Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 1 (2022), pp 99-109
DOI: 10.52547/ijmsi.17.1.99

A Functional Characterization of the Hurewicz Property

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Abstract. For a Tychonoff space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. We study a functional characterization of the covering property of Hurewicz.

Keywords: $U_{fin}(\mathcal{O}, \Omega)$, Hurewicz property, Selection principles, C_p theory, $U_{fin}(\mathcal{O}, \Gamma)$.

2010 Mathematics subject classification: 54C35, 54C05, 54C65, 54A20.

1. Introduction

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Received 8 June 2018; Accepted 27 April 2019

Many equivalences hold among these properties, and the surviving ones appear in the following Scheepers Diagram (Fig. 1) (where an arrow denotes implication) [11].

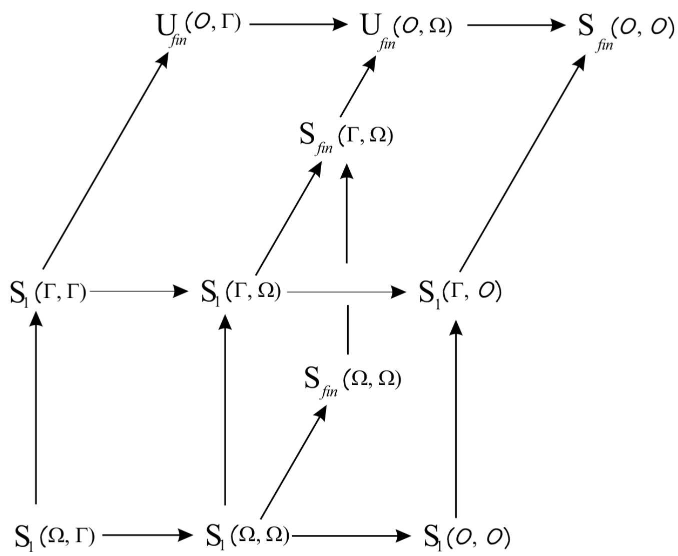

Fig. 1. The Scheepers Diagram for Lindelöf space.

The papers [11, 15, 32, 35, 36, 37, 38] have initiated the simultaneous consideration of these properties in the case where $\mathcal{A}$ and $\mathcal{B}$ are important families of open covers of a topological space X.

In papers [1-8, 12-32] (and many others) were investigated the applications of selection principles in the study of the properties of function spaces. In particular, the properties of the space $C_p(X)$ were investigated.

In [9] (see also [10]), Hurewicz introduced a covering property of a space X, nowadays called the *Hurewicz property* in this way: for each sequence $(U_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, V_n is a finite subset of U_n and each $x \in X$ belongs to $\cup V_n$ for all but finitely many n (i.e., X satisfies $U_{fin}(\mathcal{O},\Gamma)$).

In this paper we continue to study different selectors for sequences of dense sets of $C_p(X)$ and we study a functional characterization of the covering property of Hurewicz.

2. Main definitions and notation

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$. Let $\mathbb{R}$ be the real line and $\mathbb{Q}$ be the rational numbers. For a space X, we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence.

Basic open sets of $C_p(X)$ are of the form

$[x_1, ..., x_k, U_1, ..., U_k] = \{f \in C(X) : f(x_i) \in U_i,\ i = 1, ..., k\}$, where each $x_i \in X$ and each U_i is a non-empty open subset of $\mathbb{R}$. Sometimes we will write the basic neighborhood of the point f as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon\ \forall x \in A\}$, A is a finite subset of X and $\epsilon > 0$.

The symbol $\mathbf{0}$ denote the constantly zero function in $C_p(X)$. Because $C_p(X)$ is homogeneous we can work with $\mathbf{0}$ to study local properties of $C_p(X)$.

If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set of X if $X \setminus O$ is a zero-set.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is:

• an *ω-cover* if every finite subset of X is contained in a member of $\mathcal{U}$.

• a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Note that every γ-cover contains a countably γ-cover.

• a *γ_F-shrinkable* cover $\mathcal{U}$ if it is a γ-cover $\mathcal{U}$ of X by co-zero sets and there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of X by zero-sets with $F(U) \subset U$ for some $U \in \mathcal{U}$.

