

## On the WZ Factorization of the Real and Integer Matrices

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**ABSTRACT.** The *QIF* (Quadrant Interlocking Factorization) method of Evans and Hatzopoulos solves linear equation systems using *WZ* factorization. The *WZ* factorization can be faster than the *LU* factorization because, it performs the simultaneous evaluation of two columns or two rows. Here, we present a method for computing the real and integer *WZ* and *ZW* factorizations by using the null space generators of some special nested submatrices of a matrix  $A$ .

**Keywords:** Linear systems, Quadrant interlocking factorization, *WZ* factorization, *ZW* factorization, Null space generator.

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### 1. INTRODUCTION

Linear systems arise frequently in scientific and engineering computing. Various serial and parallel algorithms have been introduced for their serial solution [9, 4]. The *QIF* (Quadrant Interlocking Factorization) algorithm, introduced by Evans and Hatzopoulos, is a numerical method for finding a solution for systems of the type  $Ax = b$ , where  $A$  is a nonsingular matrix of dimensions

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$n \times n$ ,  $x$  is an unknown column vector, and  $b$  is the independent term vector provided. The *QIF* method is based on the *WZ* factorization of the coefficient matrix  $A$ . The main advantage of this factorization is that it presents a complexity order less than of the *LU* decomposition due to the fact that it performs the simultaneous evaluation of two columns or two rows. A detailed description of this algorithm for real and complex matrices can be found in [4, 5, 10, 11]. Golpar-Raboky and Mahdavi-Amiri presented new algorithms for computing the real and integer *WZ* and *ZW* matrix factorizations using *ABS* algorithms and the extended rank reduction process [6, 7, 8, 16]. Recently, some authors have considered simultaneous matrix decompositions [12, 13, 14].

The *WZ* factorization is used for solving Markovian linear systems [2] and network modeling [3], preconditioning of sparse matrices [18] and eigenvalue problems [15].

Let  $\mathbb{R}$  and  $\mathbb{R}^{m \times n}$  stand for the real number, and the set of all  $m \times n$  matrices over  $\mathbb{R}$  and  $A^T$  denotes the transpose of  $A$ . Let  $A = (a_1, \dots, a_m)^T \in \mathbb{R}^{m \times n}$ . Assume that  $a_{k_1}^T, \dots, a_{k_i}^T$  be the rows of  $A$  and  $H_1 \in \mathbb{R}^{n \times n}$  be an arbitrary nonsingular matrix. For  $j = 1, \dots, i$  update  $H_j$  by

$$H_{j+1} = H_j - \frac{H_j a_{k_j} w_j^T H_j}{w_j^T H_j a_{k_j}}, \quad (1.1)$$

where  $w_j \in \mathbb{R}^n$  such that  $w_j^T H_j a_{k_j} \neq 0$ . Then, we have

$$a_{k_i}^T H_{j+1}^T = 0, \quad i = 1, \dots, j, \quad (1.2)$$

and the linear combination of the columns of  $H_{i+1}^T$  generates the null space of  $\{a_{k_1}, \dots, a_{k_i}\}$  (see [1]).

Matrices  $H_i$  are generalizations of (oblique) projection matrices. They probably first appeared in a book by Wedderburn [19]. They have been named ***Abafians*** since the First International Conference on ABS methods (Luoyang, China, 1991) and this name will be used here.

**Notation:** Let  $A \in \mathbb{R}^{n \times n}$ . Here and subsequently  $J_n = \{j_1, \dots, j_n\}$  denotes a permutation of  $\mathcal{I}_n = \{1, 2, \dots, n\}$  and, for  $k = 1, \dots, n$ ,  $J_k = \{j_1, \dots, j_k\}$  denotes a subset of  $J_n$ . Let

$$A_{J_k} = (a_{i,j}), \quad i, j \in J_k. \quad (1.3)$$

denotes a submatrix of  $A$ , and

$$J_1 \subset J_2 \subset \dots \subset J_n, \quad (1.4)$$

and  $\{A_{J_k}\}_{k=1}^n$  be a sequence of nested submatrices of  $A$ . The following theorem describes a necessary and sufficient condition for nonsingularity  $A_{J_k}$ ,  $k = 1, \dots, n$ .

