Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 1 (2022), pp 207-217
DOI: 10.52547/ijmsi.17.1.207

Bi-concave Functions Defined by Al-Oboudi Differential Operator

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Abstract. The purpose of the present paper is to introduce a class $D^n_{\Sigma;\delta}C_0(\alpha)$ of bi-concave functions defined by Al-Oboudi differential operator. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class. Several consequences of these results are also pointed out in the form of corollaries.

Keywords: Bi-concave functions, Al-Oboudi differential operator, Coefficient estimates.

2000 Mathematics subject classification: 30C45.

1. Introduction

Let A indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ.

It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z$, $(z \in \Delta)$ and $f\left(f^{-1}(w)\right) = w$, $\left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$,

Received 15 September 2018; Accepted 27 April 2019

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots$$

(for details, see Duren [13]). A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ stand for the class of bi-univalent functions defined in the unit disk Δ. For a brief history of functions in the class Σ, see [25] (see also [10, 11, 14, 17, 20, 26, 27]). More recently, Srivastava *et al.* [25], Altınkaya and Yalcın [3] made an effort to introduce various subclasses of the bi-univalent function class Σ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ (see also [21, 15]). But determination of the bounds for the coefficients

$$|a_n|, \quad n \in \mathbb{N} \backslash \{1, 2\}; \quad \mathbb{N} = \{1, 2, 3, ...\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see, for example [4, 16, 28]).

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields (see, for example [2, 18, 19]). Recently, the interest in this area has been increasing because it permits detailed investigations of problems with physical applications. For $f \in A$, we consider the following differential operator introduced by Al-Oboudi [1],

$$\begin{aligned}
D_\delta^0 f(z) &= f(z), \\
D_\delta^1 f(z) &= (1-\delta) f(z) + \delta f'(z) \quad (\delta \geq 0), \\
&\vdots \\
D_\delta^k f(z) &= D_\delta(D_\delta^{k-1} f(z)) \quad (k \in \mathbb{N}).
\end{aligned}$$

Additionally, in view of (1.1), we deduce that

$$D_\delta^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\delta]^k a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

with $D_\delta^k f(0) = 0$.

It is of interest to note that D_1^k is the Salagean's differential operator [23].

2. Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently, Avkhadiev and Wirths [6] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : \Delta \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

[DOI: 10.52547/ijmsi.17.1.207]

- f is analytic in Δ with the standard normalization $f(0) = f'(0)-1 = 0$. In addition it satisfies $f(1) = \infty$.
- f maps Δ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(\Delta)$ at ∞ is less than or equal to $\pi\alpha,\ \alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [7], Cruz and Pommerenke [12] and references there in.

In particular, the inequality

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<0 \qquad (z\in\Delta)$$

is used - sometimes also as a definition - for concave functions $f \in C_0$ (see e.g. [22] and others).

Bhowmik et al. [9] showed that an analytic function f maps Δ onto a concave domain of angle $\pi\alpha$, if and only if $\Re(P_f(z)) > 0$, where

$$P_f(z)=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{f''(z)}{f'(z)}\right].$$

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [5], [8] and [24]).

Let us recall now the following definition required in sequel.

Definition 2.1. Let the functions $h, p : \Delta \to \mathbb{C}$ be so constrained that

$$\min\left\{\Re\left(h\left(z\right)\right),\Re\left(p\left(z\right)\right)\right\}>0$$

and

$$h\left(0\right)=p\left(0\right)=1.$$

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving Al-Oboudi differential operator D_δ^k.

Definition 2.2. A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$D^k_{\Sigma;\delta}C_0(\alpha) \qquad \left(k\in\mathbb{N}_0,\ \delta\geq 0,\ \alpha\in(1,2]\,,\ z,w\in\Delta\right)$$

if the following conditions are satisfied:

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{\left[D^k_{\Sigma;\delta}f(z)\right]''}{\left[D^k_{\Sigma;\delta}f(z)\right]'}\right]\in h(\Delta) \tag{2.1}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-w\frac{\left[D^k_{\Sigma;\delta}g(w)\right]''}{\left[D^k_{\Sigma;\delta}g(w)\right]'}\right]\in p(\Delta), \tag{2.2}$$

where $g = f^{-1}$.