For a topological space X we denote:

• $\mathcal{O}$ — the family of all open covers of X;

• Γ — the family of all countable open γ-covers of X;

• Ω — the family of all open ω-covers of X;

• Γ_F — the family of all γ_F-shrinkable covers of X.

For a topological space $C_p(X)$ we denote:

• $\mathcal{D}$ — the family of all dense subsets of $C_p(X)$;

• $\mathcal{S}$ — the family of all sequentially dense subsets of $C_p(X)$.

In the case of U_{fin} note that for any class of covers $\mathcal{B}$ of Lindelöf space X, $U_{fin}(\mathcal{O}, \mathcal{B})$ is equivalent to $U_{fin}(\Gamma, \mathcal{B})$ because given an open cover $\{U_n : n \in \mathbb{N}\}$ we may replace it by $\{\bigcup_{i<n} U_i : n \in \mathbb{N}\}$, which is a γ-cover (unless it contains a finite subcover) of X.

Recall that the i-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

Theorem 2.1. (*Noble* [18]) *A space $C_p(X)$ is separable iff $iw(X) = \aleph_0$.*

[DOI: 10.52547/ijmsi.17.1.99]

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

• $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;

• $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

So we have three types of topological properties described through the selection principles:

• local properties of the form $S_*(\Phi_x, \Psi_x)$;

• global properties of the form $S_*(\Phi, \Psi)$;

• semi-local properties of the form $S_*(\Phi, \Psi_x)$.

3. $U_{fin}(\mathcal{O}, \Omega)$

For a function space $C_p(X)$, we represent the following selection principle

$F_{fin}(\mathcal{S}, \mathcal{D})$: whenever $\mathcal{S}_1, \mathcal{S}_2, ... \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$ and a base neighborhood $\langle f, K, \epsilon\rangle$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \mathbb{N}$ such that for each $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

It is clear that the condition of the selection principle $F_{fin}(\mathcal{S}, \mathcal{D})$ can be written more briefly: whenever $\mathcal{S}_1, \mathcal{S}_2, ... \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$, $\epsilon > 0$ and $K \in [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that $\min\limits_{h \in \mathcal{F}_{n'}} \{|f(x) - h(x)|\} < \epsilon$ for each $x \in K$.

Similarly, $F_{fin}(\Gamma_0, \Omega_0)$: whenever $\mathcal{S}_1, \mathcal{S}_2, ... \in \Gamma_0$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for $\epsilon > 0$ and $K \in [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that $\min\limits_{h \in \mathcal{F}_{n'}} \{|h(x)|\} < \epsilon$ for each $x \in K$.

Theorem 3.1. *For a space X, the following statements are equivalent:*

(1) *$C_p(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$;*

(2) *X satisfies $U_{fin}(\Gamma_F, \Omega)$.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_i\}_{i \in \mathbb{N}} \subset \Gamma_F$ and let $\mathcal{U}_i = \{U_i^m\}$ for each $i \in \mathbb{N}$. We consider $\mathcal{K}_i = \{f_i^m \in C(X) : f_i^m \upharpoonright F(U_i^m) = 0$ and $f_i^m \upharpoonright (X \setminus U_i^m) = 1$ for $m \in \mathbb{N}\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \mathbb{N}\}$ is a γ-cover of X, we have that $\mathcal{K}_i$ converges to $\mathbf{0}$ for each $i \in \mathbb{N}$. Since $C_p(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$, there are finite sets $F_i = \{f_i^{m_1}, ..., f_i^{m_{s(i)}}\} \subseteq \mathcal{K}_i$ such that for a base neighborhood $O(f) = \langle f, K, \epsilon\rangle$ of $f = \mathbf{0}$ where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \mathbb{N}$ such that for each $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_{n'}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$. Note that $\{\bigcup\{U_i^{m_1}, ..., U_i^{m_{s(i)}}\} : i \in \mathbb{N}\} \in \Omega$.

[DOI: 10.52547/ijmsi.17.1.99]

(2) $\Rightarrow$ (1). Let X satisfies $U_{fin}(\Gamma_F,\Omega)$ and $A_i\in\Gamma_{\mathbf{0}}$ for each $i\in\mathbb{N}$. Consider $\mathcal{U}_i=\{U_{i,f}=f^{-1}(-\frac{1}{i},\frac{1}{i}):f\in A_i\}$ for each $i\in\mathbb{N}$. Without loss of generality we can assume that a set $U_{i,f}\neq X$ for any $i\in\mathbb{N}$ and $f\in A_i$. Otherwise there is sequence $\{f_{i_k}\}_{k\in\mathbb{N}}$ such that $\{f_{i_k}\}_{k\in\mathbb{N}}$ uniform converges to $\mathbf{0}$ and $\{f_{i_k}:k\in\mathbb{N}\}\in\Omega_{\mathbf{0}}$.