**Theorem 1.1.** (*Nested submatrices*) Let  $A \in \mathbb{R}^{n \times n}$  and  $H_1 = I$ . Then the nested submatrices  $A_{J_i}$ ,  $i = 1, \dots, n$ , are nonsingular if and only if  $e_{j_i}^T H_i a_{j_i} \neq 0$ ,  $i = 1, \dots, n$ .

*Proof.* Follow the lines of the proof for Theorem 6.5 in [1] by replacing  $i$  to  $j_i$ .  $\square$

From (1.1) and Theorem 1.1 we have the following result.

**Theorem 1.2.** Let  $A \in \mathbb{R}^{n \times n}$ ,  $H_1 = I$ , and for  $i = 1, \dots, n$ ,  $e_{j_i}^T A H_i e_{j_i} \neq 0$ . Then,

$$H_{i+1} = H_i - \frac{H_i a_{j_i} e_{j_i}^T H_i}{e_{j_i}^T H_i a_{j_i}}, \quad (1.5)$$

is well defined.

The parameter choices in Theorem 1.2, induce a structure in the matrix  $H_i$ , described by the following theorem.

**Theorem 1.3.** Let the conditions of Theorem 1.2 be satisfied and  $H_{i+1}$  defined by (1.5). Then, the following properties hold:

- (a) The  $j$ th row of  $H_{i+1}$  is zero, for  $j \in J_i$ .
- (b) The  $j$ th column of  $H_{i+1}$  is equal to the  $j$ th column of  $H_1$ , for  $j \notin J_i$ .

*Proof.* See Theorem 6.3 in [1].  $\square$

In this paper we present new algorithms for computing the  $WZ$  and  $ZW$  factorizations using null space of special submatrices of the matrix  $A$ .

The structure of this paper is organized as follows. In Section 2, we discuss our proposed algorithm for the  $WZ$  factorization of a matrix  $A$  by using null space of special submatrices of  $A$ . In Section 3, we propose a new algorithm for computing the  $WZ$  and  $ZW$  factorizations. In Section 4, we report a numerical experiment. We conclude in Section 5.

## 2. THE $WZ$ FACTORIZATION

The  $WZ$  factorization is a parallel method for solving dense linear systems of the form

$$Ax = b, \quad (2.1)$$

where  $A$  is a square  $n \times n$  matrix, and  $b$  is an  $n$ -vector.

**Definition 2.1.** Let  $s$  be a real number and denote by  $\lfloor s \rfloor$  ( $\lceil s \rceil$ ), the greatest (least) integer less (greater) than or equal to  $s$ .

**Definition 2.2.** We say that a matrix  $A$  is factorized in the form  $WZ$  if

$$A = WZ, \quad (2.2)$$

where the matrices  $W$  and  $Z$  have the following structures:

$$W = \begin{pmatrix} * & 0 & \cdots & 0 & * \\ * & * & 0 & * & * \\ * & * & * & * & * \\ * & * & 0 & * & * \\ * & 0 & \cdots & 0 & * \end{pmatrix}, Z = \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & 0 \\ \vdots & 0 & * & 0 & \vdots \\ 0 & * & * & * & 0 \\ * & * & * & * & * \end{pmatrix} \quad (2.3)$$

where stars stand for possible nonzero entries.

The matrices  $W$  and  $Z$  have two zero opposite quadrants. Then, we refer to  $W$  and  $Z$  as the interlocking quadrant factors of  $A$ . The factorization is unique if  $W$  has 1's on the main diagonal and 0's on the cross diagonal entries (see [17]).

Now, we give a characterization for the existence of the  $WZ$  factorization of  $A$ .