[DOI: 10.52547/ijmsi.17.1.207]

Remark 2.3. There are several choices of k and δ which would provide interesting subclasses of the class $D^k_{\Sigma;\delta}C_0(\alpha)$. For example,

(i) For $k=0$, it can be directly verified that the functions h and p satisfy the hypotheses of Definition 2.1. Now if $f \in C_{\Sigma;0}(\alpha)$ then

$$f \in \Sigma, \quad \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right] \in h(\Delta) \quad (z \in \Delta)$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+w}{1-w} - 1 - w\frac{g''(w)}{g'(w)}\right] \in p(\Delta) \quad (w \in \Delta),$$

where $g = f^{-1}$.

(ii) For $\delta = 1$, it can be directly verified that the functions h and p satisfy the hypotheses of Definition 2.1. Now if $f \in D^k_{\Sigma}C_0(\alpha)$ then

$$f \in \Sigma, \quad \frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z} - 1 - z\frac{\left[D^k_\Sigma f(z)\right]''}{\left[D^k_\Sigma f(z)\right]'}\right] \in h(\Delta) \quad (k \in \mathbb{N}_0,\ z \in \Delta)$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w} - 1 - w\frac{\left[D^k_\Sigma g(w)\right]''}{\left[D^k_\Sigma g(w)\right]'}\right] \in p(\Delta) \quad (k \in \mathbb{N}_0, w \in \Delta),$$

where $g = f^{-1}$.

3. Main Results and Their Consequences

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $D^k_{\Sigma;\delta}C_0(\alpha)$.

Theorem 3.1. *Let f given by (1.1) be in the class $D^k_{\Sigma;\delta}C_0(\alpha)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{(\alpha+1)^2}{4(1+\delta)^{2k}} + \frac{(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}} + \frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}}},\right.$$
$$\left.\sqrt{\frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{16\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|} + \frac{(\alpha+1)}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}}\right\} \tag{3.1}$$

and

$$|a_3| \leq \min$$

$$\left\{\frac{8(\alpha+1)^2+(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}} + \frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}} + \frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{48(1+2\delta)^k},\right.$$
$$\left.\frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|\left|h''(0)\right|+(\alpha-1)(1+\delta)^{2k}\left|p''(0)\right|}{24(1+\delta)^{2k}\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|} + \frac{\alpha+1}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}\right\}. \tag{3.2}$$

[DOI: 10.52547/ijmsi.17.1.207]

Proof. Let $f \in D^k_{\Sigma;\delta}C_0(\alpha)$ and g be the analytic extension of f^{-1} to Δ. It follows from (2.1) and (2.2) that

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-1-z\frac{\left[D^k_{\Sigma;\delta}f(z)\right]''}{\left[D^k_{\Sigma;\delta}f(z)\right]'}\right]=h\left(z\right) \tag{3.3}$$

and

$$\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1-w}{1+w}-1-w\frac{\left[D^k_{\Sigma;\delta}g(w)\right]''}{\left[D^k_{\Sigma;\delta}g(w)\right]'}\right]=p\left(w\right), \tag{3.4}$$

where h and p satisfy the conditions of Definition 2.1. Furthermore, the functions $h\left(z\right)$ and $p\left(w\right)$ have the following Taylor-Maclaurin series expansions:

$$h\left(z\right)=1+h_1z+h_2z^2+\cdots$$

and

$$p\left(w\right)=1+p_1w+p_2w^2+\cdots,$$

respectively. Now, equating the coefficients in (3.3) and (3.4), we get

$$\frac{2\left[(\alpha+1)-2(1+\delta)^k a_2\right]}{\alpha-1}=h_{1}, \tag{3.5}$$

$$\frac{2\left[(\alpha+1)+4(1+\delta)^{2k}a_2^2-6(1+2\delta)^k a_3\right]}{\alpha-1}=h_2, \tag{3.6}$$

$$-\frac{2\left[(\alpha+1)-2(1+\delta)^k a_2\right]}{\alpha-1}=p_{1}, \tag{3.7}$$

$$\frac{2\left[(\alpha+1)+4(1+\delta)^{2k}a_2^2-6(1+2\delta)^k(2a_2^2-a_3)\right]}{\alpha-1}=p_2. \tag{3.8}$$

