

## On a New Reverse Hilbert's Type Inequality

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**ABSTRACT.** In this paper, by using the Euler-Maclaurin expansion for the Riemann- $\zeta$  function, we establish an inequality of a weight coefficient. Using this inequality, we derive a new reverse Hilbert's type inequality. As an applications, an equivalent form is obtained.

**Keywords:** Hilbert's type inequality, Weight coefficient, Hölder inequality, Riemann- $\zeta$  function, Reverse.

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### 1. INTRODUCTION

If  $p, q > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $a_n \geq 0$ ,  $b_n \geq 0$ , for  $n \geq 1$ ,  $n \in N$  and  $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ ,  
 $0 < \sum_{n=1}^{\infty} b_n^q < \infty$ , then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}, \quad (1.1)$$

and

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < pq \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}}, \quad (1.2)$$

where the constant  $\frac{\pi}{\sin \frac{\pi}{p}}$  and  $pq$  is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [5], [10] and [9], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [3], Kuang and Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < \left\{ \sum_{n=1}^{\infty} [pq - G(p, n)] a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} [pq - G(q, n)] b_n^q \right\}^{\frac{1}{q}} \quad (1.3)$$

where  $G(r, n) = \frac{r + \frac{1}{3r} - \frac{4}{3}}{(2n+1)^{\frac{1}{r}}} > 0$  ( $r = p, q$ ).

In [6] and [7], Xi gave a generalization and reinforcement of inequalities (1.2) and (1.3):

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \left\{ \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{1}{3qn^{\frac{q+\lambda-2}{q}}} \right] n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \\ \times \left\{ \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{1}{3pn^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \end{aligned} \quad (1.4)$$

where  $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$ ,  $2 - \min\{p, q\} < \lambda \leq 2$ .

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda + A, n^\lambda + B\}} < \left\{ \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{1}{n^{\frac{q+\lambda-2}{q}}} \left( \frac{1}{3q} - \frac{B}{1+B} \right) \right] \right. \\ \left. \times n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{1}{n^{\frac{p+\lambda-2}{p}}} \left( \frac{1}{3p} - \frac{A}{1+A} \right) \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \end{aligned} \quad (1.5)$$

For the reverse Hilbert's type inequality, In [8], Xi and Wang gave a reverse Hilbert's type inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^2, n^2\}} > 2 \left[ \sum_{n=1}^{\infty} \left( 1 - \frac{1}{2n} \right) \frac{1}{n} a_n^p \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \frac{1}{n} b_n^q \right]^{\frac{1}{q}}. \quad (1.6)$$

In this paper, by introducing a parameter  $\lambda$  and using the Euler-Maclaurin expansion for the Riemann- $\zeta$  function, we establish an inequality of a weight coefficient. Using this inequality, we derive a reverse of the Hilbert's type inequality (1.4).

## 2. A LEMMA

First, we need the following formula of the Riemann- $\zeta$  function (see [4], [12] and [11]):

$$\begin{aligned} \zeta(\sigma) = & \sum_{k=1}^n \frac{1}{k^\sigma} - \frac{n^{1-\sigma}}{1-\sigma} - \frac{1}{2n^\sigma} - \sum_{k=1}^{l-1} \frac{B_{2k}}{2k} \binom{-\sigma}{2k-1} \frac{1}{n^{\sigma+2k-1}} \\ & - \frac{B_{2l}}{2l} \binom{-\sigma}{2l-1} \frac{\varepsilon}{n^{\sigma+2l-1}}, \end{aligned} \tag{2.1}$$

where  $\sigma > 0, \sigma \neq 1, n, l \geq 1, n, l \in N, 0 < \varepsilon = \varepsilon(\sigma, l, n) < 1$ . The numbers  $B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \dots$  are Bernoulli numbers. In particular,  $\zeta(\sigma) = \sum_{k=1}^\infty \frac{1}{k^\sigma} (\sigma > 1)$ .