Note that $\mathcal{F}_i=\{F_{i,m}\}_{m\in\mathbb{N}}=\{f^{-1}_{i,m}[-\frac{1}{i+1},\frac{1}{i+1}]:m\in\mathbb{N}\}$ is a γ-cover of X and $F_{i,m}\subset U_{i,m}$ for each $i,m\in\mathbb{N}$. It follows that $\mathcal{U}_i\in\Gamma_F$ for each $i\in\mathbb{N}$.

Since X satisfies $U_{fin}(\Gamma_F,\Omega)$, there is a sequence $\{U_{i,m(1)},U_{i,m(2)},...,U_{i,m(i)}:i\in\mathbb{N}\}$ such that for each i and $k\in\{m(1),...,m(i)\}$, $U_{i,m(k)}\in\mathcal{U}_i$, and

$\{\bigcup\{U_{i,m(1)},...,U_{i,m(i)}\}:i\in\mathbb{N}\}\in\Omega$.

Let $\langle\mathbf{0},K,\epsilon\rangle$ be a base neighborhood of $\mathbf{0}$ where $\epsilon>0$ and $K=\{x_1,...,x_s\}$ is a finite subset of X, then there is $i_1\in\mathbb{N}$ such that $\frac{1}{i_1}<\epsilon$ and $\bigcup_{k=m(1)}^{m(i_1)}U_{i_1,k}\supset K$. It follows that for each $j\in\{1,...,s\}$ there is $g\in\{f_{i_1,m(1)},...,f_{i_1,m(i_1)}\}$ such that $g(x_j)\in(-\epsilon,\epsilon)$. $\square$

Lemma 3.2. *(Lemma 6.5 in [20]) Let $\mathcal{U}=\{U_n:n\in\mathbb{N}\}$ be a γ_F-shrinkable co-zero cover of a space X. Then the set $S=\{f\in C(X):f\upharpoonright(X\setminus U_n)\equiv 1$ for some $n\in\mathbb{N}\}$ is sequentially dense in $C_p(X)$.*

Theorem 3.3. *For a space X with $iw(X)=\aleph_0$, the following statements are equivalent:*

1. *$C_p(X)$ satisfies $F_{fin}(\mathcal{S},\mathcal{D})$;*
2. *X satisfies $U_{fin}(\Gamma_F,\Omega)$;*
3. *$C_p(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})$;*
4. *$C_p(X)$ satisfies $F_{fin}(\mathcal{S},\Omega_{\mathbf{0}})$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{U}_i=\{U_i^j:j\in\mathbb{N}\}\in\Gamma_F$ for each $i\in\mathbb{N}$. Then, by Lemma 3.2, each $S_i=\{f\in C(X):f\upharpoonright(X\setminus U_i^j)\equiv 1$ for some $m\in\mathbb{N}\}$ is sequentially dense in $C_p(X)$. Since $C_p(X)$ satisfies $F_{fin}(\mathcal{S},\mathcal{D})$, there are finite sets $F_i=\{f_i^{m_1},...,f_i^{m_{s(i)}}\}\subseteq\mathcal{S}_i$ such that for each $f\in C_p(X)$ and a base neighborhood $\langle f,K,\epsilon\rangle$ of f where $\epsilon>0$ and $K=\{x_1,...,x_k\}$ is a finite subset of X, there is $n'\in\mathbb{N}$ such that for each $j\in\{1,...,k\}$ there is $g\in F_{n'}$ such that $g(x_j)\in(f(x_j)-\epsilon,f(x_j)+\epsilon)$. Note that $\{\bigcup\{U_i^{m_1},...,U_i^{m_{s(i)}}\}:i\in\mathbb{N}\}\in\Omega$.

(2) $\Rightarrow$ (3). By Theorem 3.1.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). Suppose that $C_p(X)$ satisfies $F_{fin}(\mathcal{S},\Omega_{\mathbf{0}})$.

