

## On Bernstein Type Inequalities for Complex Polynomial

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ABSTRACT. In this paper, we establish some Bernstein type inequalities for the complex polynomial. Our results constitute generalizations and refinements of some well-known polynomial inequalities.

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### 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $p$  be a polynomial of degree at most  $n$ . Then, according to a famous result known as Bernsteins inequality [8]

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \quad (1.1)$$

whereas concerning the maximum modulus of  $p$  on a large circle  $|z| = R > 1$ , we have [20]

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \quad (1.2)$$

If we restrict ourselves to the class of polynomials having no zeros in  $|z| < 1$ , then inequalities (1.1) and (1.2) can be sharpened. In fact, if  $p(z) \neq 0$  in  $|z| < 1$ , then (1.1) and (1.2) can respectively be replaced by

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$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \quad (1.3)$$

and

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|, \quad R > 1. \quad (1.4)$$

Inequality (1.3) was conjectured by Erdős and later verified by Lax [19], whereas Ankeny and Rivlin [5] used (1.3) to prove (1.4).

In the literature, there are already various generalizations and refinements of (1.3) and (1.4), for example (see Aziz [6], Bidkham et al. [9, 10, 11], Khojastehnezhad and Bidkham [17], Zireh [21], etc).

Inequalities (1.3) and (1.4) were sharpened by Dewan et.al [12, 13] proving that under the same hypothesis, for every real or complex number  $\beta$  with  $|\beta| \leq 1$ ,  $R > 1$  and  $|z| = 1$ , we have

$$|zp'(z) + \frac{n\beta}{2}p(z)| \leq \frac{n}{2} \{ (|1 + \frac{\beta}{2}| + |\frac{\beta}{2}|) \max_{|z|=1} |p(z)| - (|1 + \frac{\beta}{2}| - |\frac{\beta}{2}|) \min_{|z|=1} |p(z)| \}, \quad (1.5)$$

and

$$|p(Rz) + \beta(\frac{R+1}{2})^n p(z)| \leq \frac{1}{2} \{ (|R^n + \beta(\frac{R+1}{2})^n| + |1 + \beta(\frac{R+1}{2})^n|) \max_{|z|=1} |p(z)| - (|R^n + \beta(\frac{R+1}{2})^n| - |1 + \beta(\frac{R+1}{2})^n|) \min_{|z|=1} |p(z)| \}. \quad (1.6)$$

Also they [12] proved if  $p$  has all its zeros in  $|z| \leq 1$ , then for every real or complex number  $\beta$  with  $|\beta| \leq 1$ , we have

$$\min_{|z|=1} |zp'(z) + \frac{n\beta}{2}p(z)| \geq n|1 + \frac{\beta}{2}| \min_{|z|=1} |p(z)|. \quad (1.7)$$

In this paper, we first prove an interesting result which is a compact generalization of inequality (1.7).

**Theorem 1.1.** *If  $p$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ ,  $k > 0$ , then for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1$ ,  $|\beta| \leq 1$ ,  $R \geq r$ ,  $rR \geq k^2$  and  $|z| = 1$ , we have*

$$|p(Rz) - \alpha p(rz) + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} p(rz)| \geq \frac{1}{k^n} |R^n - \alpha r^n + \beta \{ (\frac{R+k}{r+k})^n - |\alpha| \} r^n| \min_{|z|=k} |p(z)|. \quad (1.8)$$

Assuming  $\alpha = 1$  in Theorem 1.1, we have the following result.

**Corollary 1.2.** *Let  $p$  be a polynomial of degree  $n$  such that does not vanish in  $|z| > k$ ,  $k > 0$ , then for all  $\beta \in \mathbb{C}$  with  $|\beta| \leq 1$ ,  $R > r$ ,  $rR \geq k^2$  and  $|z| = 1$ , we get*

$$|p(Rz) - p(rz) + \beta\{(\frac{R+k}{r+k})^n - 1\}p(rz)| \geq \frac{1}{k^n}|R^n - r^n + \beta\{(\frac{R+k}{r+k})^n - 1\}r^n| \min_{|z|=k} |p(z)|. \tag{1.9}$$

By dividing the two sides of the inequality (1.9) by  $(R-r)$  and letting  $R \rightarrow r$ , we get the following interesting result.

