathbb{N}$ and $m \geq 1$, the following symmetry identity holds:*

$$\sum_{l=0}^{n} \binom{n}{l} {\mathfrak{h}_1}^{l-1} {\mathfrak{h}_2}^{n-l} B^{(m-1)}_{n-l,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 y) \sum_{i=0}^{\mathfrak{h}_1-1} {}_H B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(\mathfrak{h}_2 x + \frac{\mathfrak{h}_2}{\mathfrak{h}_1} i, {\mathfrak{h}_2}^2 \mathcal{Z}\right)$$

$$= \sum_{l=0}^{n} \binom{n}{l} {\mathfrak{h}_2}^{l-1} {\mathfrak{h}_1}^{n-l} B^{(m-1)}_{n-l,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 y) \sum_{i=0}^{\mathfrak{h}_2-1} {}_H B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_2}}\left(\mathfrak{h}_1 x + \frac{\mathfrak{h}_1}{\mathfrak{h}_2} i, {\mathfrak{h}_1}^2 \mathcal{Z}\right). \quad (3.10)$$

Proof. We begin with

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda) = \left(\frac{\mathfrak{h}_1 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}} - 1}\right)^m (1+\lambda t)^{\frac{\mathfrak{h}_1\mathfrak{h}_2}{\lambda}(x+y)} (1+\lambda t^2)^{\frac{{\mathfrak{h}_1}^2 {\mathfrak{h}_2}^2}{\lambda}\mathcal{Z}}$$

$$\times \frac{(1+\lambda t)^{\frac{\mathfrak{h}_1\mathfrak{h}_2}{\lambda}} - 1}{(1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}\mathcal{Z}} - 1} \left(\frac{\mathfrak{h}_2 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}} - 1}\right)^{m-1} \frac{1}{\mathfrak{h}_1}$$

$$= \frac{1}{\mathfrak{h}_1} \left(\frac{\mathfrak{h}_1 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}} - 1}\right)^m (1+\lambda t)^{\frac{\mathfrak{h}_1\mathfrak{h}_2}{\lambda}x} (1+\lambda t^2)^{\frac{{\mathfrak{h}_1}^2 {\mathfrak{h}_2}^2}{\lambda}\mathcal{Z}}$$

$$\times \left(\sum_{i=0}^{\mathfrak{h}_1-1} (1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda} i}\right) \left(\frac{\mathfrak{h}_2 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}} - 1}\right)^{m-1} (1+\lambda t)^{\frac{\mathfrak{h}_1\mathfrak{h}_2}{\lambda}y}$$

$$= \frac{1}{\mathfrak{h}_1} \left(\frac{\mathfrak{h}_1 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_1}{\lambda}} - 1}\right)^m \sum_{i=0}^{\mathfrak{h}_1-1} (1+\lambda t)^{\frac{\mathfrak{h}_1\mathfrak{h}_2}{\lambda}x + \frac{\mathfrak{h}_2}{\lambda} i} (1+\lambda t^2)^{\frac{{\mathfrak{h}_1}^2 {\mathfrak{h}_2}^2}{\lambda}\mathcal{Z}}$$

$$\times \left(\frac{\mathfrak{h}_2 \log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{\mathfrak{h}_2}{\lambda}} - 1}\right)^{m-1} (1+\lambda t)^{\frac{\mathfrak{h}_1\mathfrak{h}_2}{\lambda}y}$$

$$= \frac{1}{\mathfrak{h}_1} \left[\sum_{i=0}^{\mathfrak{h}_1-1} \sum_{l=0}^{\infty} {}_H B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(\mathfrak{h}_2 x + \frac{\mathfrak{h}_2}{\mathfrak{h}_1} i, {\mathfrak{h}_2}^2 \mathcal{Z}\right) {\mathfrak{h}_1}^l \frac{t^l}{l!}\right]$$

$$\times \left[\sum_{n=0}^{\infty} B^{(m-1)}_{n,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 y) {\mathfrak{h}_2}^n \frac{t^n}{n!}\right]$$

$$I^{(m)}(\mathfrak{h}_1, \mathfrak{h}_2|\lambda) = \sum_{n=0}^{\infty} \left[\sum_{l=0}^{n} \binom{n}{l} {\mathfrak{h}_1}^{l-1} {\mathfrak{h}_2}^{n-l} B^{(m-1)}_{n-l,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 y)\right.$$

$$\left.\times \sum_{i=0}^{\mathfrak{h}_1-1} {}_H B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(\mathfrak{h}_2 x + \frac{\mathfrak{h}_2}{\mathfrak{h}_1} i, {\mathfrak{h}_2}^2 \mathcal{Z}\right)\right] \frac{t^n}{n!}. \quad (3.11)$$

