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Graded Prime Ideals Attached to a Group Graded Module

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Abstract. Let G be a finitely generated abelian group and M be a G-graded A-module. In general, G-associated prime ideals to M may not exist. In this paper, we introduce the concept of G-attached prime ideals to M as a generalization of G-associated prime ideals which gives a connection between certain G-prime ideals and G-graded modules over a (not necessarily G-graded Noetherian) ring. We prove that the G-attached prime ideals exist for every nonzero G-graded module and this generalization is proper. We transfer many results of G-associated prime ideals to G-attached prime ideals and give some applications of it.

Keywords: G-graded module, G-associated prime ideal, G-attached prime ideal, Weak G-attached prime ideal.

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1. Introduction

The theory of associated prime ideals plays an important role in the study of modules over Noetherian rings. However, associated prime ideals may not exist in the case of modules over non-Noetherian rings. For example, let $A = F^{\mathbb{N}}$ (Direct product), where F is a field and consider the ideal $I = F^{(\mathbb{N})}$ (Direct sum) of A. Then the A-module A/I has no associated prime ideal. In [4], P. Dutton introduced the notion of attached prime ideals to a module over a (not necessarily Noetherian) ring as a generalization of associated prime ideals. A prime ideal P of a commutative ring A is said to be attached prime to an A-module M if every finitely generated ideal of A_P which is contained in PA_P annihilates a nonzero element of M_P, [4]. He showed that the set $Att_A(M)$ of attached prime ideals to M is the best choice for a notion of associated prime ideals over a (not necessarily Noetherian) ring A by proving many useful properties of modules such as zero-divisors on M and radical of $Ann(M)$ in terms of attached prime ideals, which were enjoyed by associated prime ideals only under Noetherian conditions. In [6], J. Iroz and D. E. Rush gave several other notions of associated prime ideals to a module over a (not necessarily Noetherian) ring. Results of [4], [6], [9] and [10] show that the attached prime ideals are useful alternatives for associated prime ideals in the absence of Noetherian conditions. But it seems that the graded analogous of this concept is not found anywhere. This paper tries to fill this gap by systematically developing the theory of G-attached and weak G-attached prime ideals as a generalization of attached prime ideals to the graded case for an arbitrary finitely generated abelian group G. One of the main reasons to discuss G-attached and weak G-attached prime ideals is that for a G-graded non-Noetherian modules, the G-graded primary decomposition may not exist. Consequently, a lot of properties of G-graded non-Noetherian rings and modules which depend on G-graded primary decomposition are left. We study the properties of G-graded non-Noetherian modules more closely by using G-attached and weak G-attached prime ideals. Recent works in this direction (i.e., concepts of G-associated prime ideals and graded primary decomposition) can be found in [1], [2], [7] and [11].

In this paper, we start with an example of a G-graded module for which G-associated prime ideal does not exist. For a G-graded A-module M, we define G-attached and weak G-attached prime ideals to M and prove the existence of both. We give a number of examples to distinguish the graded and non-graded cases of attached prime ideals. We demonstrate that this concept is nothing but a generalization of G-associated prime ideals (Theorem 3.12). We denote the set of all G-attached and weak G-attached prime ideals to M by $Att^G_A(M)$ and W-$Att^G_A(M)$, respectively. We prove some important properties of a graded module M such as graded zero-divisors on M and graded radical of

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$Ann(M)$ in terms of G-attached and weak G-attached prime ideals. The two theories of G-associated prime ideals and G-graded primary decomposition are closely related under graded Noetherian conditions, for example, the set of all G-prime ideals associated to a reduced G-graded primary decomposition of the zero submodule of a G-graded A-module M is the same as the set of all G-associated prime ideals to M, [7, Theorem 2.4]. One of the main aim of this paper is to prove Theorem 4.7 which gives a connection between $Att^G_A(M)$ and the set of all G-prime ideals associated to a reduced G-graded primary decomposition of the zero submodule of a G-graded A-module M, which is a generalization of [7, Theorem 2.4]. Further, it is well known that the set of all G-associated prime ideals to a finitely generated G-graded Noetherian A-module M is finite. We show by an example that the set $Att^G_A(M)$ may be infinite (Example 5.1), and then we characterize those graded modules for which the set $Att^G_A(M)$ is finite (Theorem 5.6).

In this paper, section 3, 4 and 5 contain our results.

2. Preliminaries

Let G be a finitely generated abelian group. If not stated otherwise, all rings are assumed to be commutative rings with identity.

A ring A is said to be G-graded if $A = \bigoplus_{g\in G} A_g$ for additive subgroups A_g and $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. The grading is trivial if $A_g = 0$ for every non-identity $g \in G$. The elements of the set $h(A) = \bigcup_{g\in G} A_g$ are said to be homogeneous. An ideal I of A is said to be graded if $I = \bigoplus_{g\in G}(I \cap A_g)$. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring and M an A-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ for additive subgroups M_g and $A_gM_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of $h(M) = \bigcup_{g\in G} M_g$ are said to be homogeneous. Let N be a submodule of a G-graded A-module M. Then N is said to be graded if $N = \bigoplus_{g\in G}(N \cap M_g)$. Moreover, M/N becomes a G-graded A-module with $(M/N)_g = (M_g + N)/N$ for all $g \in G$. Let M and M' be G-graded A-modules. Then an A-module homomorphism $f : M \longrightarrow M'$ is said to be G-graded if $f(M_g) \subseteq M'_g$ for every $g \in G$.

Let A be a G-graded ring and M be a G-graded A-module. A proper graded ideal P of A is said to be G-prime if for $a, b \in h(A)$, $ab \in P$ implies either $a \in P$ or $b \in P$. A proper graded ideal of A is said to be G-maximal if it is not contained properly in any proper graded ideal of A. The set of all G-prime ideals of A is denoted by $Spec^G(A)$. A is called a G-graded integral domain if for $a, b \in h(A)$, $ab = 0$ implies that either $a = 0$ or $b = 0$. A is called a G-graded field if each nonzero homogeneous element of A has a multiplicative inverse. A is called a G-graded Noetherian ring if it satisfies the ACC on the graded ideals of A. An ideal generated by the set of all nilpotent homogeneous elements of A is called G-graded nil-radical of A and is denoted by $N^G(A)$.