**Corollary 1.3.** *Let  $p$  be a polynomial of degree  $n$  such that does not vanish in  $|z| > k$ ,  $k > 0$ . Then for all  $\beta \in \mathbb{C}$  with  $|\beta| \leq 1$ ,  $r \geq k$  and  $|z| = 1$ , we get*

$$|zp'(rz) + \frac{n\beta}{r+k}p(rz)| \geq \frac{n}{k^n}|r^{n-1} + \frac{\beta}{r+k}r^n| \min_{|z|=k} |p(z)|. \tag{1.10}$$

Assuming  $k = 1$ ,  $r = 1$  in Corollary (1.3), we have the inequality (1.7). Using Theorem 1.1, we prove the following theorem, which provides a compact generalization of inequalities (1.5), (1.6).

**Theorem 1.4.** *Let  $p$  be a polynomial of degree  $n$  such that it does not vanish in  $|z| < k$ ,  $k > 0$ . Then for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1$ ,  $|\beta| \leq 1$ ,  $R \geq r$ ,  $rR \geq \frac{1}{k^2}$  and  $|z| = 1$ ,*

$$|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \frac{1}{2}\{[k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \max_{|z|=k} |p(z)| - [k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| - |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \min_{|z|=k} |p(z)|\}. \tag{1.11}$$

Equality holds for the polynomials  $az^n + bk^n$ ,  $|a| = |b|$ .

Assuming  $\alpha = 1$  in Theorem 1.4, we have the following result.

**Corollary 1.5.** *Let  $p$  be a polynomial of degree  $n$  such that does not vanish in  $|z| < k$ ,  $k > 0$ . Then for every  $\beta \in \mathbb{C}$  with  $|\beta| \leq 1$ ,  $R \geq r$ ,  $rR \geq \frac{1}{k^2}$  and  $|z| = 1$ , we get*

$$|p(Rk^2z) - p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}p(rk^2z)| \leq \frac{1}{2}\{ [k^n|R^n - r^n + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}r^n| + |\beta\{(\frac{Rk+1}{rk+1})^n - 1\}| \} \max_{|z|=k} |p(z)| - [k^n|R^n - r^n + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}r^n| - |\beta\{(\frac{Rk+1}{rk+1})^n - 1\}| \} \min_{|z|=k} |p(z)|\}. \tag{1.12}$$

By dividing the two sides of the inequality (1.12) by  $(R - r)$  and letting  $R \rightarrow r$ , we get the following interesting result.

**Corollary 1.6.** *Let  $p$  be a polynomial of degree  $n$  such that does not vanish in  $|z| < k$ ,  $k > 0$ . Then for every  $\beta \in \mathbb{C}$  with  $|\beta| \leq 1$ ,  $r \geq \frac{1}{k}$  and  $|z| = 1$ , we have*

$$|k^2 zp'(rk^2 z) + \frac{n\beta k}{rk+1} p(rk^2 z)| \leq \frac{n}{2} \left\{ [k^n |r^{n-1} + \frac{\beta k}{rk+1} r^n| + |\frac{\beta k}{rk+1}|] \max_{|z|=k} |p(z)| - [k^n |r^{n-1} + \frac{\beta k}{rk+1} r^n| - |\frac{\beta k}{rk+1}|] \min_{|z|=k} |p(z)| \right\}. \quad (1.13)$$

*Remark 1.7.* Assuming  $k = 1$  and  $r = 1$  in Corollary 1.6 we have the inequality (1.5).