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Likewise, we get

$$I^{(m)}(\mathfrak{h}_1,\mathfrak{h}_2|\lambda)=\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}{\mathfrak{h}_2}^{l-1}{\mathfrak{h}_1}^{n-l}B^{(m-1)}_{n-l,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 y)\right.$$
$$\left.\times\sum_{i=0}^{\mathfrak{h}_2-1}{}_HB^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_2}}\left(\mathfrak{h}_1x+\frac{\mathfrak{h}_1}{\mathfrak{h}_2}i,{\mathfrak{h}_1}^2\mathcal{Z}\right)\right]\frac{t^n}{n!}. \tag{3.12}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (3.11) and (3.12), we obtain the required symmetry identity.

□

Remark 3.3. On putting $\mathcal{Z}=0$ in (3.6) and (3.10), the following results involving degenerate Bernoulli polynomials are deduced (see [16]).

Corollary 3.4. *For $\mathfrak{h}_1,\mathfrak{h}_2\in\mathbb{N}$, $n\geq 0$ and $m\geq 1$, one has*

$$\sum_{j=0}^{n}\binom{n}{j}{\mathfrak{h}_1}^{n-j-1}{\mathfrak{h}_2}^{j}\sum_{k=0}^{j}\binom{j}{k}B^{(m)}_{n-j,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2x)S_k\left(\mathfrak{h}_1-1|\frac{\lambda}{\mathfrak{h}_2}\right)B^{(m-1)}_{j-k,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1y)$$
$$=\sum_{j=0}^{n}\binom{n}{j}{\mathfrak{h}_2}^{n-j-1}{\mathfrak{h}_1}^{j}\sum_{k=0}^{j}\binom{j}{k}B^{(m)}_{n-j,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1x)S_k\left(\mathfrak{h}_2-1|\frac{\lambda}{\mathfrak{h}_1}\right)B^{(m-1)}_{j-k,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2y). \tag{3.13}$$

and

$$\sum_{l=0}^{n}\binom{n}{l}{\mathfrak{h}_1}^{l-1}{\mathfrak{h}_2}^{n-l}B^{(m-1)}_{n-l,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1y)\sum_{i=0}^{\mathfrak{h}_1-1}B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(\mathfrak{h}_2x+\frac{\mathfrak{h}_2}{\mathfrak{h}_1}i\right)$$
$$=\sum_{l=0}^{n}\binom{n}{l}{\mathfrak{h}_2}^{l-1}{\mathfrak{h}_1}^{n-l}B^{(m-1)}_{n-l,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2y)\sum_{i=0}^{\mathfrak{h}_2-1}B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_2}}\left(\mathfrak{h}_1x+\frac{\mathfrak{h}_1}{\mathfrak{h}_2}i\right). \tag{3.14}$$

Remark 3.5. On taking $m=1$, in (3.6) and (3.10), we have the following corollary.

Corollary 3.6. *For $\mathfrak{h}_1,\mathfrak{h}_2\in\mathbb{N}$, we have*

$$\sum_{j=0}^{n}\binom{n}{j}{\mathfrak{h}_1}^{n-j-1}{\mathfrak{h}_2}^{j}\sum_{k=0}^{j}\binom{j}{k}{}_HB_{n-j,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2x,{\mathfrak{h}_2}^2\mathcal{Z})$$
$$\times S_k\left(\mathfrak{h}_1-1|\frac{\lambda}{\mathfrak{h}_2}\right)B_{j-k,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1y)$$
$$=\sum_{j=0}^{n}\binom{n}{j}{\mathfrak{h}_2}^{n-j-1}{\mathfrak{h}_1}^{j}\sum_{k=0}^{j}\binom{j}{k}{}_HB_{n-j,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1x,{\mathfrak{h}_1}^2\mathcal{Z})$$
$$\times S_k\left(\mathfrak{h}_2-1|\frac{\lambda}{\mathfrak{h}_1}\right)B_{j-k,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2y). \tag{3.15}$$