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Let S be a multiplicatively closed subset of $h(A)$ containing 1_A. Then the ring of fraction $S^{-1}A$ is a graded ring called a G-graded ring of fraction. Indeed $S^{-1}A = \bigoplus_{g\in G}(S^{-1}A)_g$, where $(S^{-1}A)_g = \{\frac{a}{s} : a \in h(A), s \in S$ and $g = (degs)^{-1}(dega)\}$. Let P be a G-prime ideal of A and $S = h(A)\setminus P$. We denote $S^{-1}A$ by A_P^G and call it the G-graded localization of A at P. This ring A_P^G has a unique G-maximal ideal $S^{-1}P$ which is denoted by PA_P^G. Also, $S^{-1}M$ is denoted by M_P^G called the G-graded localization of M at P. We denote by $h(A_P^G)$, $h(M_P^G)$, $Ann(M)$, $Ann_{A_P^G}(M_P^G)$, the set of all homogeneous elements of A_P^G, M_P^G, annihilator of M in A, annihilator of M_P^G in A_P^G respectively. Let I be a graded ideal of A. Then the graded radical of I is denoted by $Gr(I)$ and defined as $Gr(I) = \{a = \sum_{g\in G} a_g \in A :$ for every $g \in G$, there exists an integer $n_g \geq 1$ such that $a_g^{n_g} \in I$ $\}$.

For more definitions and properties of graded rings and graded modules one can see [5] and [8] .

Definition 2.1. [7] Let M be a G-graded A-module and N be a graded submodule of M. Then

(1) The graded radical of N is denoted by $Gr(N)$ and defined as the ideal of A generated by the set $\{a \in h(A) : a^nM \subseteq N$ for some positive integer $n\}$.
(2) $N \neq M$ is called a G-graded primary submodule of M if for each $a \in h(A)$ and $x \in h(M)$, $ax \in N$ implies that either $x \in N$ or $a \in Gr(N)$.

For a G-graded primary submodule N of M, $Gr(N) = P$ where P is a G-prime ideal of A and we call N a G-graded P-primary submodule, [7].

Definition 2.2. [7] Let M be a G-graded A-module and N be its graded submodule. Then a decomposition of the type $N = N_1 \cap N_2 \cap \cdots \cap N_r$, where each N_i is a G-graded P_i-primary submodule, is called a G-graded primary decomposition of N in M.

A G-graded primary decomposition of N is said to be reduced if

(1) $N_i \not\supseteq \bigcap_{j\neq i} N_j$, for all $i = 1, 2, \cdots, r$.
(2) N_i are G-graded P_i-primary with all the P_i distinct.

We can always obtain a reduced G-graded primary decomposition from a G-graded primary decomposition.

Definition 2.3. [11] Let M be a G-graded A-module. Then

(1) A graded submodule N of M is said to be G-decomposable if it has a G-graded primary decomposition in M.
(2) A G-prime ideal P of A is said to be G-associated prime to M if $P = Ann(x)$ for some nonzero $x \in h(M)$ where $Ann(x)$ denotes the annihilator of x. We denote the set of all G-associated prime ideals to M by $Ass_A^G(M)$.

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(3) The graded support of M is denoted by $Supp_A^G(M)$ and defined as $Supp_A^G(M) = \{P \in Spec^G(A) : M_P^G \neq 0\}$.

Theorem 2.4. [7, Theorem 2.4] *Let A be a G-graded Noetherian ring and M be a finitely generated G-graded A-module. Suppose $0 = N_1 \cap N_2 \cap \cdots \cap N_r$ is a reduced G-graded primary decomposition of the zero submodule in M, where N_i is G-graded P_i-primary. Then $Ass_A^G(M) = \{P_1, P_2, \ldots, P_r\}$. In particular, $Ass_A^G(M)$ is a finite set.*

Theorem 2.5. [2, Corollary 4.3] *Let M and N be G-graded A-modules and $f : M \to M'$ be a G-graded A-module homomorphism. Then the following are equivalent.*

(1) *f is surjective (injective).*
(2) *$f_P : M_P^G \to N_P^G$ is surjective (injective), for every $P \in Spec^G(A)$.*
(3) *$f_m : M_m^G \to N_m^G$ is surjective (injective), for every G-maximal ideals m of A.*

3. Properties of G-attached prime ideals

Let M be a G-graded A-module. It is well known that if A is G-graded Noetherian, then $Ass_A^G(M) \neq \emptyset$ if and only if $M \neq \{0\}$. But in general, G-associated prime ideals may not exist. For example, consider the trivially G-graded ring $A = \prod_{i\in\mathbb{N}} F_i$ where each $F_i = F$ is a field and $\mathbb{N}$ denotes the set of all positive integers. Consider the ideal $I = \bigoplus_{i\in\mathbb{N}} F_i$ of A. Let $M = A/I$ as an A-module. We claim that $Ass_A^G(M) = \emptyset$. Contrary suppose P $\in Ass_A^G(M)$. Then $P = Ann(x + I)$ for some nonzero $x + I \in M$. Clearly $x \notin I$ and we can find two elements $y, z \in A$ such that $xy, xz \notin I$ but $yz = 0$. Consequently, $yz \in P$ but $y, z \notin P$, absurd. Hence $Ass_A^G(M) = \emptyset$.

Now we define the notions of G-attached and weak G-attached prime ideals. Our definition of G-attached prime ideals is the graded analogue of attached prime ideals which can be found in [4].

Definition 3.1. Let M be a G-graded A-module. A G-prime ideal P of A is said to be G-attached to M if every finitely generated proper graded ideal of A_P^G annihilates a nonzero homogeneous element of M_P^G. We denote the set of all G-attached prime ideals to M by $Att_A^G(M)$.

Definition 3.2. Let M be a G-graded A-module. A G-prime ideal P of A is said to be weak G-attached to M if each non-unit of $h(A_P^G)$ annihilates a nonzero homogeneous element of M_P^G. We denote the set of all weak G-attached prime ideals to M by W-$Att_A^G(M)$.