**Theorem 2.1.** Let  $A \in \mathbb{R}^{n \times n}$  be a nonsingular matrix.  $A$  has quadrant interlocking factorization  $QIF$ ,  $A=WZ$ , if and only if for every  $k$ ,  $1 \leq k \leq s$ , where  $s = \lfloor n/2 \rfloor$  if  $n$  is even and  $s = \lceil n/2 \rceil$  if  $n$  is odd, the  $2k \times 2k$  submatrix

$$\Delta_k = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} & a_{1,n-k+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & a_{k,n-k+1} & \cdots & a_{k,n} \\ a_{n-k+1,1} & \cdots & a_{n-k+1,k} & a_{n-k+1,n-k+1} & \cdots & a_{n-k+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,k} & a_{n,n-k+1} & \cdots & a_{n,n} \end{pmatrix} \quad (2.4)$$

of  $A$  is invertible. Moreover, the factorization is unique.

*Proof.* See Theorem 2 in [17].  $\square$

If  $A \in \mathbb{R}^{n \times n}$  is nonsingular, then the  $WZ$  factorization with pivoting can always be carried out. Whenever  $\Delta_k$  is nonsingular, it is always possible to interchange the rows  $k \leq i \leq (n-k+1)$ . These row interchanges can be viewed in a matrix form as premultiplication by a permutation matrix. Thus, we have the following result.

**Theorem 2.2.** If  $A \in \mathbb{R}^{n \times n}$  is nonsingular, then the with pivoting  $WZ$  factorization can always be carried out, that is, a row permutation matrix  $P$  and the factors  $W$  and  $Z$  exist so that,  $PA = WZ$ .

*Proof.* See [17]. □

Let  $A \in \mathbb{R}^{n \times n}$  and there exists a WZ factorization without pivoting of  $A$ . Let,  $n$  be an even number. Here, we present a new algorithm for computing the WZ factorization of  $A$  using null space of the sequence submatrices

$$\Delta_1 \subset \Delta_2 \subset \dots \subset \Delta_{n/2}. \tag{2.5}$$

For  $k = 1, \dots, s$ , where  $s = \frac{n}{2}$  consider  $\Delta_k$  defined by (2.4). Let, the rows of  $H_k$  generate the null space of  $\Delta_k$  expect the  $k$ th and the  $(k+1)$ th rows. Let  $e_j \in \mathbb{R}^{2k}$  be the  $j$ th unit vector, then we have,

$$e_i^T \Delta_k H_k^T = 0, \quad i \neq k, k + 1, \tag{2.6}$$

and

$$(e_k^T, e_{k+1}^T) \Delta_k H_k^T \neq 0. \tag{2.7}$$

Therefore, there exist  $1 \leq j_1, j_2 \leq 2k$  such that,

$$\alpha_1 = e_{j_1}^T H_k \Delta_k^T e_k \neq 0, \alpha_2 = e_{j_2}^T H_k \Delta_k^T e_{k+1} \neq 0. \tag{2.8}$$

Let  $\mathcal{T}_k = (t_1, \dots, t_{2k}) = H_k^T e_{j_1} / \alpha_1 \in \mathbb{R}^{2k}$  and  $\mathcal{Y}_k = (y_1, \dots, y_{2k}) = H_k^T e_{j_2} / \alpha_2 \in \mathbb{R}^{2k}$ . Then, we have

$$\Delta_k \mathcal{T}_k = (\underbrace{0 \dots 0}_{k-1}, 1, \underbrace{0, \dots, 0}_k)^T, \quad \Delta_k \mathcal{Y}_k = (\underbrace{0 \dots 0}_k, 1, \underbrace{0, \dots, 0}_{k-1})^T. \tag{2.9}$$