## 2. LEMMAS

To prove of these theorems, we need the following lemmas. The first lemma is due to Aziz and Zargar [7].

**Lemma 2.1.** *Let  $p$  be a polynomial of degree  $n$  having all its zeros in  $|z| \leq k$ ,  $k > 0$ . Then for every  $R \geq r$  and  $rR \geq k^2$ , we have*

$$|p(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |p(rz)|, \quad |z| = 1. \quad (2.1)$$

**Lemma 2.2.** *Let  $p$  be a polynomial of degree  $n$  such that does not vanish in  $|z| < k$ ,  $k > 0$ , and  $q(z) = z^n p(\frac{1}{z})$ . Then for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1$ ,  $|\beta| \leq 1$ ,  $R \geq r$ ,  $rR \geq \frac{1}{k^2}$  and  $|z| = 1$ , we have*

$$|p(Rk^2 z) - \alpha p(rk^2 z) + \beta \left\{ \left(\frac{Rk+1}{rk+1}\right)^n - |\alpha| \right\} p(rk^2 z)| \leq k^n |q(Rz) - \alpha q(rz) + \beta \left\{ \left(\frac{Rk+1}{rk+1}\right)^n - |\alpha| \right\} q(rz)|. \quad (2.2)$$

*Proof.* Based on the hypotheses that the polynomial  $p$  has no zeros in  $|z| < k$ , therefore the polynomial  $q(z) = z^n p(\frac{1}{z})$  has all its zeros in  $|z| \leq \frac{1}{k}$ . Since  $\frac{1}{k^n} |p(k^2 z)| = |q(z)|$  for  $|z| = \frac{1}{k}$ , therefore the function  $\phi(z) = \frac{p(k^2 z)}{k^n q(z)}$  is analytic in the disc  $|z| \geq \frac{1}{k}$  and  $|\phi(z)| = 1$  on  $|z| = \frac{1}{k}$ . Hence based on the the maximum modulus principle  $|\phi(z)| < 1$  for  $|z| > \frac{1}{k}$ , or equivalently

$$|p(k^2 z)| \leq k^n |q(z)|, \quad |z| \geq \frac{1}{k}. \quad (2.3)$$

Since  $\frac{1}{k^n} |p(k^2 z)| = |q(z)|$  for  $|z| = \frac{1}{k}$ , therefore for every real or complex number  $\delta$  with  $|\delta| < 1$  and  $|z| = \frac{1}{k}$ ,  $|\delta p(k^2 z)| < |k^n q(z)|$ . Now using Rouché's theorem it follows that all the zeros of  $H(z) := k^n q(z) + \delta p(k^2 z)$  lie in  $|z| \leq \frac{1}{k}$ . While applying Lemma 2.1, we have

$$|H(Rz)| \geq \left(\frac{Rk+1}{rk+1}\right)^n |H(rz)| > |H(rz)|, \quad |z| = 1, \quad (2.4)$$

where  $R > r$ ,  $rR \geq \frac{1}{k^2}$ .

It follows that for every  $\alpha \in \mathbb{C}$  with  $|\alpha| \leq 1$ , we get

$$|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha| |H(rz)| \geq \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} |H(rz)|, \quad |z| = 1$$

i.e.

$$|H(Rz) - \alpha H(rz)| \geq \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} |H(rz)| \quad \text{for } |z| = 1. \quad (2.5)$$

Since  $H(Rz)$  has all its zeros in  $|z| \leq \frac{1}{Rk} < 1$ , and  $|H(rz)| < |H(Rz)|$  for  $|z| = 1$ , a direct application of Rouché's theorem shows that the polynomial  $H(Rz) - \alpha H(rz)$  has all its zeros in  $|z| < 1$ . Using Rouché's theorem again, it follows that for every  $\beta \in \mathbb{C}$  with  $|\beta| < 1$  and  $R > r$ ,  $rR \geq \frac{1}{k^2}$ , all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} H(rz)$$

lie in  $|z| < 1$ .