Since  $\zeta(0) = -1/2$ , then the formula of the Riemann- $\zeta$  function (2.1) is also true for  $\sigma = 0$ .

**Lemma 2.1.** *If  $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - p < \lambda \leq 2, n \geq 1$  and  $n \in N$ , then*

$$\omega(n, \lambda, q) = \sum_{k=1}^\infty \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} > \frac{qn^{1-\lambda}}{q + \lambda - 2}, \tag{2.2}$$

and

$$\begin{aligned} \omega(n, \lambda, p) = & \sum_{k=1}^\infty \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ < & n^{1-\lambda} \left[ \kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}} \right], \end{aligned} \tag{2.3}$$

where  $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$ .

*Proof.* Equalities (2.2) and (2.3) define the weight coefficient. When  $2 - p < \lambda \leq 2$ , taking  $\sigma = \frac{2-\lambda}{p} \geq 0, l = 1$ , in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^n \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}} \varepsilon_1, \tag{2.4}$$

where  $0 < \varepsilon_1 < 1$ .

Since  $\frac{2}{q} + \frac{\lambda}{p} = 2 + \frac{\lambda-2}{p} = \frac{2p+\lambda-2}{p} > 0 (p+\lambda-2 > 0)$ . Taking  $\sigma = \frac{\lambda}{p} + \frac{2}{q}, l = 1$ , we obtain

$$\zeta\left(\frac{2}{q} + \frac{\lambda}{p}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{q} + \frac{\lambda}{p}}} + \frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q} + \frac{\lambda}{p}}} + \frac{q\lambda + 2p}{12pqn^{1+\frac{2}{q} + \frac{\lambda}{p}}} \varepsilon_2, \tag{2.5}$$

where  $0 < \varepsilon_2 < 1$ .

Since  $\frac{2}{p} + \frac{\lambda}{q} = 2 + \frac{\lambda-2}{q} = \frac{2q+\lambda-2}{q} > 0 (q+\lambda-2 < 0, q < 0)$ . Taking  $\sigma = \frac{2}{p} + \frac{\lambda}{q}, l = 1$ , we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p} + \frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p} + \frac{\lambda}{q}}} + \frac{p\lambda + 2q}{12pqn^{1+\frac{2}{p} + \frac{\lambda}{q}}} \varepsilon_3, \tag{2.6}$$

where  $0 < \varepsilon_3 < 1$ .

In addition,

$$\begin{aligned}\omega(n, \lambda, q) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &= \sum_{k=1}^n \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &= \sum_{k=1}^n \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &= \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^n \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p} + \frac{2}{q}}} \\ &> \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p} + \frac{2}{q}}}.\end{aligned}$$

By (2.5) and  $\frac{2}{q} + \frac{\lambda}{p} = \frac{q\lambda+2p}{pq} > 0$

$$\begin{aligned}\omega(n, \lambda, q) &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[ \frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q} + \frac{\lambda}{p}}} + \frac{q\lambda+2p}{12pqn^{1+\frac{2}{q} + \frac{\lambda}{p}}} \varepsilon_2 \right] \\ &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[ \frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q} + \frac{\lambda}{p}}} \right] \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + \frac{qn^{1-\lambda}}{q+\lambda-2} + \frac{1}{2n^\lambda} \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{2n^\lambda} + \frac{qn^{1-\lambda}}{q+\lambda-2} \\ &> \frac{qn^{1-\lambda}}{q+\lambda-2}.\end{aligned}$$

Using the last result and the inequality for  $\omega(n, \lambda, q)$  above, we obtain (2.2).

$$\begin{aligned}\omega(n, \lambda, p) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \sum_{k=1}^n \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \sum_{k=1}^n \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^n \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p} + \frac{\lambda}{q}}}.\end{aligned}$$