Now, let

$$\bar{z}_k = (t_1, \dots, t_k, \underbrace{0, \dots, 0}_{n-2k}, t_{k+1}, \dots, t_{2k})^T, \tag{2.10}$$

and

$$\bar{z}_{n-k+1} = (y_1, \dots, y_k, \underbrace{0, \dots, 0}_{n-2k}, y_{k+1}, \dots, y_{2k})^T, \tag{2.11}$$

then, we have

$$w_k = A \bar{z}_k = (\underbrace{0, \dots, 0}_{k-1}, 1, w_{k+1,k}, \dots, w_{n-k,k}, \underbrace{0, \dots, 0}_k)^T \tag{2.12}$$

and

$$w_{n-k+1} = A \bar{z}_{n-k+1} = (\underbrace{0, \dots, 0}_k, w_{k+1,n-k+1}, \dots, w_{n-k,n-k+1}, 1, \underbrace{0, \dots, 0}_{k-1})^T. \tag{2.13}$$

$$\tilde{Z} = (\tilde{z}_1, \dots, \tilde{z}_n), \quad W = (w_1, \dots, w_n), \quad (2.14)$$

then, we have,

$$A\bar{Z} = W \Rightarrow A = WZ, \quad Z = \bar{Z}^{-1}.$$

Here, we are ready to present the  $WZ$  algorithm. Without loss of generality we assume that  $A$  is an even order matrix.

**Algorithm 1.  $WZ$  algorithm**

- (1) Let  $A^{(0)} = A$ ,  $k = 1$ ,  $s = n/2$ .
- (2) Compute  $P_k$ ,  $A^{(k)} = P_k A^{(k-1)}$  where,  $P_k$  is a permutation matrix and  $\Delta_k$  is nonsingular.
- (3) Compute  $H_k^k$ , so that the rows of  $H_k^k$  present the null space of the rows of  $\Delta_k$  except the  $k$ th and  $(k+1)$ th rows.
- (4) Determine  $1 \leq j_1, j_2 \leq 2k$  such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_k \neq 0, \quad \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{k+1} \neq 0. \quad (2.15)$$

- (5) Compute,

$$\mathcal{T}_k = (t_1, \dots, t_{2k}) = H_k^T e_{j_1} / \alpha_1 \quad \text{and} \quad \mathcal{Y}_k = (y_1, \dots, y_{2k}) = H_k^T e_{j_2} / \alpha_2.$$

- (6) Compute,

$$\bar{z}_k = (t_1, \dots, t_k, \underbrace{0, \dots, 0}_{n-2k}, t_{k+1}, \dots, t_{2k})^T, \quad (2.16)$$

and

$$\bar{z}_{n-k+1} = (y_1, \dots, y_k, \underbrace{0, \dots, 0}_{n-2k}, y_{k+1}, \dots, y_{2k})^T, \quad (2.17)$$

- (7) **If**  $k < s$  **then**  $k=k+1$  and **go to** (2).

- (8) Compute

$$PA = WZ,$$

where,  $P = P_s \cdots P_1$ ,  $\bar{Z} = (\bar{z}_1, \cdots, \bar{z}_n)$ ,  $W = PA\bar{Z}$  and  $Z = \bar{Z}^{-1}$ .

(9) **Stop.**

The integer WZ factorization of an integer matrix, can be calculated as the real case if it exists. Here, we present the conditions for existence of the integer WZ factorization of an integer matrix.

**Definition 2.3.**  $A \in \mathbb{Z}^{n \times n}$  is a unimodular matrix if and only if  $|\det(A)| = 1$ .

If  $A$  is unimodular, then  $A^{-1}$  is also unimodular.

**Definition 2.4.** We say that a matrix  $A$  is factorized in an integer WZ form if

$$A = WZ, \quad (2.18)$$

where the matrices  $W$  and  $Z$  are matrices with integer entries defined by (2.3).

According to Theorem 2.1, we have the following result.

**Theorem 2.3.** Let  $A \in \mathbb{Z}^{n \times n}$  and the submatrices  $\Delta_k$  defined by (2.4) be unimodular, then  $A$  has an integer WZ factorization.