Let $D=\{d_n:n\in\mathbb{N}\}$ be a dense subspace of $C_p(X)$ and $S_i\in\mathcal{S}$ for each $i\in\mathbb{N}$. Given a sequence of sequentially dense subspace of $C_p(X)$, enumerate it as $\{S_{n,m}:n,m\in\mathbb{N}\}$. For each $n\in\mathbb{N}$, pick

$\mathcal{F}_{n,m}=\{d_{n,m,1},...,d_{n,m,k(n,m)}\}\subset S_{n,m}$ so that for a base neighborhood $\langle d_n,K,\epsilon\rangle$ of d_n where $\epsilon>0$ and $K=\{x_1,...,x_k\}$ is a finite subset of X, there is $m'\in\mathbb{N}$ such that for each $j\in\{1,...,k\}$ there is $g\in\mathcal{F}_{n,m'}$ such that $g(x_j)\in(d_n(x_j)-\epsilon,d_n(x_j)+\epsilon)$. It follows that $C_p(X)$ satisfies $F_{fin}(\mathcal{S},\mathcal{D})$. $\square$

[DOI: 10.52547/ijmsi.17.1.99]

Theorem 3.4. *For a space X the following statements are equivalent:*

(1) *X is Lindelöf and X satisfies $U_{fin}(\Gamma_F, \Omega)$;*
(2) *X satisfies $U_{fin}(\mathcal{O}, \Omega)$.*

Proof. It is proved similarly to the proof of Theorem 4.1. □

Theorem 3.5. *For a separable metrizable space X, the following statements are equivalent:*

(1) *$C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \mathcal{D})$;*
(2) *X satisfies $U_{fin}(\mathcal{O}, \Omega)$;*
(3) *$C_p(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$;*
(4) *$C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \Omega_{\mathbf{0}})$.*

4. $U_{fin}(\mathcal{O}, \Gamma)$ - Hurewicz property

Theorem 4.1. *For a space X the following statements are equivalent:*

(1) *X satisfies $U_{fin}(\Gamma_F, \Gamma)$ and is Lindelöf;*
(2) *X has the Hurewicz property.*

Proof. (1) $\Rightarrow$ (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For every n, $U \in \mathcal{U}_n$ and $x \in X$ we find co-zero sets $W_{0,n,U,x}$ and $W_{2,n,U,x}$, and, a zero-set $W_{1,n,U,x}$ such that $x \in W_{0,n,U,x} \subset W_{1,n,U,x} \subset W_{2,n,U,x} \subset U$. Since X is Lindelöf, there is a sequence $(x^n_k : k \in \mathbb{N})$ such that X is covered be $\{W_{0,n,U,x^n_k} : k \in \mathbb{N}\}$. Look at the cover $\mathcal{W}_n$ of X consisting of sets $W^n_k = \bigcup_{i \leq k} W_{2,n,U,x^n_i}$, $k \in \mathbb{N}$. Note that $\mathcal{W}_n \in \Gamma_F$ because $\bigcup_{i \leq k} W_{1,n,U,x^n_i}$ is a zero-set contained in W^n_k, and $\{\bigcup_{i \leq k} W_{1,n,U,x^n_i} : k \in \mathbb{N}\}$ is a γ-cover of X because even $\{\bigcup_{i \leq k} W_{0,n,U,x^n_i} : k \in \mathbb{N}\}$ is a γ-cover of X.

Now use the property $U_{fin}(\Gamma_F, \Gamma)$ to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ together with the fact that $\mathcal{W}_n$ is a finer cover that $\mathcal{U}_n$ for all n. □

For a function space $C_p(X)$, we represent the following selection principle $F_{fin}(\mathcal{S}, \mathcal{S})$: whenever $\mathcal{S}_1, \mathcal{S}_2, ... \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$ there is $\{\mathcal{F}_{n_k} : k \in \mathbb{N}\}$ such that for a base neighborhood $\langle f, K, \epsilon \rangle$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_m\}$ is a finite subset of X, there is $k' \in \mathbb{N}$ such that for each $k > k'$ and $\forall\ j \in \{1, ..., m\}$ there is $g \in \mathcal{F}_{n_k}$ such that $g(x_j) \in (f(x_j) - \epsilon, f(x_j) + \epsilon)$.