By (2.4) and (2.6)

$$\begin{aligned}
 \omega(n, \lambda, p) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \left[ \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2-\lambda}{p}}} \right] - \frac{1}{n^\lambda} \\
 &+ n^{\frac{2-\lambda}{p}} \left[ \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}} \right] \\
 &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pn^{1-\lambda}}{p+\lambda-2} + \frac{1}{2n^\lambda} - \frac{1}{n^\lambda} + \frac{qn^{1-\lambda}}{q+\lambda-2} \\
 &+ \frac{1}{2n^\lambda} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} \\
 &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \zeta\left(\frac{2-\lambda}{p}\right) + \frac{pq\lambda n^{1-\lambda}}{(p+\lambda-2)(q+\lambda-2)} + \frac{p\lambda+2q}{12pqn^{1+\lambda}} \\
 &= n^{1-\lambda} \left\{ \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)} - \frac{1}{n^{\frac{p+\lambda-2}{p}}} \left[ -\zeta\left(\frac{2-\lambda}{p}\right) \right. \right. \\
 &\quad \left. \left. - \frac{p\lambda+2q}{12pqn^{\frac{p-\lambda+2}{p}}} \right] \right\}.
 \end{aligned}$$

In (2.4), taking  $n = 1$ , by  $2 - p < \lambda \leq 2$ , we obtain

$$\begin{aligned}
 \zeta\left(\frac{2-\lambda}{p}\right) &= 1 - \frac{p}{p+\lambda-2} - \frac{1}{2} + \frac{(2-\lambda)\varepsilon_1}{12p} \\
 &< \frac{1}{2} - \frac{p}{p+\lambda-2} + \frac{2-\lambda}{12p} \\
 &= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} \\
 &< 0.
 \end{aligned}$$

So for  $n \geq 1$ ,  $n \in N$ ,  $2 - p < \lambda \leq 2$ , we have

$$\begin{aligned}
 &-\zeta\left(\frac{2-\lambda}{p}\right) + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}} \\
 &\leq -\zeta\left(\frac{2-\lambda}{p}\right) + \frac{2-\lambda}{12p} \\
 &= \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)} + \frac{2-\lambda}{12p}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(\lambda - 2 - 3p)(\lambda - 2 - 2p) + (2 - \lambda)(p + \lambda - 2)}{12p(p + \lambda - 2)} \\
&= \frac{(\lambda - 2)^2 + (-5p - p - \lambda + 2)(\lambda - 2) + 6p^2}{12p(p + \lambda - 2)} \\
&= \frac{6p(2 - \lambda) + 6p^2}{12p(p + \lambda - 2)} \\
&= \frac{(2 - \lambda) + p}{2(p + \lambda - 2)} \\
&\leq \frac{2 + p}{2(p + \lambda - 2)}.
\end{aligned}$$

Using the last result and the inequality for  $\omega(n, \lambda, p)$  above, we obtain (2.3).  $\square$

### 3. MAIN RESULTS

**Theorem 3.1.** *If  $0 < p < 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $2 - p < \lambda \leq 2$ ,  $a_n \geq 0$ ,  $b_n \geq 0$ , for  $n \geq 1, n \in N$  and  $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ ,  $0 < \sum_{n=1}^{\infty} b_n^q < \infty$ , then*

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &> \left\{ \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \quad (3.1)
\end{aligned}$$

where  $\kappa(\lambda) = \frac{p q \lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$ .

*Proof.* By the reverse Hölder's inequality [2], we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \frac{a_m}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{p}}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}} \right] \\
&\times \left[ \frac{b_n}{\max\{m^\lambda, n^\lambda\}^{\frac{1}{q}}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}} \right] \\
&\geq \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \frac{a_m^p}{\max\{m^\lambda, n^\lambda\}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}} \right] \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \frac{b_n^q}{\max\{m^\lambda, n^\lambda\}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}} \right] \right\}^{\frac{1}{q}} \\
&= \left\{ \sum_{m=1}^{\infty} \omega(m, \lambda, q) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega(n, \lambda, p) b_n^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

