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Hyers–Ulam Stability of Non–Linear Volterra Integro–Delay Dynamic System with Fractional Integrable Impulses on Time Scales

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Abstract. This manuscript presents Hyers–Ulam stability and Hyers–Ulam–Rassias stability results of non–linear Volterra integro–delay dynamic system on time scales with fractional integrable impulses. Picard fixed point theorem is used for obtaining existence and uniqueness of solutions. By means of abstract Grönwall lemma, Grönwall's inequality on time scales, we establish Hyers–Ulam stability and Hyers–Ulam–Rassias stability results. There are some primary lemmas, inequalities and relevant assumptions that helps in our stability results.

Keywords: Hyers–Ulam stability, Time scale, Impulses, Delay dynamic system, Grönwall's inequality, Abstract Grönwall lemma, Banach fixed point theorem.

2000 Mathematics subject classification: 34N05, 34G20, 34A37, 35B35, 45J05.

1. Introduction

In 1940, in a talk in front of the mathematics club at the university of Wisconsin, Ulam [24, 25] presented a famed question related to the stability of

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homomorphisms: "*With which requirements does an additive mapping near an approximate additive mapping exists?*".

This question was answered by Hyers [11] for the case when G_1 and G_2 are assumed to be Banach spaces by using direct method. Since then, this interesting stability, initiated by Ulam and Hyers, is called Hyers–Ulam stability. In 1978, Rassias [21] extended Hyers–Ulam stability concept by introducing new function variables, so this kind of stability is known as the Hyers–Ulam–Rassias stability. In fact, the most interesting result was of Rassias [21] that weakens the condition for the bound of the norm of Cauchy difference $f(x+y)-f(x)-f(y)$. For further details and discussions, we recommend the book by Jung [13].

At the end of 19th century, a large number of researchers contributed to the stability idea of Ulam's type for various types of differential equations. There are many advantages of Ulam's type stability in tackling problems related to optimization techniques, numerical analysis, control theory and many more, in such situations to get an exact solution is challenging. For more details on Hyers–Ulam stability, see [12, 14, 18, 19, 23, 29, 31, 32, 34, 35, 37, 38].

There are several implications for simple differential equations. Anyhow, the circumstances rather change when a real world process undergoes with unexpected variations, like significant mechanical processes, blood flows, heart beats, changes in population, radio physics, pharmacokinetics, mathematical economy, chemical technology, electrical technology, chemistry, different engineering fields, control theory and so on, see [4, 5, 17]. Such circumstances generate a differential equation, which is known as impulsive differential equation. More precisely, there are three parts of differential equations with impulse impact: an instantaneous impulsive differential equation, in which the impulse action is defined at certain discrete points; non–instantaneous impulsive differential equation, it establishes the effect of impulse on an interval; and the third one is an impulse rule, in which we define a distinct and well defined collection of impulse events having an active impulse equation.

Fractional differential and integral equations play a key role not only in mathematics but also in the modeling of various physical phenomena in physics, control systems and dynamical systems. In fact, fractional order derivatives and integrals are assumed to be more realistic and practical than derivatives and integrals of integral order. These are excellent tools to model genetic transformation and memory retention qualities of several systems and products.

It is to be noted that, the pioneers of the Ulam's type stability for impulsive ordinary differentiable equations are Wang *et al.* [26]. Following their own work, in 2014, they proved the Hyers–Ulam–Rassias stability and generalized Hyers–Ulam–Rassias stability of impulsive evolution equations on a compact interval [27] which then they extended for infinite impulses in the same paper. Wang and Zhang [28], initially studied nonlinear differential equations

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having fractional integrable impulses, which are more interesting. They presented four Bielecki–Ulam's type stabilities for this class of differential equations. Also Lin *et al.* [15] discussed the existence and stability results for impulsive integro-differential equations. The work of Wang *et al.* [28] was extended by Zada *et al.* [32] in which they discussed Hyers–Ulam stability of higher-order nonlinear differential equations with fractional integrable impulses. They established Bielecki–Ulam–Hyers–Rassias stability, generalized Bielecki–Ulam–Hyers–Rassias stability and Bielecki–Ulam–Hyers stability for this class of differential equations on a compact interval. Recently, Zada *et al.* [36] obtained very interesting results about the Hyers–Ulam stability of nonlinear impulsive Volterra integro–delay dynamic system on time scales.

However, despite the situations where only impulsive factor is involved or delay effects happened, we have a wide variety of evolutionary processes with both delay and impulsive effects. To model such phenomena which are subject to impulsive perturbations as the time delays, an impulsive delay differential equation is used.