Replacing  $H(z)$  by  $k^n q(z) + \delta p(k^2 z)$ , we conclude that all the zeros of

$$T(z) = k^n [q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz)] + \delta \{ p(Rk^2 z) - \alpha p(rk^2 z) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2 z) \} \quad (2.6)$$

lie in  $|z| < 1$ , for every  $R > r$ ,  $rR \geq \frac{1}{k^2}$ ,  $|\alpha| \leq 1$ ,  $|\beta| < 1$  and  $|\delta| < 1$ . We now show that (2.6) implies (2.2). Indeed, suppose otherwise. Then, there is a point  $z = z_0$  with  $|z_0| = 1$  such that

$$|p(Rk^2 z_0) - \alpha p(rk^2 z_0) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2 z_0)| > k^n |q(Rz_0) - \alpha q(rz_0) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz_0)|.$$

We take

$$\delta = - \frac{k^n [q(Rz_0) - \alpha q(rz_0) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz_0)]}{p(Rk^2 z_0) - \alpha p(rk^2 z_0) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2 z_0)},$$

then  $|\delta| < 1$  and with this choice of  $\delta$ , we have,  $T(z_0) = 0$  for  $|z_0| = 1$ . But this contradicts that  $T$  has all its zeros in  $|z| < 1$ . For the case  $\beta$ , with  $|\beta| = 1$ , (2.2) follows by continuity. For  $R = r$  inequality (2.2) follows by inequality (2.3). This completes the proof of Lemma 2.2.  $\square$

**Lemma 2.3.** *Let  $p$  be a polynomial of degree  $n$ . Then for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1, |\beta| \leq 1, R \geq r, rR \geq k^2, k > 0$  and  $|z| = 1$ , we have*

$$|p(Rz) - \alpha p(rz) + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} p(rz)| \leq \frac{1}{k^n} |R^n - \alpha r^n + \beta \left\{ \left( \frac{R+k}{r+k} \right)^n - |\alpha| \right\} r^n| \max_{|z|=k} |p(z)|. \quad (2.7)$$

*Proof.* Let  $M = \max_{|z|=k} |p(z)|$ , then for  $\delta$  with  $|\delta| > 1$ , we can conclude from Rouché's theorem that all zeros of polynomial  $H(z) = p(z) - \delta M (\frac{z}{k})^n$  lie in the closed disk  $|z| \leq k$ ,  $k > 0$ . Using Lemma 2.1, we have

$$|H(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |H(rz)| > |H(rz)|, \quad |z| = 1, \quad (2.8)$$

where  $R > r$ ,  $rR \geq k^2$ .

It follows that for every  $\alpha \in \mathbb{C}$  with  $|\alpha| \leq 1$ , we get

$$|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha| |H(rz)| \geq \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} |H(rz)|, \quad |z| = 1,$$

i.e.

$$|H(Rz) - \alpha H(rz)| \geq \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} |H(rz)|, \quad |z| = 1. \quad (2.9)$$

Since  $H(Rz)$  has all its zeros in  $|z| \leq \frac{k}{R} < 1$ , and  $|H(rz)| < |H(Rz)|$ , a direct application of Rouché's theorem shows that the polynomial  $H(Rz) - \alpha H(rz)$  has all its zeros in  $|z| < 1$ . Using Rouché's theorem again, implies that for every  $\beta \in \mathbb{C}$  with  $|\beta| < 1$  and  $R > r$ ,  $rR \geq k^2$ , all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} H(rz)$$

lie in  $|z| < 1$ .