It is clear that the condition of the selection principle $F_{fin}(\mathcal{S}, \mathcal{S})$ can be written more briefly: whenever $\mathcal{S}_1, \mathcal{S}_2, ... \in \mathcal{S}$ there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for each $f \in C_p(X)$, $\epsilon > 0$ and $K \in [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that for every $n > n'$ $\min\limits_{h \in \mathcal{F}_n} \{|f(x) - h(x)|\} < \epsilon$ for each $x \in K$.

Similarly, $F_{fin}(\Gamma_0, \Gamma_0)$: whenever $\mathcal{S}_1, \mathcal{S}_2, ... \in \Gamma_0$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{S}_n$, $n \in \mathbb{N}$, such that for $\epsilon > 0$ and $K \in [X]^{<\omega}$, there is $n' \in \mathbb{N}$ such that for every $n > n'$ $\min\limits_{h \in \mathcal{F}_n} \{|h(x)|\} < \epsilon$ for each $x \in K$.

Theorem 4.2. *For a space X, the following statements are equivalent:*

[DOI: 10.52547/ijmsi.17.1.99]

(1) $C_p(X)$ *satisfies* $F_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
(2) X *satisfies* $U_{fin}(\Gamma_F, \Gamma)$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_i\}_{i\in\mathbb{N}} \subset \Gamma_F$, $\mathcal{U}_i = \{U_i^m\}_{m\in\mathbb{N}}$ for each $i \in \mathbb{N}$. We consider a subset $\mathcal{S}_i$ of $C_p(X)$ where

$\mathcal{S}_i = \{f_i^m \in C(X) : f_i^m \restriction F(U_i^m) = 0$ and $f_i^m \restriction (X \setminus U_i^m) = 1$ for $m \in \mathbb{N}\}$.

Since $\mathcal{F}_i = \{F(U_i^m) : m \in \mathbb{N}\}$ is a γ-cover of X, we have that $\mathcal{S}_i$ converges to $\mathbf{0}$, i.e. $\mathcal{S}_i \in \Gamma_0$ for each $i \in \mathbb{N}$.

Since $C(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$, there is a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}} = \{f_i^{m_1}, ..., f_i^{m_{k(i)}} : i \in \mathbb{N}\}$ such that for each i, $\mathcal{F}_i \subseteq \mathcal{S}_i$, and for a base neighborhood $\langle \mathbf{0}, K, \epsilon\rangle$ of $\mathbf{0}$ where $\epsilon > 0$ and $K = \{x_1, ..., x_k\}$ is a finite subset of X, there is $n' \in \mathbb{N}$ such that for each $n > n'$ and $j \in \{1, ..., k\}$ there is $g \in \mathcal{F}_n$ such that $g(x_j) \in (-\epsilon, \epsilon)$.

Consider the sequence $\{W_i\}_{i\in\mathbb{N}} = \{U_i^{m_1}, ..., U_i^{m_{k(i)}} : i \in \mathbb{N}\}$.

(a). $W_i \subset \mathcal{U}_i$ for each $i \in \mathbb{N}$.

(b). $\{\bigcup W_i : i \in \mathbb{N}\}$ is a γ-cover of X.

Let $K = \{x_1, ..., x_s\}$ be a finite subset of X and $\langle \mathbf{0}, K, \frac{1}{2}\rangle$ be a base neighborhood of $\mathbf{0}$, then there exists $i_0 \in \mathbb{N}$ such that for each $i > i_0$ and

$j \in \{1, ..., s\}$ there is $g \in \mathcal{F}_i$ such that $g(x_j) \in (-\frac{1}{2}, \frac{1}{2})$.

It follows that $K \subset \bigcup\limits_{j=1}^{k(i)} U_i^{m_j}$ for $i > i_0$. We thus get that X satisfies $U_{fin}(\Gamma_F, \Gamma)$.

(2) $\Rightarrow$ (1). Fix $\{S_i : i \in \mathbb{N}\} \subset \Gamma_0$ where $S_i = \{f_k^i : k \in \mathbb{N}\}$ for each $i \in \mathbb{N}$.

For each $i, k \in \mathbb{N}$, we put $U_{i,k} = \{x \in X : |f_k^i(x)| < \frac{1}{i}\}$, $Z_{i,k} = \{x \in X : |f_k^i(x)| \leq \frac{1}{i+1}\}$.