Proposition 3.3. *Let M be a G-graded A-module. Then*

$$Ass_A^G(M) \subseteq Att_A^G(M) \subseteq W - Att_A^G(M) \subseteq Supp_A^G(M).$$

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Proof. If $M = \{0\}$, then all the inclusions trivially hold. Assume $M \neq \{0\}$. Suppose $P \in Ass_A^G(M)$. Then $P = Ann(x)$ for some $0 \neq x \in h(M)$, whence $\frac{x}{1}$ is a nonzero homogeneous element of M_P^G. Clearly $\frac{x}{1}$ is annihilated by PA_P^G, and so every finitely generated proper graded ideal of A_P^G annihilates $\frac{x}{1}$. Hence, $P \in Att_A^G(M)$. The other inclusions are obvious. □

We will see shortly that the converse of the second inclusion is not true in general with the help of an example. The aim to define a weak G-attached prime ideal is that it is easy to compute the set $W\text{-}Att_A^G(M)$ in comparison to $Att_A^G(M)$ and later we will give some properties of graded modules in terms of weak G-attached prime ideals.

A G-attached prime ideal need not be a G-associated prime ideal. For this, consider the following example.

Example 3.4. Let $A = \mathbb{Z} + x\mathbb{Q}[x]$ and $G = \mathbb{Z}$. Then A is a G-graded ring with grading $A = \bigoplus_{n\in\mathbb{Z}} A_n$, where $A_0 = \mathbb{Z}$, $A_n = \mathbb{Q}x^n$ if $n \geq 1$ and $A_n = 0$ if $n < 0$. Let $I = xA = \{a_1x + a_2x^2 + \cdots + a_nx^n : a_1 \in \mathbb{Z}, a_2, a_3, \ldots, a_n \in \mathbb{Q}, n \geq 1\}$. Then I is a graded ideal of A. Consider $M = A/I$ as an A-module. Then M is a G-graded A-module with grading $M = \bigoplus_{n\in\mathbb{Z}} M_n$, where $M_n = (A/I)_n = \{f + I : f \in A_n\}$ if $n \geq 0$ and $M_n = 0$ if $n < 0$. Let $P = x\mathbb{Q}[x]$. Then P is a G-prime ideal of A such that $I \subseteq P$. We show that $P \in Att_A^G(M)$ but $P \notin Ass_A^G(M)$.

Let J be a finitely generated proper graded ideal of A_P^G. Then $J \subseteq PA_P^G$. Let $\{h_1, h_2, \ldots, h_r\}$ be a set of homogeneous generators of J. Then $h_j = \frac{u_j x^{i_j}}{t_j}$ for some integer $i_j \geq 1$, where $t_j \in h(A)\setminus P$ and $u_j \in \mathbb{Q}$. Write $u_j = \frac{b_j}{c_j}$ for some $b_j, c_j \in \mathbb{Z}$ with $c_j \neq 0$ for $j = 1, 2, \ldots, r$. Then each h_j annihilates a nonzero homogeneous element $\frac{c_j+I}{1}$ of M_P^G. Let $c = c_1c_2\ldots c_r$. Then J annihilates $\frac{c+I}{1}$, which is nonzero and homogeneous element of M_P^G. Hence $P \in Att_A^G(M)$.

On contrary suppose $P \in Ass_A^G(M)$. Then there exists a nonzero $f + I \in h(M)$ such that $P = Ann(f + I)$. Now, $f \notin I$ implies that $f + I = a + I$ or $f+I = bx+I$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Q}\setminus\mathbb{Z}$. Assume $f+I = a+I$. Write $x = d\frac{x}{d}$ where $d \in \mathbb{Z}$ such that d does not divide a. Then $x \in Ann(f+I)$ but neither d nor $\frac{x}{d}$ belongs to $P = Ann(f+I)$, a contradiction. Assume $f+I = bx+I$. Write $b = \frac{\alpha}{\beta}$ for some nonzero $\alpha, \beta \in \mathbb{Z}$. Then $\beta \in Ann(f + I) = P$, a contradiction. Hence $P \notin Ass_A^G(M)$, as desired.

The next example shows that there is an attached prime ideal which is not a G-attached prime ideal.

Example 3.5. Consider $A = \{f : f : [-1,1] \longrightarrow \mathbb{R}$ is a function$\}$, which is a ring under the usual definition of point wise addition and multiplication of functions. Let $M = A$ as an A-module graded by $G = \mathbb{Z}_2$ as follows: $A = A_{\bar{0}} \oplus A_{\bar{1}}$, where $A_{\bar{0}}$ denotes the set of all even functions and $A_{\bar{1}}$ denotes the set of all odd functions. Consider the maximal ideals of A of the form

$P = P_c = \{f \in A : f(c) = 0\}$ for some $c \in [-1, 1]$. Clearly P is a principal ideal generated by $f \in A$, where f is defined to be 0 if $x = c$ and 1 otherwise. Now since PA_P is generated by $\frac{f}{1}$, which is clearly a zero element of A_P as there exists $t \in A \setminus P$ defined by $t(x) = 1$ if $x = c$ and 0 otherwise, such that $tf = 0$. Hence $PA_P = \{0\}$, and so every finitely generated proper ideal of A_P is $\{0\}$ which annihilates a nonzero element of M_P obviously. Thus $P = P_c$ is an attached prime ideal to M but not a G-attached prime ideal since P is not a graded ideal as f is not a homogeneous element of A.

Now we present an example of a G-attached prime ideal which is not an attached prime ideal.

Example 3.6. Let $A = \mathbb{R}[x]$ and $M = A/I$, where I is an ideal generated by $x^2 - 1$ in A. Consider the group $G = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. Then M is a G-graded A-module with grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$, $M = M_{\bar{0}} \oplus M_{\bar{1}}$, where $A_{\bar{0}} = \mathbb{R}+\mathbb{R}x^2+\mathbb{R}x^4+\cdots$, $A_{\bar{1}} = \mathbb{R}x+\mathbb{R}x^3+\mathbb{R}x^5+\cdots$, $M_{\bar{0}} = (A/I)_{\bar{0}} = \{f+I : f \in A_{\bar{0}}\}$ and $M_{\bar{1}} = (A/I)_{\bar{1}} = \{f+I : f \in A_{\bar{1}}\}$. Consider the G-prime ideal $P = (x^2-1)$ of A which is clearly not a prime ideal of A. Clearly $P = Ann(1+I)$, where $1+I$ is a nonzero homogeneous element of M. Consequently, P is a G-associated prime ideal to M, and so by Proposition 3.3, P is a G-attached prime ideal to M. On the other hand, P is not an attached prime ideal to M since P is not a prime ideal of A.