Since  $0 < p < 1$  and  $q < 0$ . By (2.2), (2.3), we obtain (3.1). Theorem 3.1 is proved.  $\square$

**Theorem 3.2.** *If  $0 < p < 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $2 - p < \lambda \leq 2$ ,  $a_n \geq 0$ ,  $b_n \geq 0$ , for  $n \geq 1, n \in N$  and  $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty$ , then*

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \left( \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \\ > \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{p + 2}{2(p + \lambda - 2)n^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q. \end{aligned} \tag{3.2}$$

where  $\kappa(\lambda) = \frac{p q \lambda}{(p+\lambda-2)(q+\lambda-2)} > 0$ .

Inequalities (3.2) and (3.1) are equivalent.

*Proof.* Let

$$a_n = \left( \frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \left[ \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right]^{q-1}, \quad n \in N.$$

By (3.1), we have

$$\begin{aligned} \left\{ \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p \right\}^q &= \left\{ \sum_{n=1}^{\infty} \left( \frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \right. \\ &\quad \times \left. \left( \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \right\}^q \\ &= \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} \right\}^q \\ &> \left\{ \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p \right\}^{q-1} \left\{ \sum_{n=1}^{\infty} [\kappa(\lambda) \right. \\ &\quad \left. - \frac{p + 2}{2(p + \lambda - 2)n^{\frac{p+\lambda-2}{p}}} n^{1-\lambda} b_n^q] \right\}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q}{q + \lambda - 2} n^{1-\lambda} a_n^p &= \sum_{n=1}^{\infty} \left( \frac{q}{q + \lambda - 2} n^{1-\lambda} \right)^{1-q} \left( \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \\ &> \sum_{n=1}^{\infty} \left[ \kappa(\lambda) - \frac{p + 2}{2(p + \lambda - 2)n^{\frac{p+\lambda-2}{p}}} \right] n^{1-\lambda} b_n^q. \end{aligned}$$

On the other-hand, by the reverse Hölder 's inequality [2], we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} &= \sum_{n=1}^{\infty} \left[ \left( \frac{q}{q+\lambda-2} n^{1-\lambda} \right)^{-\frac{1}{p}} \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right] \\ &\quad \times \left[ \left( \frac{q}{q+\lambda-2} n^{1-\lambda} \right)^{\frac{1}{p}} a_n \right] \\ &\geq \left[ \sum_{n=1}^{\infty} \left( \frac{q}{q+\lambda-2} n^{1-\lambda} \right)^{1-q} \left( \sum_{m=1}^{\infty} \frac{b_m}{\max\{m^\lambda, n^\lambda\}} \right)^q \right]^{\frac{1}{q}} \\ &\quad \times \sum_{n=1}^{\infty} \left[ \frac{q}{q+\lambda-2} n^{1-\lambda} a_n^p \right]^{\frac{1}{p}}. \end{aligned}$$

From (3.2), it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{\max\{m^\lambda, n^\lambda\}} &> \sum_{n=1}^{\infty} \left[ \frac{q}{q+\lambda-2} n^{1-\lambda} a_n^p \right]^{\frac{1}{p}} \\ &\quad \times \left[ \sum_{n=1}^{\infty} \left( \kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}} \right) n^{1-\lambda} b_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Then, (3.2) and (3.1) are equivalent. Theorem 3.2 is proved.  $\square$

In inequality (3.1), taking  $\lambda = 2$ , we obtain:

**Corollary 3.3.** *If  $0 < p < 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $a_n \geq 0$ ,  $b_n \geq 0$ , for  $n \geq 1, n \in N$  and  $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ ,  $0 < \sum_{n=1}^{\infty} b_n^q < \infty$ , then*

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^2, n^2\}} &> \left\{ \sum_{n=1}^{\infty} \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \left[ 1 - \frac{p+2}{2pn} \right] \frac{1}{n} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (3.3)$$

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