and

$$\sum_{l=0}^{n}\binom{n}{l}\mathfrak{h}_1{}^{l-1}\mathfrak{h}_2{}^{n-l}B_{n-l,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 y)\sum_{i=0}^{\mathfrak{h}_1-1}{}_HB_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(\mathfrak{h}_2 x+\frac{\mathfrak{h}_2}{\mathfrak{h}_1}i,\mathfrak{h}_2{}^2\mathcal{Z}\right)$$

$$=\sum_{l=0}^{n}\binom{n}{l}\mathfrak{h}_2{}^{l-1}\mathfrak{h}_1{}^{n-l}B_{n-l,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 y)\sum_{i=0}^{\mathfrak{h}_2-1}{}_HB_{l,\frac{\lambda}{\mathfrak{h}_2}}\left(\mathfrak{h}_1 x+\frac{\mathfrak{h}_1}{\mathfrak{h}_2}i,\mathfrak{h}_1{}^2\mathcal{Z}\right). \quad (3.16)$$

For the proof of these identities refer to Haroon and Khan [6].

Remark 3.7. Further, on taking $y=0$ in (3.13) and (3.14), we get the following pair of identities:

Corollary 3.8. *For $\mathfrak{h}_1,\mathfrak{h}_2\in\mathbb{N}$, we have*

$$\sum_{j=0}^{n}\binom{n}{j}\mathfrak{h}_1{}^{n-j-1}\mathfrak{h}_2{}^{j}B^{(m)}_{n-j,\frac{\lambda}{\mathfrak{h}_1}}(\mathfrak{h}_2 x)S_k\left(\mathfrak{h}_1-1|\frac{\lambda}{\mathfrak{h}_2}\right)$$

$$=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{h}_2{}^{n-j-1}\mathfrak{h}_1{}^{j}B^{(m)}_{n-j,\frac{\lambda}{\mathfrak{h}_2}}(\mathfrak{h}_1 x)S_k\left(\mathfrak{h}_2-1|\frac{\lambda}{\mathfrak{h}_1}\right). \quad (3.17)$$

and

$$\mathfrak{h}_1{}^{l-1}\sum_{i=0}^{\mathfrak{h}_1-1}B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(\mathfrak{h}_2 x+\frac{\mathfrak{h}_2}{\mathfrak{h}_1}i\right)=\mathfrak{h}_2{}^{l-1}\sum_{i=0}^{\mathfrak{h}_2-1}B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_2}}\left(\mathfrak{h}_1 x+\frac{\mathfrak{h}_1}{\mathfrak{h}_2}i\right). \quad (3.18)$$

Remark 3.9. Moreover, if we take $\mathfrak{h}_2=1$ in the resultant identities (3.17) and (3.18), then we get the following equalities:

Corollary 3.10. *We have, for $\mathfrak{h}_1\in\mathbb{N}$*

$$B^{(m)}_{n,\lambda}(\mathfrak{h}_1 x)=\sum_{j=0}^{n}\binom{n}{j}\mathfrak{h}_1{}^{n-j-1}B^{(m)}_{n-j,\frac{\lambda}{\mathfrak{h}_1}}(x)S_j(\mathfrak{h}_1-1|\lambda). \quad (3.19)$$

and

$$\sum_{i=0}^{\mathfrak{h}_1-1}B^{(m)}_{l,\frac{\lambda}{\mathfrak{h}_1}}\left(x+\frac{1}{\mathfrak{h}_1}i\right)=\mathfrak{h}_1{}^{1-l}B^{(m)}_{l,\lambda}(\mathfrak{h}_1 x). \quad (3.20)$$

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Concluding Remarks

Several results in the above sections have corroborated the application of p-adic invariant integrals. Furthermore, quite a number of results for higher order special polynomials can be investigated through the aid of this method which can in turn be extended to derive novel relations for the generalized and conventional polynomials.

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