For computing an integer WZ factorization (if there exists), in the  $k$ th step  $H_k$  generates the integer null space of  $\Delta_k$  except the  $k$ th and the  $(k+1)$ th rows. Furthermore, in (2.8) we choose two integer vectors  $j_1$  and  $j_2$  such that

$$\alpha_1 = e_{j_1}^T H_k \Delta_k^T e_k = \gcd(H_k \Delta_k^T e_k), \alpha_2 = e_{j_2}^T H_k \Delta_k^T e_{k+1} = \gcd(H_k \Delta_k^T e_{k+1}), \quad (2.19)$$

where,  $\gcd(x)$  is the greatest common divisor of entries of  $x$ .

**Definition 2.5.** A matrix  $A \in \mathbb{Z}^{n \times n}$  is called totally unimodular if each square submatrix of  $A$  has determinant equal to 0, +1, or -1. In particular, each entry of a totally unimodular matrix is 0, +1, or -1.

**Corollary 2.1.** Every totally unimodular symmetric positive definite matrix has an integer WZ factorization.

### 3. THE ZW FACTORIZATION

**Definition 3.1.** We say that a matrix  $A$  is factorized in the form ZW if

$$A = ZW, \quad (3.1)$$

where the matrices  $W$  and  $Z$  are defined as (2.3)

where the empty bullets stand for zero and the other bullets stand for possible nonzero entries.

The factorization is unique if  $Z$  has 1's on the main diagonal and 0's on the

cross diagonal.

Without loss of generality, suppose that  $n$  be an even number and  $s = \frac{n}{2}$ . Here, we present a new algorithm for computing the  $ZW$  factorization of  $A$  using null space of the sequence of submatrices

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{n/2} \quad (3.2)$$

where

$$\Lambda_k = \begin{pmatrix} a_{s-k+1, s-k+1} & \cdots & a_{s-k+1, s+k} \\ \vdots & \ddots & \vdots \\ a_{s+k, s-k+1} & \cdots & a_{s+k, s+k} \end{pmatrix}_{2k, 2k}. \quad (3.3)$$

**Theorem 3.1.** *Let  $A \in \mathbb{R}^{n \times n}$  be a nonsingular matrix.  $A$  has a  $ZW$  factorization, if and only if for every  $k$ ,  $1 \leq k \leq n/2$ , the submatrix  $\Lambda_k$  defined by (3.3) be invertible.*

*Proof.* The proof follows the lines of the proof for Theorem 2 in [17] replacing  $\Delta_i$  by  $\Lambda_i$ .  $\square$

Let  $\Lambda_k$  be nonsingular, for  $k = 1, \dots, n/2$ . Let, the rows of  $H_k$  generates the null space of  $\Lambda_k$  except the first and the last rows. Let  $e_i \in \mathbb{R}^{2k}$  be the  $i$ th unit vector (i.e. the  $i$ th element is 1, otherwise 0). Then, we have,

$$e_i^T \Lambda_k H_k^T = 0, \quad i \neq 1, 2k, \quad (e_1^T, e_{2k}^T) \Lambda_k H_k^T \neq 0, \quad (3.4)$$

then there exists  $1 \leq j_1, j_2 \leq 2k$  such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 \neq 0, \quad \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} \neq 0. \quad (3.5)$$

Let  $\mathcal{T}_k = (t_1, \dots, t_{2k}) = H_k^T e_{j_1} / \alpha_1$  and  $\mathcal{Y}_k = (y_1, \dots, y_{2k}) = H_k^T e_{j_2} / \alpha_2$ . Then, we have

$$\Lambda_k t = (1, \underbrace{0 \dots 0}_{2k-1})^T, \quad \Lambda_k y = (\underbrace{0 \dots 0}_{2k-1}, 1)^T. \quad (3.6)$$