Now we prove the existence of G-attached and weak G-attached prime ideals. First we prove the following two lemmas used in our theorem.

Lemma 3.7. *Let M be a nonzero G-graded A-module. Then $Supp_A^G(M)$ is contained in the set $\{P \in Spec^G(A) : P \supseteq Ann(M)\}$. Moreover, if M is finitely generated, then*

$$Supp_A^G(M) = \{P \in Spec^G(A) : P \supseteq Ann(M)\}.$$

Proof. Let $P \in Supp_A^G(M)$. Then $M_P^G \neq \{0\}$, and so there exists a nonzero $x \in h(M)$ such that $\frac{x}{1}$ is a nonzero homogeneous element of M_P^G. Let $a \in Ann(M)$ be a homogeneous element. Then $ax = 0$, and therefore $a \in P$. This shows every homogeneous element of $Ann(M)$ is in P, whence $P \supseteq Ann(M)$.

Further, suppose M is a finitely generated G-graded A-module. Let $P \in Spec^G(A)$ such that $P \supseteq Ann(M)$. Write $M = Ax_1 + Ax_2 + \cdots + Ax_n$, where each $x_i \in h(M)$. Then

$$P \supseteq Ann(M) = \bigcap_{i=1}^{n} Ann(x_i).$$

Now since P is a G-prime ideal and each $Ann(x_i)$ is a graded ideal of A, we have $P \supseteq Ann(x_i)$ for some i. Consequently, $\frac{x_i}{1}$ is a nonzero homogeneous element of M_P^G which implies that $P \in Supp_A^G(M)$. Hence $Supp_A^G(M) = \{P \in Spec^G(A) : P \supseteq Ann(M)\}$. □

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Lemma 3.8. *Let M be a nonzero G-graded A-module. Then the set $X = \{P \in Spec^G(A) : P \supseteq Ann(M)\}$ has a minimal element.*

Proof. By Lemma 3.7, $Supp_A^G(M) \subseteq X$, and so $X \neq \emptyset$. Then by Zorn's lemma, X has a minimal element with respect to inclusion, as desired. □

Theorem 3.9. *Let M be a G-graded A-module. Then $Att_A^G(M) \neq \emptyset$ if and only if $M \neq \{0\}$.*

Proof. First we assume that M is a nonzero finitely generated G-graded A-module. Then by Lemma 3.7,

$$Supp_A^G(M) = \{P \in Spec^G(A) : P \supseteq Ann(M)\}.$$

Also, by Lemma 3.8, a minimal G-prime ideal containing $Ann(M)$ exists in $Supp_A^G(M)$ say P. Evidently, $M_P^G \neq \{0\}$. We claim that $P \in Att_A^G(M)$. Since M is finitely generated, therefore $P \supseteq Ann(M)$ implies that $PA_P^G \supseteq Ann_{A_P^G}(M_P^G)$. Also, P is a minimal G-prime ideal containing $Ann(M)$ implies that PA_P^G is the unique G-prime ideal of A_P^G containing $Ann_{A_P^G}(M_P^G)$. Consequently,

$$PA_P^G = Gr(Ann_{A_P^G}(M_P^G)).$$

Now, let I be a finitely generated proper graded ideal of A_P^G. Then I is generated by a set of homogeneous elements say $\{a_1, a_2, \dots, a_n\}$. This implies that $a_i^{r_i} M_P^G = \{0\}$ for some positive integer r_i for each i, whence $I^r M_P^G = \{0\}$ for some positive integer r. Choose r minimum so that $I^r M_P^G = \{0\}$ but $I^{r-1} M_P^G \neq \{0\}$. Then I annihilates a nonzero homogeneous element of M_P^G. Hence $P \in Att_A^G(M)$.

Now assume that M is any nonzero G-graded A-module and x be a nonzero homogeneous element of M. Consider the graded submodule $N = Ax$, which is clearly nonzero (as $1 \in A$) and finitely generated. Hence there exists a G-prime ideal P such that $P \in Att_A^G(N) \subseteq Att_A^G(M)$, and so $Att_A^G(M) \neq \emptyset$. The converse is straightforward.

□

Corollary 3.10. *Let M be a nonzero G-graded A-module. Then M possesses a weak G-attached prime ideal.*

Proof. Follows from Theorem 3.9 and Proposition 3.3. □

The aim of introducing the notion of G-attached prime ideal is that both the sets $Ass_A^G(M)$ and $Att_A^G(M)$ should be the same under G-graded Noetherian conditions. We shall now prove this one. We begin with the following lemma.

Lemma 3.11. *Let P be a finitely generated G-prime ideal of A and M be a G-graded A-module such that $PA_P^G \in Ass_{A_P^G}^G(M_P^G)$. Then $P \in Ass_A^G(M)$.*

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Proof. By assumption, there exists a nonzero homogeneous element $\frac{x}{s} \in M_P^G$ such that $PA_P^G = Ann(\frac{x}{s})$. Let $\{a_1, a_2, \dots, a_n\}$ be a set of homogeneous generators of P. Then $\frac{a_i}{1}\frac{x}{s} = 0$, and hence there exists $t_i \in h(A) \setminus P$ such that $t_i a_i x = 0$ for all $i = 1, 2, \dots, n$. Let $t = t_1 t_2 \dots t_n$. Then $t \in h(A) \setminus P$ as a product of homogeneous elements is homogeneous. Now, let $a \in P$. Then $tax = 0$, whence $P \subseteq Ann(tx)$. On the other hand, if $b \in Ann(tx)$, then $tbx = 0$. Consequently, $\frac{bx}{1} = 0$ in M_P^G, whence $\frac{b}{1} \in Ann(\frac{x}{s}) = PA_P^G$, and so $b \in P$. This implies that $Ann(tx) \subseteq P$. Thus $P = Ann(tx)$ where tx is a non zero homogeneous element of M, whence $P \in Ass_A^G(M)$, as required. □

Theorem 3.12. *Let M be a G-graded module over a G-graded Noetherian ring A. Then $Ass_A^G(M) = Att_A^G(M)$.*

Proof. Let $P \in Att_A^G(M)$. Since P is finitely generated, PA_P^G is a finitely generated graded ideal of A_P^G, and so annihilates a nonzero homogeneous element $\frac{x}{s}$ of M_P^G, where $x \in h(M)$ and $s \in h(A) \setminus P$. Now, since PA_P^G is the unique G-maximal ideal of A_P^G, we have $PA_P^G = Ann(\frac{x}{s})$. This implies that $PA_P^G \in Ass_{A_P^G}^G(M_P^G)$, and so by Lemma 3.11, $P \in Ass_A^G(M)$. Hence $Att_A^G(M) \subseteq Ass_A^G(M)$. The reverse containment follows from Proposition 3.3. Hence $Ass_A^G(M) = Att_A^G(M)$. □

Now we prove an important property of the G-attached prime ideals in the next theorem which is a generalization of [4, Theorem 2].