Replacing  $H(z)$  by  $p(z) - \delta M (\frac{z}{k})^n$ , we conclude that all the zeros of

$$\begin{aligned} T(z) = & [p(Rz) - \alpha p(rz) + \beta \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} p(rz)] + \\ & \delta \frac{Mz^n}{k^n} \{R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} r^n\} \end{aligned} \quad (2.10)$$

lie in  $|z| < 1$ , for every  $R > r$ ,  $rR \geq k^2$ ,  $|\alpha| \leq 1$ ,  $|\beta| < 1$  and  $|\delta| > 1$ . This implies

$$\begin{aligned} |p(Rz) - \alpha p(rz) + \beta \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} p(rz)| \leq \\ |R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{r+k}\right)^n - |\alpha| \right\} r^n| \frac{M}{k^n}, \end{aligned} \quad (2.11)$$

where  $|z| = 1$ .

For  $\beta$ , with  $|\beta| = 1$ , (2.11) follows by continuity. For  $R = r$  inequality (2.11) reduces to  $|p(rz)| \leq \frac{r^n}{k^n} \max_{|z|=k} |p(z)|$  which it follows by taking  $p(kz)$  and  $|z| = \frac{r}{k}$  where  $\frac{r}{k} \geq 1$  in inequality (1.2). This completes the proof of Lemma 2.3.  $\square$

**Lemma 2.4.** *If  $p$  is a polynomial of degree  $n$ , then for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1, |\beta| \leq 1, R \geq r, rR \geq \frac{1}{k^2}$  and  $|z| = 1$ ,*

$$\begin{aligned} &|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| + \\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq \\ &\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}\} \max_{|z|=k} |p(z)|, \end{aligned} \tag{2.12}$$

where  $q(z) = z^n \overline{p(1/\bar{z})}$ .

*Proof.* Assume that  $M = \max_{|z|=k} |p(z)|$ . Then, for  $\delta$  with  $|\delta| > 1$ , we can conclude from Rouché's theorem that the polynomial  $G(z) = p(z) - \delta M$  does not vanish in  $|z| < k$ . If we take  $H(z) = z^n \overline{G(1/\bar{z})}$ , then  $|G(k^2z)| = k^n |H(z)|$  for  $|z| = \frac{1}{k}$ . Using Lemma 2.2, for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1, |\beta| \leq 1, R \geq r, rR \geq \frac{1}{k^2}$  and  $|z| = 1$ , we have

$$\begin{aligned} &|G(Rk^2z) - \alpha G(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}G(rk^2z)| \leq \\ &k^n |H(Rz) - \alpha H(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}H(rz)|. \end{aligned} \tag{2.13}$$

Therefore, by using the equality

$$\begin{aligned} H(z) &= z^n \overline{G(\frac{1}{\bar{z}})} = z^n \overline{p(\frac{1}{\bar{z}}) - \delta M z^n} \\ &= q(z) - \bar{\delta} M z^n, \end{aligned}$$

we get

$$\begin{aligned} &|\{p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)\} - \\ &\delta\{1 - \alpha + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}\}M| \leq \\ &k^n |\{q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)\} - \\ &\bar{\delta}\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}M|. \end{aligned} \tag{2.14}$$

Since  $\frac{1}{k^n} |p(k^2z)| = |q(z)|$  for  $|z| = \frac{1}{k}$ , therefore

$$\begin{aligned} \max_{|z|=\frac{1}{k}} |q(z)| &= \frac{1}{k^n} \max_{|z|=k} |p(z)|, \\ \max_{|z|=\frac{1}{k}} |q(z)| &= \frac{M}{k^n}. \end{aligned}$$

Now by applying Lemma 2.3 to  $q(z)$  for  $\frac{1}{k} > 0$ , we have

$$|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)| \leq \\ \{R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n\}k^n \max_{|z|=\frac{1}{k}} |q(z)|.$$

i.e.

$$|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)| \leq \\ \{R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n\}M.$$

Now by suitable choice of argument of  $\delta$ , we get

$$|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)| - \\ \delta\{R^n - \alpha r^n + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}r^n\}M = \\ |\delta|\{R^n - \alpha r^n + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\}r^n\}M - \\ |q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)|. \quad (2.15)$$