Now, let

$$\bar{w}_{\frac{n}{2}-k+1} = (\underbrace{0, \dots, 0}_{(n-2k)/2}, t_1, \dots, t_{2k}, \underbrace{0, \dots, 0}_{(n-2k)/2})^T, \quad (3.7)$$

and

$$\bar{w}_{\frac{n}{2}+k} = (\underbrace{0, \dots, 0}_{(n-2k)/2}, y_1, \dots, y_{2k}, \underbrace{0, \dots, 0}_{(n-2k)/2})^T, \quad (3.8)$$

then, we have

$$z_{\frac{n}{2}-k+1} = A\bar{w}_{\frac{n}{2}-k+1} = \left( z_1, \dots, z_{\frac{(n-2k)}{2}, \frac{n}{2}-k+1}, \underbrace{1, 0, \dots, 0}_{2k-1}, z_{\frac{(n+2k)}{2}+1, \frac{n}{2}-k+1}, \dots, z_n, \frac{n}{2}-k+1 \right)^T \tag{3.9}$$

and

$$z_{\frac{n}{2}+k} = A\bar{w}_{\frac{n}{2}+k} = \left( z_1, \dots, z_{\frac{(n-2k)}{2}, \frac{n}{2}+k}, \underbrace{0, \dots, 0}_{2k-1}, 1, z_{\frac{(n+2k)}{2}+1, \frac{n}{2}+k}, \dots, z_n, \frac{n}{2}+k \right)^T. \tag{3.10}$$

$$\bar{W} = (\bar{w}_1, \dots, \bar{w}_n), \quad Z = (z_1, \dots, z_n),$$

then, we have,

$$A\bar{W} = Z \Rightarrow A = ZW, \quad W = \bar{W}^{-1}$$

Here, we are ready to present the  $ZW$  algorithm. Without loss of generality we assume that  $A$  is an even order matrix.

**Algorithm 2.  $ZW$  algorithm**

- (1) Let  $A^{(0)} = A, k = 1, s = n/2$ .
- (2) Compute  $P_k, A^{(k)} = P_k A^{(k-1)}$  where,  $P_k$  is a permutation matrix and  $\Lambda_k$  is nonsingular.
- (3) Let the rows of  $H_k$  generate the null space of  $\Lambda_k$  expect the first and the last rows.
- (4) Determine  $1 \leq j_1, j_2 \leq 2k$  such that,

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 \neq 0, \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} \neq 0. \tag{3.11}$$

- (5) Compute,

$$\mathcal{T}_k = (t_1, \dots, t_{2k}) = H_k^T e_{j_1} / \alpha_1 \text{ and } \mathcal{Y}_k = (y_1, \dots, y_{2k}) = H_k^T e_{j_2} / \alpha_2.$$

- (6) Compute,

$$\bar{w}_{\frac{n}{2}-k+1} = \left( \underbrace{0, \dots, 0}_{(n-2k)/2}, t_1, \dots, t_{2k}, \underbrace{0, \dots, 0}_{(n-2k)/2} \right)^T, \tag{3.12}$$

and

$$\bar{w}_{\frac{n}{2}+k} = \underbrace{(0, \dots, 0)}_{(n-2k)/2}, y_1, \dots, y_{2k}, \underbrace{(0, \dots, 0)}_{(n-2k)/2}^T, \quad (3.13)$$

(7) **If**  $k < s$  **then**  $k=k+1$  and **go to** (2).

(8) Compute

$$PA = ZW,$$

where,  $P = P_s \cdots P_1$ ,  $\bar{W} = (\bar{w}_1, \dots, \bar{w}_n)$ ,  $Z = PA\bar{W}$  and  $W = \bar{W}^{-1}$ .

(9) **Stop.**

We can also calculate the integer  $ZW$  factorization of an integer matrix  $A$ . The existence conditions are the same as Theorem 2.3 by replacing  $\Delta$  by  $\Lambda$ .