Theorem 3.13. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of G-graded A-modules. Then*

$$Att_A^G(M) \subseteq Att_A^G(M') \cup Att_A^G(M'').$$

Proof. Suppose $P \in Att_A^G(M)$ and $P \notin Att_A^G(M')$. Let I be a finitely generated proper graded ideal of A_P^G. Then $Ann_{M_P^G}(I) \neq \{0\}$ where $Ann_{M_P^G}(I)$ denotes the annihilator of I in M_P^G given by

$$Ann_{M_P^G}(I) = \{\tfrac{x}{s} \in M_P^G : \tfrac{a}{t}\tfrac{x}{s} = 0 \text{ for all } \tfrac{a}{t} \in I\}.$$

Also, since $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of G-graded A-modules, then by Theorem 2.5,

$$0 \to M_P'^G \xrightarrow{f_P} M_P^G \xrightarrow{g_P} M_P''^G \to 0$$

is an exact sequence of G-graded A_P^G-modules. In order to prove $P \in Att_A^G(M'')$, it is enough to prove $Ann_{M''^G_P}(I) \neq \{0\}$. For let $\frac{x}{t} \in Ann_{M_P^G}(I)$, where $x \in h(M)$, $t \in h(A) \setminus P$. Then certainly I annihilates $g_P(\frac{x}{t})$. This implies $g_P(Ann_{M_P^G}(I)) \subseteq Ann_{M_P''^G}(I)$. Now it is sufficient to show that

$$g_P(Ann_{M_P^G}(I)) \neq 0.$$

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Contrary suppose $g_P(Ann_{M_P^G}(I)) = 0$. Then $Ann_{M_P^G}(I) \subseteq Ker\ g_P = Image\ f_P$. Choose a nonzero homogeneous element $\frac{x}{t} \in Ann_{M_P^G}(I)$, then there exists a nonzero homogeneous element $\frac{y}{s} \in M_P'^G$ where $y \in h(M')$ and $s \in h(A) \setminus P$, such that $\frac{x}{t} = f_P(\frac{y}{s})$ since f_P is a G-graded homomorphism which sends homogeneous elements to homogeneous element. Since f_P is injective, then I annihilates a nonzero homogeneous elements $\frac{y}{s} \in M_P'^G$, whence $Ann_{M_P'^G}(I) \neq \{0\}$. Consequently, $P \in Att_A^G(M')$, which is a contradiction. Hence $P \in Att_A^G(M'')$, as required. □

Corollary 3.14. *Let $M_1, M_2, \ldots, M_r$ be G-graded A-modules. Then*

$$Att_A^G(\bigoplus_{i=1}^r M_i) = \bigcup_{i=1}^r Att_A^G(M_i).$$

Proof. Let $M = \bigoplus_{i=1}^r M_i$. Then by Theorem 3.13, we have

$$Att_A^G(M) \subseteq \bigcup_{i=1}^r Att_A^G(M_i).$$

For the reverse containment, let $P \in \bigcup_{i=1}^r Att_A^G(M_i)$. Then $P \in Att_A^G(M_i)$ for some i. This implies that each finitely generated proper graded ideal of A_P^G annihilates a nonzero homogeneous element of $(M_i)_P^G$, and so annihilates a nonzero homogeneous element of M_P^G since $(M_i)_P^G$ is a graded submodule of M_P^G. Consequently, $P \in Att_A^G(M)$. Hence $\bigcup_{i=1}^r Att_A^G(M_i) \subseteq Att_A^G(M)$, as required. □

Remark 3.15. It is easy to show that if M is a G-graded A-module and P is a G-prime ideal of A, then

$$Att_{A_P^G}^G(M_P^G) = \{PA_P^G : P \in Att_A^G(M)\}.$$

Now we give an example of a weak G-attached prime ideal which is not a G-attached prime ideal.

Example 3.16. Let $A = \mathbb{Z}[x_1, x_2, x_3, \ldots, x_n, \ldots]$ be the polynomial ring in infinitely many indeterminates over $\mathbb{Z}$. Then A is a $G = \mathbb{Z}$-graded ring with $deg(x_i) = 1$ and support $\mathbb{N}$ (see [5, Example 1.1.9]). Consider the graded ideal $I = (x_1^2, x_2^2, x_3^2, \ldots, x_n^2, \ldots)$ of A. Let M be A/I as an A-module naturally graded by $G = \mathbb{Z}$. Then $Ass_A^G(M) = \emptyset$ since annihilator of any nonzero homogeneous element of M contains x_i^2 for all i but not x_j for some j. On the other hand, consider the G-prime ideal P of A generated by $x_1, x_2, \ldots, x_n$. Clearly, each x_i annihilates a nonzero homogeneous element $x_i + I$ of M which implies that every homogeneous element of PA_P^G annihilates a nonzero homogeneous element of M_P^G. Consequently, $P \in W - Att_A^G(M)$. Now, on contrary suppose $P \in Att_A^G(M)$. Since PA_P^G is finitely generated, there exists a nonzero homogeneous element $\frac{a}{s} \in M_P^G$, where $a \in h(M)$ and $s \in h(A) \setminus P$ such that $PA_P^G = Ann(\frac{a}{s})$, whence $PA_P^G \in Ass_{A_P^G}^G(M_P^G)$. Therefore by Lemma 3.11, $P \in Ass_A^G(M)$, a contradiction. Hence $P \notin Att_A^G(M)$.