Combining right hand sides of (2.14) and (2.15) we can obtain

$$|p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)| - \\ |\delta|\{1 - \alpha + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}\}M \leq \\ |\delta|\{k^n|R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n\}M| - \\ k^n|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)|,$$

which implies

$$|p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)| + \\ k^n|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)| \leq \\ |\delta|\{k^n|R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n\} + |1 - \alpha + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}\}M.$$

Making  $|\delta| \rightarrow 1$ , we have the result.  $\square$

**Lemma 2.5.** *Let  $p$  be a polynomial of degree  $n$  having no zeros in  $|z| < k$ ,  $k > 0$ . Then for all  $\alpha, \beta \in \mathbb{C}$  with  $|\alpha| \leq 1$ ,  $|\beta| \leq 1$ ,  $R \geq r$ ,  $Rr \geq \frac{1}{k^2}$  and  $|z| = 1$ ,*

we have

$$\begin{aligned}
 &|p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)| \leq \frac{1}{2} \\
 &\{k^n|R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n| + |1 - \alpha + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}|\} \max_{|z|=k} |p(z)|
 \end{aligned}
 \tag{2.16}$$

*Proof.* Since  $p$  does not vanish in  $|z| < k$ ,  $k > 0$ , Lemma 2.2, yields

$$\begin{aligned}
 &|p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)| \leq \\
 &k^n|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)|,
 \end{aligned}
 \tag{2.17}$$

Now by combining the inequalities (2.12) and (2.17), we have

$$\begin{aligned}
 &2|p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)| \leq \\
 &|p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)| + \\
 &k^n|q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)| \leq \\
 &\{k^n|R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n| + |1 - \alpha + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}|\} \max_{|z|=k} |p(z)|.
 \end{aligned}
 \tag{2.18}$$

This gives the result. □

### 3. PROOFS OF THE THEOREMS

**Proof of Theorem 1.1.** If  $p$  has a zero on  $|z| = k$ , then inequality is trivial. Therefore, we assume that  $p(z)$  has all its zeros in  $|z| < k$ . If  $m = \min_{|z|=k} |p(z)|$ , then  $m > 0$  and  $|p(z)| \geq m$  for  $|z| = k$ . If  $|\lambda| < 1$ , then it follows by Rouché's theorem that the polynomial  $p(z) - \lambda m \left(\frac{z}{k}\right)^n$ , has all its zeros in  $|z| < k$ ,  $k > 0$ . Proceeding similarly as in the proof of Lemma 2.3, it follows that all the zeros of

$$\begin{aligned}
 &p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz) + \\
 &\lambda m \left(\frac{z}{k}\right)^n \{R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n\}
 \end{aligned}
 \tag{3.1}$$

lie in  $|z| < 1$ , for every  $R \geq r$ ,  $Rr \geq k^2$ ,  $|\alpha| \leq 1$ ,  $|\beta| < 1$  and  $|\lambda| < 1$ . This implies

$$\begin{aligned}
 &\frac{m}{k^n} |R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}r^n| \leq \\
 &|p(Rz) - \alpha p(rz) + \beta\left\{\left(\frac{R+k}{r+k}\right)^n - |\alpha|\right\}p(rz)|,
 \end{aligned}
 \tag{3.2}$$

where  $|z| = 1$ . This completes the proof.  $\square$

**Proof of Theorem 1.4.** If  $p(z)$  has a zero on  $|z| = k$ , then  $\min_{|z|=k} |p(z)| = 0$  and in this case the result follows from Lemma 2.5. Hence we assume that  $p(z) \neq 0$  in  $|z| \leq k$ . In this case we have  $m = \min_{|z|=k} |p(z)| > 0$  and for  $\gamma$  with  $|\gamma| < 1$ , we get  $|\gamma m| < m \leq |p(z)|$ , where  $|z| = k$ . Now we conclude from Rouché's theorem that the polynomial  $G(z) = p(z) - \gamma m$  does not vanish in  $|z| < k$ . If we take  $H(z) = z^n \overline{G(1/\bar{z})}$ , then by using the polynomials  $G(z)$  and  $H(z)$  in Lemma 2.2, we have

$$\begin{aligned} |G(Rk^2z) - \alpha G(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}G(rk^2z)| \leq \\ k^n |H(Rz) - \alpha H(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}H(rz)|. \end{aligned} \quad (3.3)$$