**Theorem 3.2.** *Let  $A \in \mathbb{Z}^{n \times n}$  and the submatrices  $\Lambda_k$  be unimodular, then  $A$  has an integer  $ZW$  factorization.*

For computing an integer  $ZW$  factorization (if there exists), in the  $k$ th step  $H_k$  generates the integer null space of  $\Lambda_k$  except the first and the last rows. Furthermore, in (3.5) we choose two integer vectors  $j_1$  and  $j_2$  such that

$$\alpha_1 = e_{j_1}^T H_k \Lambda_k^T e_1 = \gcd(H_k \Lambda_k^T e_1), \alpha_2 = e_{j_2}^T H_k \Lambda_k^T e_{2k} = \gcd(H_k \Lambda_k^T e_{2k}). \quad (3.14)$$

**Corollary 3.1.** *Every totally unimodular symmetric positive definite matrix has an integer  $ZW$  factorization.*

#### 4. EXAMPLES

In this section, we present some numerical illustrations of our proposed algorithms to compute the  $WZ$  and  $ZW$  factorizations of real and integer matrices.

EXAMPLE 4.1. Consider the following matrix,

$$A = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

Upon an application of Algorithm 1 for computing the  $WZ$  factorization, we obtain the following results:

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7895 & 1 & 0 & 0.0526 \\ 0.1053 & 0 & 1 & 0.4737 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 0 & 1.7895 & 0.1053 & 0 \\ 0 & 0.1053 & 2.9474 & 0 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

EXAMPLE 4.2. Consider the following matrix

$$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 & 2.5 & 2.5 \\ 3 & 3 & 3.5 & 2.5 & 3 & 2.5 \\ 1.5 & 3.5 & 1 & 2.5 & 2 & 2.5 \\ 2 & 2.5 & 2.5 & 4 & 1.5 & 3 \\ 2.5 & 3 & 2 & 1.5 & 2 & 2.5 \\ 2.5 & 2.5 & 2.5 & 3 & 2.5 & 1 \end{pmatrix}.$$

By applying Algorithm 2 for computing the  $ZW$  factorization we have

$$Z = \begin{pmatrix} 1 & 0.3219 & 1.5 & 0.5 & 0.6452 & -0.8439 \\ 0 & 1 & 3.5 & 0.6250 & 1.6613 & 0 \\ 0 & 0 & 1 & 0.6250 & 0 & 0 \\ 0 & 0 & 2.5 & 1 & 0 & 0 \\ 0 & 0.7055 & 2 & 0.3750 & 1 & 0 \\ -1.5774 & 0.3425 & 2.5000 & 0.7500 & 0.7419 & 1.0000 \end{pmatrix},$$

and

$$W = \begin{pmatrix} -0.4855 & 0 & 0 & 0 & 0 & 0 \\ 6.2826 & 8.1111 & 0 & 0 & 0 & 8.5331 \\ -0.4444 & -3.4444 & 1 & 0 & -1.8889 & -1.1111 \\ 3.1111 & 11.1111 & 0 & 4 & 6.2222 & 5.7778 \\ -2.2100 & 0 & 0 & 0 & 3.4444 & -3.4644 \\ 0 & 0 & 0 & 0 & 0 & -0.9075 \end{pmatrix}.$$

EXAMPLE 4.3. Consider the following integer real matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 2 & 0 & 3 & 1 & 1 \\ -1 & 0 & 5 & -1 & 7 & 2 \\ 1 & 3 & -1 & 8 & 2 & 1 \\ -1 & 1 & 7 & 2 & 15 & 4 \\ -1 & 1 & 2 & 1 & 4 & 2 \end{pmatrix}.$$

Upon an application of Algorithm 1 for computing the integer  $WZ$  factorization, we obtain the following results:

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -1 & -2 & 0 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 \end{pmatrix}.$$

EXAMPLE 4.4. Consider the following matrix

$$A = \begin{pmatrix} 5 & 4 & -1 & 1 & -3 & -2 \\ 4 & 7 & -1 & 1 & -4 & 0 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ -3 & -4 & 1 & -1 & 3 & 1 \\ -2 & 0 & 1 & 0 & 1 & 3 \end{pmatrix}.$$

By applying Algorithm 2 for computing the integer  $ZW$  factorization we have

$$Z = \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

## 5. CONCLUSION

Parallel implicit matrix elimination schemes for the solution of linear systems were introduced by Evans. In this paper we showed how to compute the real (integer)  $WZ$  and  $ZW$  factorizations by using the null space generators of particular submatrices of a given matrix  $A$ .