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It seems reasonable to ask for which G-graded A-modules M, the two sets $W\text{-}Att_A^G(M)$ and $Att_A^G(M)$ will be equal. To answer this, first we give the following definitions which are the graded analogues of valuation domains and Prüfer domains.

Definition 3.17. [1, Definition 3.1] Let A be a G-graded integral domain. Then

(1) A is said to be a G-graded valuation domain if for any $x, y \in h(A)$, either x divides y or y divides x.
(2) A is said to be a G-graded Prüfer domain if its graded localization A_p^G is a G-graded valuation domain for each G-prime ideal P of A.

Proposition 3.18. [1, Proposition 3.2] *Every finitely generated graded ideal of a G-graded valuation domain is principal.*

Theorem 3.19. *Let M be a G-graded module over a G-graded Prüfer domain A. Then $Att_A^G(M) = W\text{-}Att_A^G(M)$.*

Proof. Let $P \in W - Att_A^G(M)$ and I be a finitely generated proper graded ideal of A_P^G. Since A is a G-graded Prüfer domain, A_P^G is a G-graded valuation domain. Then by Proposition 3.18, I is a principal ideal generated by a homogeneous element $\frac{a}{t}$ say, and so there exists a nonzero homogeneous element $\frac{x}{s}$ of M_P^G such that $\frac{a}{t}\frac{x}{s} = 0$. This implies that I annihilates a nonzero homogeneous element of M_P^G. Hence $P \in Att_A^G(M)$, and so $W\text{-}Att_A^G(M) \subseteq Att_A^G(M)$. The reverse containment follows from Proposition 3.3. □

Remark 3.20. It is straight forward to show that if M is a torsion free G-graded module over a G-graded integral domain A, then M_P^G is a torsion free A_P^G-module for every $P \in Spec^G(A)$ and

$$Att_A^G(M) = W - Att_A^G(M) = Ass_A^G(M) = \{0\}.$$

4. Applications of G-attached prime ideals

In this section, we discuss some applications of the theory of G-attached and weak G-attached prime ideals developed in this paper.

Let M be a G-graded A-module. Following [7], an element $a \in h(A)$ is said to be a graded zero-divisor on M if there exists a nonzero $x \in h(M)$ such that $ax = 0$. The set of all graded zero-divisors of M is denoted by Z^G(M). Note that a zero-divisor need not be a graded zero-divisor.

Proposition 4.1. *Let M be a G-graded A-module. Then*

$$Z^G(M) = \bigcup_{P \in Att_A^G(M)} P^G$$

where P^G denotes the set of all homogeneous elements of P.

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Proof. Let $a \in Z^G(M)$. Then there exists a nonzero $x \in h(M)$ such that $ax = 0$. Let P be a minimal G-prime ideal of A containing $Ann(x)$. Then $a \in P^G$ and $P \in Att_A^G(Ax) \subseteq Att_A^G(M)$ as in proof of Theorem 3.9. Consequently, $a \in \bigcup_{P \in Att_A^G(M)} P^G$. On the other hand, let $a \in P^G$ for some $P \in Att_A^G(M)$. Then there exists a nonzero homogeneous element $\frac{y}{s}$ of M_P^G such that $\frac{a}{1}\frac{y}{s} = 0$, and so there exists $t \in h(A) \setminus P$ such that $aty = 0$, where ty is a nonzero homogeneous element of M. Consequently, $a \in Z^G(M)$, as required. □

Corollary 4.2. *Let M be a G-graded A-module. Then*

$$Z^G(M) = \bigcup_{P \in W-Att_A^G(M)} P^G$$

where P^G denotes the set of all homogeneous elements of P.

Proof. It follows from the fact that $\bigcup_{P \in Att_A^G(M)} P^G = \bigcup_{P \in W-Att_A^G(M)} P^G$. □

Proposition 4.3. *Let M be a G-graded A-module. Let $Gr(Ann(M))$ be the graded radical of $Ann(M)$. Then $Gr(Ann(M)) = \bigcap_{P \in Att_A^G(M)} P$.*

Proof. Let $a \in Gr(Ann(M))$ be a homogeneous element and $P \in Att_A^G(M)$. Then by Proposition 3.3, $P \in Supp_A^G(M)$, and so there exists a nonzero $x \in h(M)$ such that $P \supseteq Ann(x)$. Since $a \in Gr(Ann(M))$, $a^n x = 0$ for some positive integer n. This implies that $a \in P$, whence $Gr(Ann(M)) \subseteq \bigcap_{P \in Att_A^G(M)} P$. On the other hand, let $a \in P$ for every $P \in Att_A^G(M)$. On contrary suppose $a \notin Gr(Ann(M))$. Then there exists $y \in h(M)$ such that $a^n y \neq 0$ for all positive integers n. Consider the multiplicatively closed subset $T = \{a^k : k \geq 0\}$. Then $Ann(y) \cap T = \emptyset$. We may choose a minimal G-prime ideal L containing $Ann(y)$ such that $L \cap T = \emptyset$. This implies $L \in Att_A^G(Ay) \subseteq Att_A^G(M)$. Since $a^k \notin L$ for all $k \geq 0$, we conclude that $a \notin L$, a contradiction. Hence $a \in Gr(Ann(M))$, as required. □

Corollary 4.4. *Let A be a G-graded ring and $N^G(A)$ be the G-graded nilradical of A. Then*

$$N^G(A) = \bigcap_{P \in Att_A^G(M)} P.$$

Proof. Follows from Proposition 4.3 for $M = A$ as an A-module. □

Corollary 4.5. *Let M be a G-graded A-module. Then*

$$Gr(Ann(M)) = \bigcap_{P \in W-Att_A^G(M)} P.$$

Proof. Follows from the fact that $\bigcap_{P \in Att_A^G(M)} P = \bigcap_{P \in W-Att_A^G(M)} P$. □

In [7], it was shown that for a finitely generated G-graded module M over a G-graded Noetherian ring A, if $0 = N_1 \cap N_2 \cap \cdots \cap N_r$ is a reduced G-graded primary decomposition, where N_i is G-graded P_i-primary. Then $Ass_A^G(M) = \{P_1, P_2, \ldots, P_r\}$ (Theorem 2.4). We show how this phenomena figure in a more general situation. We need the following lemma in the proof of the Theorem 4.7.