Each $U_{i,k}$ (resp., $Z_{i,k}$) is a cozero-set (resp., zero-set) in X with $Z_{i,k} \subset U_{i,k}$. Let $\mathcal{U}_i = \{U_{i,k} : k \in \mathbb{N}\}$ and let $\mathcal{Z}_i = \{Z_{i,k} : k \in \mathbb{N}\}$. So without loss of generality, we may assume $U_{i,k} \neq X$ for each $i, k \in \mathbb{N}$. We can easily check that the condition $f_k^i \to \mathbf{0}$ $(k \to \infty)$ implies that $\mathcal{Z}_i$ is a γ-cover of X.

Since X satisfies $U_{fin}(\Gamma_F, \Gamma)$ there is a sequence $\{\mathcal{F}_i\}_{i\in\mathbb{N}} = (U_{i,k_1}, ..., U_{i,k_i} : i \in \mathbb{N})$ such that for each i, $\mathcal{F}_i \subset \mathcal{U}_i$, and $\{\bigcup \mathcal{F}_i : i \in \mathbb{N}\}$ is an element of Γ.

Let $K = \{x_1, ..., x_s\}$ be a finite subset of X, $\epsilon > 0$, and $\langle \mathbf{0}, K, \epsilon\rangle$ be a base neighborhood of $\mathbf{0}$, then there exists $i' \in \mathbb{N}$ such that for every $i > i'$ $K \subset \bigcup \mathcal{F}_i$. It follow that for every $i > i'$ and $j \in \{1, ..., s\}$ there is $g \in \mathcal{S}_i$ such that $g(x_j) \in (-\epsilon, \epsilon)$. So $C_p(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$. □

Theorem 4.3. *For a Lindelöf space X, the following statements are equivalent:*

(1) $C_p(X)$ *satisfies* $F_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
(2) X *has the Hurewicz property.*

A space X has *Velichko property* $(X \models V)$, if there exists a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space X on a separable metric space Y, such that $f(U)$ is an F_σ-set of Y for any cozero-set U of X.

[DOI: 10.52547/ijmsi.17.1.99]

Theorem 4.4. *(Velichko [40]). Let X be a space. A space $C_p(X)$ is sequentially separable iff $X \models V$.*

Theorem 4.5. *For a space X with $X \models V$, the following statements are equivalent:*

(1) *$C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \mathcal{S})$;*
(2) *X satisfies $U_{fin}(\Gamma_F, \Gamma)$;*
(3) *$C_p(X)$ satisfies $F_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
(4) *$C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \Gamma_{\mathbf{0}})$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{U}_i = \{U_i^j : j \in \mathbb{N}\} \in \Gamma_F$ for each $i \in \mathbb{N}$. Then, by Lemma 3.2, each $S_i = \{f \in C(X) : f \upharpoonright (X \setminus U_i^j) \equiv 1$ for some $m \in \mathbb{N}\}$ is sequentially dense in $C_p(X)$.

Since $C(X)$ satisfies $U_{fin}(\mathcal{S}, \mathcal{S})$, there is a sequence $\{\mathcal{F}_i\} = \{f_i^{m_1}, ..., f_i^{m_s} : i \in \mathbb{N}\}$ such that for $f = \mathbf{0}$ there is $\{F_{i_k} : k \in \mathbb{N}\}$ such that for a base neighborhood $\langle f, K, \epsilon\rangle$ of f where $\epsilon > 0$ and $K = \{x_1, ..., x_m\}$ is a finite subset of X, there is $k' \in \mathbb{N}$ such that for each $k > k'$ and $j \in \{1, ..., m\}$ there is $g \in F_{i_k}$ such that $g(x_j) \in (-\epsilon, \epsilon)$.

Let $\epsilon = \frac{1}{2}$ and $\mathbb{N}' = \mathbb{N} \setminus \{k'\}$. Consider a sequence $\{Q_k\}_{k \in \mathbb{N}'} = \{U_{i_k}^{m_1}, ..., U_{i_k}^{m_s} : k \in \mathbb{N}'\}$ for corresponding to $\{F_{i_k}\} = \{f_{i_k}^{m_1}, ..., f_{i_k}^{m_s} : k \in \mathbb{N}'\}$.

(a). $Q_k \subset \mathcal{U}_{i_k}$ for $k \in \mathbb{N}'$.