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## REFERENCES

1. J. Abaffy, E. Spedicato, *ABS Projection Algorithms, Mathematical Techniques for Linear and Nonlinear Equations*, Prentice-Hall, 1989.
2. B. Bylina, The Inverse Iteration with the WZ Factorization Used to the Markovian Models, *Annales UMCS Informatica AI*, **2**, (2004), 15-23
3. B. Bylina, J. Bylina, Solving Markov Chains with the WZ Factorization for Modelling Networks., *Proceedings of Aplimat 2004*, Bratislava, 307-312.

4. D. J. Evans, 17 Parallel Numerical Algorithms for Linear Systems , in: *Parallel Processing Systems: An Advanced Course*, (1982), 357.
5. D. J. Evans, Implicit Matrix Elimination Schemes, *International Journal of Computer Mathematics*, **48**, (1993), 229-237.
6. E. Golpar-Raboky, ABS Algorithms for Integer WZ Factorization, *Malaysian journal of mathematical sciences*, **8**(1), (2014), 69-85.
7. E. Golpar-Raboky, N. Mahdavi-Amiri, A New Interpretation of the Integer and Real WZ Factorization Using Block Scaled ABS Algorithms, *STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING*, **2**(3), (2014), 243-256.
8. E. Golpar-Raboky, N. Mahdavi-Amiri, WZ Factorization via Abaffy-Broyden-Spedicato Algorithms, *Bulletin of the Iranian Mathematical Society*, **40**(2), (2014), 399-411.
9. G. H. Golub, C. F. Van Loan, *Matrix Computations*, John Hopkins University Press, 2013.
10. D. J. Evans, M. Hatzopoulos, The Solution of Certain Banded Systems of Linear Equations Using the Folding Algorithm, *The Computer Journal*, **19**(2), (1976), 184-186.
11. D. J. Evans, M. Hatzopoulos, A Parallel Linear System Solver, *International Journal of Computer Mathematics*, **7**(3), (1979), 227-238.
12. Z. H. He, O. M. Agudelo, Q. W. Wang, B. De Moor, Two-sided Coupled Generalized Sylvester Matrix Equations Solving Using a Simultaneous Decomposition for Fifteen Matrices, *Linear Algebra and its Applications*, **496**, (2016), 549-593.
13. Z. H. He, Q. W. Wang, Y. Zhang, Simultaneous Decomposition of Quaternion Matrices Involving  $\eta$ -Hermiticity with Applications, *Applied Mathematics and Computation*, **298**, (2017), 13-35.
14. Z. H. He, Q. W. Wang, Y. Zhang, The Complete Equivalence Canonical Form of Four Matrices Over an Arbitrary Division Ring, *Linear and Multilinear Algebra*, **66**(1), (2018), 74-95.
15. S. Li, G. Zhou, D.Chen, The WZ Algorithm for the Eigenvalue Problem of Complex Matrix, *Applied Mathematics and Computation*, **165**(2), (2005), 347-353
16. N. Mahdavi-Amiri, E. Golpar-Raboky, Real and Integer Wedderburn Rank Reduction Formulas for Matrix Decompositions, *Optimization methods and software*, **30**(4), (2015), 864-879.
17. S. C. S. Rao, Existence and Uniqueness of WZ Factorization, *Parallel Computing*, **23**(8), (1997), 1129-1139.
18. S. C. S. Rao, R.kamara, A Hybrid Parallel Algorithm for Large Sparse Linear Systems, *Numerical Linear Algebra with Applications*, **25**(6), (2018) pe2210.
19. J. H. M. Wedderburn, Lectures on Matrices, Colloquium Publications, *American Mathematical Society, New York*, 1934.