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Lemma 4.6. *Let M be a G-graded A-module such that the zero submodule is G-graded P-primary. Let N be a graded submodule of M such that $P \in Supp^G_A(N)$. Then $P \in Att^G_A(N)$.*

Proof. Consider a finitely generated proper graded ideal IA^G_P of A^G_P, I being a graded ideal of A. Since $P = Gr(Ann(M)) \subseteq Gr(Ann(N))$, $I^k N = \{0\}$ for some positive integer k. Therefore $(IA^G_P)^k N^G_P = \{0\}$. Since $N^G_P \neq \{0\}$, this implies that IA^G_P annihilates a nonzero homogeneous element of N^G_P, whence $P \in Att^G_A(N)$. □

Theorem 4.7. *Let M be a G-graded A-module. Suppose the zero submodule of M is reduced G-decomposable with $0 = N_1 \cap N_2 \cap \cdots \cap N_r$, where N_i is a G-graded P_i-primary submodule. Then*

$$Att^G_A(M) = \{P_1, P_2, \ldots, P_r\}.$$

Proof. Suppose $P \in Att^G_A(M)$. Then by Proposition 4.3, we have

$$P \supseteq Gr(Ann(M)) = P_1 \cap P_2 \cap \cdots \cap P_r.$$

Consequently, $P \supseteq P_i$ for some i. By rearranging P_i's, we may assume that there exists k such that $P \supseteq P_i$ for all $i(1 \leq i \leq k)$ and $P \not\supseteq P_j$ for all $j(k+1 \leq j \leq r)$. Now

$$\{0\} = (N_1)^G_P \cap (N_2)^G_P \cap \cdots \cap (N_k)^G_P$$

is a reduced G-graded primary decomposition of the zero submodule of M^G_P. Since $P \in Att^G_A(M)$, it follows from Remark 3.15,

$$PA^G_P \in Att^G_{A^G_P}(M^G_P).$$

Now let $a \in PA^G_P$ be a homogeneous element, then there exists a nonzero homogeneous element $\frac{x}{s} \in (M^G_P)^G_P$, where $x \in h(M^G_P)$ and $s \in h(A^G_P) \setminus PA^G_P$ such that $\frac{a}{1}\frac{x}{s} = 0$. This implies that there exists $t \in h(A^G_P) \setminus PA^G_P$ such that $ay = 0$ where $y = tx$. If $y = 0$, then by Proposition 4.1, we have

$$t \in Z^G(M^G_P) = \bigcup_{Q \in Att^G_{A^G_P}(M^G_P)} Q^G.$$

But, since PA^G_P is a unique G-maximal ideal of A^G_P, we have $t \in PA^G_P$, which is a contradiction. Therefore $y \neq 0$, and so there exists a homogeneous component $y_g \neq 0$ of y such that $ay_g = 0$. Choose $j(1 \leq j \leq k)$ such that $y_g \notin (N_j)^G_P$. This implies that $a \in Gr((N_j)^G_P) = P_j A^G_P$. Thus each homogeneous element of PA^G_P belongs to $P_j A^G_P$, whence $PA^G_P = P_j A^G_P$. Consequently, $P = P_j$ for some $j(1 \leq j \leq k)$. Hence $Att^G_A(M) \subseteq \{P_1, P_2, \ldots, P_r\}$.

For the reverse containment, let $P = P_i$ for some $i(1 \leq i \leq r)$. Let us denote

$$N^i = N_1 \cap N_2 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r.$$

We claim that $(N^i)^G_P \neq \{0\}$. On contrary suppose $(N^i)^G_P = \{0\}$. Then

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$$(N_1)_P^G \cap (N_2)_P^G \cap \cdots \cap (N_{i-1})_P^G \cap (N_{i+1})_P^G \cap \cdots \cap (N_r)_P^G = (N^i)_P^G = 0$$

may be refined to give a reduced G-graded primary decomposition of the zero submodule of M_P^G, and therefore the set $\{P_j A_P^G : 1 \leq j \leq r\}$ is contained in the set $\{P_j A_P^G : 1 \leq j \leq r, j \neq i\}$. This implies that $PA_P^G = P_i A_P^G \in \{P_j A_P^G : 1 \leq i \leq r, j \neq i\}$, a contradiction. Thus $P \in Supp_A^G(N^i)$. Next since $N_i \cap N^i = 0$, so N^i is isomorphic to a graded submodule of M/N_i. Therefore by applying Lemma 4.6 to the graded A-module M/N_i, we get $P \in Att_A^G(N^i) \subseteq Att_A^G(M)$. Hence $\{P_1, P_2, \ldots, P_r\} \subseteq Att_A^G(M)$, as required. □

5. Graded modules having only finitely many G-attached prime ideals

It is known that a finitely generated G-graded Noetherian module possesses only a finite number of G-associated prime ideals (Theorem 2.4). However, there may be infinitely many G-attached prime ideals. For this, consider the following example.

Example 5.1. Let $A = \prod_{i\in\mathbb{N}} F_i$ where each $F_i = F$ is a field. Consider the ideal $I = \bigoplus_{i\in\mathbb{N}} F_i$ of A. Let $M = A/I$ as an A-module trivially graded by G. Let $n \in \mathbb{N}$. Consider the graded localization A_P^G at G-maximal ideal $P = P_n = \{(a_i)_{i\in\mathbb{N}} : a_n = 0\}$ of A. Let $a \in P$. Let $t = (t_i)_{i\in\mathbb{N}}$ where $t_i = 1$ if $i = n$ and $t_i = 0$ otherwise. Then $t \in h(A) \setminus P$ such that $ta = 0$. Consequently, $PA_P^G = \{0\}$, and so $P = P_n \in Att_A^G(M)$. Thus M has infinitely many G-attached prime ideals.

Now we characterize those graded modules which have only finitely many G-attached prime ideals.

Definition 5.2. Let M be a G-graded A-module. We say that M is G-factorizable if there exists a chain

$$\{0\} = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M \qquad (*)$$

of graded submodules of M such that $Att_A^G(M_i/M_{i-1}) = \{P_i\}$, where each P_i is a G-attached prime ideal to M. We call $(*)$ a G-factorization of M of length r, and $P_1, P_2, \ldots, P_r$ are called G-prime ideals associated to G-factorization $(*)$.