From (3.5) and (3.7), we find that

$$h_1=-p_1. \tag{3.9}$$

Also, from (3.5), we can write

$$a_2=\frac{\alpha+1}{2(1+\delta)^k}-\frac{h_1(\alpha-1)}{4(1+\delta)^k}. \tag{3.10}$$

Next, by using (3.5), (3.7), (3.9) and (3.10), we get

$$a_2^2=\frac{(\alpha+1)^2}{4(1+\delta)^{2k}}+\frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32(1+\delta)^{2k}}-\frac{\left(\alpha^2-1\right)\left(h_1-p_1\right)}{8(1+\delta)^{2k}}. \tag{3.11}$$

By adding (3.6) to (3.8), we get

$$a_2^2=\frac{(\alpha-1)\left(h_2+p_2\right)}{8\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}-\frac{\alpha+1}{2\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}. \tag{3.12}$$

Therefore, we find from the equations (3.11) and (3.12) that

$$|a_2|^2\leq\frac{(\alpha+1)^2}{4(1+\delta)^{2k}}+\frac{(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}}+\frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}}$$

[DOI: 10.52547/ijmsi.17.1.207]

and

$$|a_2|^2 \le \frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}+\frac{(\alpha+1)}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}.$$

Similarly, subtracting (3.8) from (3.6), we have

$$a_3 = a_2^2 - \frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}. \tag{3.13}$$

Then, upon substituting the value of in view of a_2^2 from (3.11) and (3.12) into (3.13), it follows that

$$a_3 = \frac{(\alpha+1)^2}{4(1+\delta)^{2k}}+\frac{(\alpha-1)^2\left(h_1^2+p_1^2\right)}{32(1+\delta)^{2k}}-\frac{\left(\alpha^2-1\right)(h_1-p_1)}{8(1+\delta)^{2k}}-\frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}$$

and

$$a_3 = \frac{(\alpha-1)(h_2+p_2)}{8\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}-\frac{\alpha+1}{2\left[2(1+\delta)^{2k}-3(1+2\delta)^k\right]}-\frac{(\alpha-1)(h_2-p_2)}{24(1+2\delta)^k}.$$

Consequently, we have

$$|a_3| \le \frac{8(\alpha+1)^2+(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{32(1+\delta)^{2k}}+\frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{8(1+\delta)^{2k}}+\frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{48(1+2\delta)^k}$$

and

$$|a_3| \le \frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|\left|h''(0)\right|+(\alpha-1)(1+\delta)^{2k}\left|p''(0)\right|}{24(1+\delta)^{2k}\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}+\frac{\alpha+1}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}.$$

This completes the proof of the theorem. □

It is easily seen that, by specializing the functions h and p involved in the Theorem, several coefficient estimates can be obtained as special cases.

Corollary 3.2. *If we set*

$$h(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1+2\gamma z+2\gamma^2z^2+\ldots \qquad (0<\gamma\le 1),$$

$$p(z) = \left(\frac{1-z}{1+z}\right)^\gamma = 1-2\gamma z+2\gamma^2z^2+\ldots \qquad (0<\gamma\le 1),$$

then inequalities (3.1) and (3.2) become

$$|a_2| \le \min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4(1+\delta)^{2k}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}}\right\}$$

and

$$\begin{aligned}|a_3| \quad \le \quad \min\Bigg\{ & \frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4(1+\delta)^{2k}}+\frac{(\alpha-1)\gamma^2}{6(1+2\delta)^k}, \\ & \frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|\gamma^2+(\alpha-1)(1+\delta)^{2k}\gamma^2}{6(1+\delta)^{2k}\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}+\frac{\alpha+1}{2\left|2(1+\delta)^{2k}-3(1+2\delta)^k\right|}\Bigg\}.\end{aligned}$$