(b). $\{\bigcup Q_k : k \in \mathbb{N}'\}$ is a γ-cover of X. We thus get X satisfies $U_{fin}(\Gamma_F, \Gamma)$.

(2) $\Rightarrow$ (3). By Theorem 4.2.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). For each $n \in \mathbb{N}$, let S_n be a sequentially dense subset of $C_p(X)$ and let $\{h_n : n \in \mathbb{N}\}$ be sequentially dense in $C_p(X)$. Take a sequence $\{f_n^m : m \in \mathbb{N}\} \subset S_n$ such that $f_n^m \mapsto h_n$ ($m \mapsto \infty$). Then $f_n^m - h_n \mapsto \mathbf{0}$ ($m \mapsto \infty$). Hence, there exist $\mathcal{F}_n = \{f_n^{m_1}, ..., f_n^{m_{k(n)}}\} \subset S_n$ such that $\{\bigcup\{f_n^{m_1} - h_n, ..., f_n^{m_{k(n)}} - h_n\} : n \in \mathbb{N}\} \in \Gamma_0$, i.e. for a base neighborhood $\langle f, K, \epsilon\rangle$ of $f = \mathbf{0}$ where $\epsilon > 0$ and $K = \{x_1, ..., x_m\}$ is a finite subset of X, there is $n' \in \mathbb{N}$ such that for each $n > n'$ and $\forall\, j \in \{1, ..., m\}$ there is $g \in \{f_n^{m_1} - h_n, ..., f_n^{m_{k(n)}} - h_n\}$ such that $g(x_j) \in (-\epsilon, \epsilon)$.

Let $h \in C_p(X)$ and take a sequence $\{h_{n_j} : j \in \mathbb{N}\} \subset \{h_n : n \in \mathbb{N}\}$ converging to h. Let $K = \{x_1, ..., x_m\}$ be a finite subset of X and $\epsilon > 0$. Consider a base neighborhood $\langle h, K, \epsilon\rangle$ of h. Then there is $j' \in \mathbb{N}$ such that $h_{n_j} \in \langle h, K, \frac{\epsilon}{2}\rangle$ and $\forall\, s \in \{1, ..., m\}$ there is $g \in \{f_{n_j}^{m_1} - h_{n_j}, ..., f_{n_j}^{m_{k(n_j)}} - h_{n_j}\}$ such that $g(x_s) \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ for $j > j'$. It follows that for each $s \in \{1, ..., m\}$ there is $l(j) \in \overline{1, k(n_j)}$ such that $((f_{n_j}^{m_{l(j)}} - h_{n_j}) + (h_{n_j} - h))(x_s) \in (-\epsilon, \epsilon)$ for $j > j'$. Hence $C_p(X)$ satisfies $F_{fin}(\mathcal{S}, \mathcal{S})$. $\square$

Theorem 4.6. *For a separable metrizable space X, the following statements are equivalent:*

[DOI: 10.52547/ijmsi.17.1.99]

(1) $C_p(X)$ *satisfies* $F_{fin}(\mathcal{S}, \mathcal{S})$;
(2) X *satisfies* $U_{fin}(\mathcal{O}, \Gamma)$ [*Hurewicz property*];
(3) $C_p(X)$ *satisfies* $F_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
(4) $C_p(X)$ *satisfies* $F_{fin}(\mathcal{S}, \Gamma_{\mathbf{0}})$.

Recall that a space X is said to be Rothberger [27] (or, [17]) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(V_n : n \in \mathbb{N})$ such that for each n, $V_n \in \mathcal{U}_n$, and $\{V_n : n \in \mathbb{N}\}$ is an open cover of X.

A space X is said to be Menger [9] (or, [30]) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a cover of X.

Every σ-compact space is Menger, and a Menger space is Lindelöf.

In [21], we gave the functional characterizations of Rothberger and Menger properties.

Recall that if $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic, uniformly homeomorphic), we say that the spaces X and Y are t-equivalent (l-equivalent, u-equivalent). The properties preserved by t-equivalence (l-equivalence, u-equivalence) we call t-invariant (l-invariant, u-invariant) [2].

Question 1. Is the Hurewicz (Rothberger, Menger) property t-invariant? l-invariant? u-invariant?

Acknowledgments

The author would like to thank the reviewers for their thoughtful comments and efforts to improve the manuscript.

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[DOI: 10.52547/ijmsi.17.1.99]