Using the fact that

$$H(z) = z^n \overline{G\left(\frac{1}{\bar{z}}\right)} = z^n \overline{p\left(\frac{1}{\bar{z}}\right)} - \bar{\gamma} m z^n = q(z) - \bar{\gamma} m z^n,$$

or

$$H(z) = q(z) - \bar{\gamma} m z^n,$$

and substituting  $G(z)$  and  $H(z)$  in (3.3), we get

$$\begin{aligned} & |\{p(Rk^2z) - \alpha p(rk^2z) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}p(rk^2z)\} - \\ & \gamma\{1 - \alpha + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}m\}| \leq \\ & k^n |\{q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)\} - \\ & \bar{\gamma}\{R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n\}mz^n|. \end{aligned} \quad (3.4)$$

Since the polynomial  $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$  has all zeros in  $|z| \leq \frac{1}{k}$  and  $m = \min_{|z|=k} |p(z)| = k^n \min_{|z|=\frac{1}{k}} |q(z)|$ , hence by applying Theorem 1.1 for the polynomial  $q(z)$  with  $\frac{1}{k}$ , we obtain

$$\begin{aligned} |R^n - \alpha r^n + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}r^n| m \leq \\ |q(Rz) - \alpha q(rz) + \beta\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\alpha|\right\}q(rz)|. \end{aligned}$$

Therefore, by suitable choice of argument of  $\gamma$ , we get

$$\begin{aligned} & \left| \{q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz) \} - \right. \\ & \quad \left. \bar{\gamma} \{ R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} r^n m \} \right| = \\ & \left| q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz) \right| - \\ & \quad |\gamma| |R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} r^n m|. \end{aligned} \tag{3.5}$$

Now combining (3.4) and (3.5), we get

$$\begin{aligned} & \left| p(Rk^2z) - \alpha p(rk^2z) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2z) \right| - \\ & \quad |\gamma| |1 - \alpha + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} m| \leq \\ & k^n \left| q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz) \right| - \\ & \quad |\gamma| |k^n |R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} r^n m|. \end{aligned}$$

This implies

$$\begin{aligned} & \left| p(Rk^2z) - \alpha p(rk^2z) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2z) \right| \leq \\ & k^n \left| q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz) \right| - \\ & |\gamma| \left\{ |k^n |R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} r^n m| - |1 - \alpha + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} m| \right\}. \end{aligned}$$

Letting  $|\gamma| \rightarrow 1$ , we have

$$\begin{aligned} & \left| p(Rk^2z) - \alpha p(rk^2z) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2z) \right| \leq \\ & k^n \left| q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz) \right| - \\ & \left\{ |k^n |R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} r^n m| - |1 - \alpha + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} m| \right\}. \end{aligned} \tag{3.6}$$

On the other hand, based on Lemma 2.4, we have

$$\begin{aligned} & \left| p(Rk^2z) - \alpha p(rk^2z) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} p(rk^2z) \right| + \\ & k^n \left| q(Rz) - \alpha q(rz) + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} q(rz) \right| \leq \\ & \left\{ |k^n |R^n - \alpha r^n + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} r^n m| + |1 - \alpha + \beta \left\{ \left( \frac{Rk+1}{rk+1} \right)^n - |\alpha| \right\} m| \right\} \max_{|z|=k} |p(z)|. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we get (1.11) and this completes the proof of Theorem 1.4.  $\square$

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