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Corollary 3.3. *If we let*

$$h(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0\le\beta<1),$$

$$p(z)=\frac{1-(1-2\beta)z}{1+z}=1-2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0\le\beta<1),$$

then inequalities (3.1) and (3.2) become

$$|a_2|\le\min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{4(1+\delta)^{2k}}},\sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}}\right\}$$

and

$$\begin{aligned}|a_3| \quad\le\quad \min\Bigg\{&\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2\left(\alpha^2-1\right)(1-\beta)}{4(1+\delta)^{2k}}+\frac{(\alpha-1)(1-\beta)}{6(1+2\delta)^k},\\ &\frac{\left|3(\alpha-1)(1+2\delta)^k-(\alpha-1)(1+\delta)^{2k}\right|(1-\beta)+(\alpha-1)(1+\delta)^{2k}(1-\beta)}{6(1+\delta)^{2k}|2(1+\delta)^{2k}-3(1+2\delta)^k|}+\frac{\alpha+1}{2|2(1+\delta)^{2k}-3(1+2\delta)^k|}\Bigg\}.\end{aligned}$$

Theorem 3.4. *Let f given by (1.1) be in the class $C_{\Sigma;0}(\alpha)$. Then*

$$\begin{aligned}|a_2| \quad\le\quad \min\Bigg\{&\sqrt{\frac{(\alpha+1)^2}{4}+\frac{(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32}+\frac{\left(\alpha^2-1\right)\left(|h'(0)|+|p'(0)|\right)}{8}},\\ &\sqrt{\frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16}+\frac{(\alpha+1)}{2}}\Bigg\} \qquad (3.14)\end{aligned}$$

and

$$\begin{aligned}|a_3| \quad\le\quad \min\Bigg\{&\frac{8(\alpha+1)^2+(\alpha-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32}+\frac{\left(\alpha^2-1\right)\left(|h'(0)|+|p'(0)|\right)}{8}+\frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{48}, \qquad (3.15)\\ &\frac{\left|3(\alpha-1)(1+2\delta)^n-(\alpha-1)(1+\delta)^{2n}\right|\left|h''(0)\right|+(\alpha-1)(1+\delta)^{2n}\left|p''(0)\right|}{24}+\frac{\alpha+1}{2}\Bigg\}.\end{aligned}$$

Corollary 3.5. *If we set*

$$h(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+\ldots \qquad (0<\gamma\le1),$$

$$p(z)=\left(\frac{1-z}{1+z}\right)^{\gamma}=1-2\gamma z+2\gamma^2z^2+\ldots \qquad (0<\gamma\le1),$$

then inequalities (3.14) and (3.15) become

$$|a_2|\le\min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4}},\sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2}}\right\}$$

[DOI: 10.52547/ijmsi.17.1.207]

and

$$|a_3| \leq \min\left\{ \frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{4} + \frac{(\alpha-1)\gamma^2}{6}, \right.$$
$$\left. \frac{\gamma^2(\alpha-1)+(\alpha+1)}{2} \right\}.$$

Corollary 3.6. *If we let*

$$h(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0 \leq \beta < 1),$$

$$p(z) = \frac{1-(1-2\beta)z}{1+z} = 1-2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0 \leq \beta < 1),$$

then inequalities (3.14) and (3.15) become

$$|a_2| \leq \min\left\{ \sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{4}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2}} \right\}$$

and

$$|a_3| \leq \min\left\{ \frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2\left(\alpha^2-1\right)(1-\beta)}{4} + \frac{(\alpha-1)(1-\beta)}{6}, \right.$$
$$\left. \frac{(1-\beta)(\alpha-1)+(\alpha+1)}{2} \right\}.$$