Example 5.3. Every finitely generated G-graded module over a G-graded Noetherian ring is G-factorizable.

A G-factorization of a graded module need not be unique. To see this, let us consider an example.

Example 5.4. Take $G = \mathbb{Z}_2$. Consider the Klein-four group V_4 as a $\mathbb{Z}$-module with grading $V_4 = \{e, a\} \oplus \{e, b\}$. Then $\{e\} \subset V_4$, $\{e\} \subset \{e, a\} \subset V_4$ and $\{e\} \subset \{e, b\} \subset V_4$ are three G-factorization as $Att_{\mathbb{Z}}^{\mathbb{Z}_2}(\{e, a\}) = Att_{\mathbb{Z}}^{\mathbb{Z}_2}(\{e, b\}) = Att_{\mathbb{Z}}^{\mathbb{Z}_2}(V_4) = \{2\mathbb{Z}\}$.

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Lemma 5.5. *Let M be a G-graded A-module. Suppose P is a minimal element of $Att_A^G(M)$. Then there exists a graded submodule N of M such that $Att_A^G(M/N) = \{P\}$ and $Att_A^G(N) = Att_A^G(M) \setminus \{P\}$.*

Proof. Let N denotes the kernel of the canonical mapping $M \to M_P^G$, i.e., $N = \{x \in M : \frac{x}{1} = 0\}$ which is a graded submodule of M. Clearly, $N_P^G = \{0\}$. Since P is minimal in $Att_A^G(M)$, by Remark 3.15, we have $\{PA_P^G\} = Att^G_{A_P^G}(M_P^G) = Att^G_{A_P^G}(M_P^G/N_P^G) = Att^G_{A_P^G}((M/N)_P^G)$. Consequently, P is minimal in $Att_A^G(M/N)$. Now suppose $L \in Att_A^G(M/N)$ and $a \in L$ be a homogeneous element. Then a annihilates a nonzero homogeneous element $x + N$ of M/N, i.e., $x \in M \setminus N$ such that $ax \in N$. Then there exists $t \in h(A) \setminus P$ such that $tax = 0$. This implies that $ta \in P$, and so $a \in P$. Thus $L \subseteq P$, and the minimality of P implies that $L = P$. Thus $Att_A^G(M/N) = \{P\}$. Next, Theorem 3.13 shows that $Att_A^G(M) \subseteq Att_A^G(M/N) \cup Att_A^G(N)$. Since $P \notin Att_A^G(N)$ and $Att_A^G(N) \subseteq Att_A^G(M)$, we have $Att_A^G(N) = Att_A^G(M) \setminus \{P\}$. □

Now we characterize those graded modules for which the set $Att_A^G(M)$ is finite.

Theorem 5.6. *Let M be a G-graded A-module. Then M has only finitely many G-attached prime ideals if only if M is G-factorizable.*

Proof. Suppose $\{0\} = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ is a G-factorization of a G-graded A-module M. For each i, write $Att_A^G(M_i/M_{i-1}) = \{P_i\}$. Theorem 3.13 shows that $Att_A^G(M) \subseteq Att_A^G(M/N) \cup Att_A^G(N)$, whence $Att_A^G(M) \subseteq \{P_1, P_2, \ldots, P_r\}$ which implies that $Att_A^G(M)$ is a finite set.

Conversely, suppose $|Att_A^G(M)| = r$. Choose a minimal element of $Att_A^G(M)$ say P_r. By Lemma 5.5, there exists a graded submodule M_{r-1} of M such that $Att_A^G(M/M_{r-1}) = \{P_r\}$ and $Att_A^G(M_{r-1}) = Att_A^G(M) \setminus \{P_r\}$. We now choose P_{r-1} to be minimal in $Att_A^G(M_{r-1})$ and continuing this process to give M_{r-2}, M_{r-3}, and so on. For each $k (1 \leq k < r)$, we have $|Att_A^G(M_{k+1}/M_k)| = 1$ and $|Att_A^G(M_k)| = |Att_A^G(M)| - (r - k) = k$. Consequently, $Att_A^G(M_1)$ is singleton. Thus $\{0\} = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ is a G-factorization of M, and so M is G-factorizable. □

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References

1. A. U. Ansari, B. K. Sharma, S. D. Kumar, Different Types of G-Prime Ideals Associated to a Graded Module and Graded Primary Decomposition in a Graded Prüfer Domain, *Int. Electron. J. Algebra*, **28**, (2020), 141-155.

[DOI: 10.52547/ijmsi.17.2.59]

2. S. Behara, S. D. Kumar, Group Graded Associated Ideals with Flat Base of Change of Rings and Short Exact Sequences, *Proc. Math. Soc.*, **121**(8), (2011), 1-10.
3. N. Bourbaki, *Commutative Algebra*, Berlin-Heidelberg- New York: Springer-Verlag, 1989.
4. P. Dutton, Prime Ideals Attached to a Module, *Quart. J. Math. Oxford Ser. (2)*, **29**, (1978), 403-413.
5. R. Hazrat, *Graded Rings and Graded Grothendieck Groups*, London Math. Soc., Lecture Notes Series 435, Cambridge University Press, 2016.
6. J. Iroz, D.E. Rush, Associated Prime Ideals in Non-Noetherian Rings, *Canad. J. Math.*, **36**(2), (1984), 344-360.
7. S. D. Kumar, S. Behara, Uniqueness of Graded Primary Decomposition of Modules Graded over Finitely Generated Abelian Groups, *Comm. Algebra*, **39**(7), (2011), 2607-2614.
8. C. Năstăsescu, F. Van Oystaeyen, *Methods of Graded Rings*, LNM (1836), Berlin-Heidelberg: Springer-Verlag, 2004.
9. D. G. Northcott, Remarks on the Theory of Attached Prime Ideals, *Quart. J. Math. Oxford Ser. (2)*, **33**, (1982), 239-245.
10. D. G. Northcott, *Finite Free Resolutions*, Cambridge Tracts in Mathematics No. 71, 1976.
11. M. Perling, S. D. Kumar, Primary Decomposition over Rings Graded by Finitely Generated Abelian Groups, *J. Algebra*, **318**, (2007), 553-561.

[DOI: 10.52547/ijmsi.17.2.59]