Theorem 3.7. *Letf given by (1.1) be in the class $D^n_\Sigma C_0(\alpha)$. Then*

$$|a_2| \leq \min\left\{ \sqrt{\frac{(\alpha+1)^2}{2^{2k+2}} + \frac{(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{2^{2k+5}} + \frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{2^{2k+3}}}, \right.$$
$$\left. \sqrt{\frac{(\alpha-1)\left(|h''(0)|+|p''(0)|\right)}{16(3^{k+1}-2^{2k+1})} + \frac{(\alpha+1)}{2(3^{k+1}-2^{2k+1})}} \right\} \tag{3.16}$$

and

$$|a_3| \leq \min\left\{ \frac{8(\alpha+1)^2+(\alpha-1)^2\left(\left|h'(0)\right|^2+\left|p'(0)\right|^2\right)}{2^{2k+5}} + \frac{\left(\alpha^2-1\right)\left(\left|h'(0)\right|+\left|p'(0)\right|\right)}{2^{2k+3}} + \frac{(\alpha-1)\left(\left|h''(0)\right|+\left|p''(0)\right|\right)}{16.3^{k+1}}, \right.$$
$$\left. \frac{(\alpha-1)\left(3^{k+1}-2^{2k}\right)\left|h''(0)\right|+(\alpha-1)2^{2k}\left|p''(0)\right|}{3.2^{2k+3}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})} \right\}. \tag{3.17}$$

Corollary 3.8. *If we set*

$$h(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1+2\gamma z+2\gamma^2z^2+\ldots \qquad (0 < \gamma \leq 1),$$

$$p(z) = \left(\frac{1-z}{1+z}\right)^\gamma = 1-2\gamma z+2\gamma^2z^2+\ldots \qquad (0 < \gamma \leq 1),$$

[DOI: 10.52547/ijmsi.17.1.207]

then inequalities (3.16) and (3.17) become

$$|a_2| \le \min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{2^{2k+2}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)\gamma^2}{2(3^{k+1}-2^{2k+1})}}\right\}$$

and

$$\begin{aligned} |a_3| \quad \le \quad & \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2\gamma^2+2\left(\alpha^2-1\right)\gamma}{2^{2k+2}} + \frac{(\alpha-1)\gamma^2}{2.3^{k+1}},\right. \\ & \left.\frac{(\alpha-1)\left(3^{k+1}-2^{2k}\right)\gamma^2+(\alpha-1)2^{2k}\gamma^2}{3.2^{2k+1}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})}\right\}. \end{aligned}$$

Corollary 3.9. *If we let*

$$h(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0 \le \beta < 1),$$

$$p(z) = \frac{1-(1-2\beta)z}{1+z} = 1-2(1-\beta)z+2(1-\beta)z^2+\cdots \qquad (0 \le \beta < 1),$$

then inequalities (3.16) and (3.17) become

$$|a_2| \le \min\left\{\sqrt{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2(\alpha^2-1)(1-\beta)}{2^{2k+2}}}, \sqrt{\frac{(\alpha+1)+(\alpha-1)(1-\beta)}{2(3^{k+1}-2^{2k+1})}}\right\}$$

and

$$\begin{aligned} |a_3| \quad \le \quad & \min\left\{\frac{(\alpha+1)^2+(\alpha-1)^2(1-\beta)^2+2\left(\alpha^2-1\right)(1-\beta)}{2^{2k+2}} + \frac{(\alpha-1)(1-\beta)}{2.3^{k+1}},\right. \\ & \left.\frac{(\alpha-1)\left(3^{k+1}-2^{2k}\right)(1-\beta)+(\alpha-1)2^{2k}(1-\beta)}{3.2^{2k+1}(3^{k+1}-2^{2k+1})} + \frac{\alpha+1}{2(3^{k+1}-2^{2k+1})}\right\}. \end{aligned}$$

Acknowledgments

The authors would like to thank the referee for useful and helpful comments and suggestions.

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[DOI: 10.52547/ijmsi.17.1.207]

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