The theory of dynamic equations on time scales has been rising fast and has acknowledged a lot of interest in recent years. This theory was introduced by Hilger [10] in 1988, with the inspiration to provide a unification of continuous and discrete calculus. For more details on time scales, see [1, 2, 3, 6, 7, 8, 9, 16, 20, 23, 30, 33, 35].

As far as we know, not too many results of stability of delay dynamic equations with impulses are analysed by researchers. Although, to the extent of our knowledge, the stability observations of Ulam's type of nonlinear Volterra integro–delay dynamic system having integral impulsions of fractional order are not yet investigated.

Motivated by the work done in [36], the utmost purpose of this manuscript is to find different Hyers–Ulam and Hyers–Ulam–Rassias outcomes of stability for the following nonlinear Volterra integro–delay dynamic system with integrable impulses having fractional order

$$\begin{cases} \omega^{\Delta}(t) = M(t)\omega(t) + \displaystyle\int_{t_0}^{t} \mathcal{K}(t, s, \omega(s), \omega(h(s)))\Delta s, \\ t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \ldots, m, \\ \omega(t) = I^{\alpha}_{t_i,t} g_i(t, \omega(t), \omega(h(t))),\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \ldots, m, \\ \omega(t) = \alpha(t),\ t \in [s_0 - \lambda, s_0] \cap T_S, \\ \omega(t_0) = \alpha(t_0) = \omega_0, \end{cases} \tag{1.1}$$

where $\lambda > 0$, T_S is a time scale, $M(t)$ is a piecewise continuous regressive square matrix, $t_i, s_i \in T_S$ are right–dense points with $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots t_m \leq s_m \leq t_{m+1} = t_f$, $\mathcal{K}(t, s, \omega(s), \omega(h(s)))$ is piecewise continuous operator on $\Gamma = \{(t, s, \omega) : t_0 \leq s \leq t \leq t_f,\ \omega \in \mathbb{R}^n\}$, $g_i : (t_i, s_i] \cap T_S \times \mathbb{R}^n \times \mathbb{R}^n \to$

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$\mathbb{R}^n$, $i = 1, 2, \ldots, m$ are continuous functions, $\phi : [s_0 - \lambda, s_0] \cap T_S \to \mathbb{R}^n$ is history function and $I^{\alpha}_{t_i,t} g_i$ are the so called Riemann–Liouville integrals having fractional order $\alpha \in (0,1)$, with the representation:

$$I^{\alpha}_{t_i,t} g_i(t, \omega(t), \omega(h(t))) = \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-s)^{\alpha-1} g_i(s, \omega(s), \omega(h(s)))\Delta s.$$

Moreover, $(s_i, t_{i+1}] \cap T_S$, $(t_i, s_i] \cap T_S$, $[s_0 - \lambda, s_0] \cap T_S$ are non-empty sets and $h : [s_0 - \lambda, t_f] \cap T_S \to (s_i, t_{i+1}] \cap T_S$ is a delay function with the consumption of continuity, additionally $h(t) \leq t$.

2. Preliminaries

In this section, we recall the main definitions and some basic notations of time scales calculus.

An arbitrary non–empty closed subset of real numbers T_S is called a time scale. The forward jump operator $\Theta : T_S \to T_S$, backward jump operator $\rho : T_S \to T_S$ and graininess operator $\mu : T_S \to [0, \infty)$, are defined by:

$$\Theta(s) = \inf\{t \in T_S : t > s\}, \ \rho(s) = \sup\{t \in T_S : t < s\}, \ \mu(s) = \Theta(s) - s,$$

respectively. An arbitrary $t \in T_S$ is called left scattered (resp. left dense) when $t < \rho(t)$ (resp. $t = \rho(t)$). While, in case of $t < \Theta(t)$ (resp. $\Theta(t) = t$), we call t right scattered (resp. right dense). For a time scale T_S, the set of all limiting points ${T_S}^z$ is called the derived set and illustrated as follows:

$${T_S}^z = \begin{cases} T_S \backslash (\rho(\sup T_S), \sup T_S], & \text{if } \sup T_S < \infty, \\ T_S, & \text{if } \sup T_S = \infty. \end{cases}$$

The function $\mathcal{W} : T_S \to \mathbb{R}$ is called right–dense continuous if it is continuous at every right dense point on T_S and its left sided limit exists at every left dense point on T_S. The function $\mathcal{W} : T_S \to \mathbb{R}$ is called regressive (resp. positively regressive) if $1 + \mu(t)\mathcal{W}(t) \neq 0$, (resp. $1 + \mu(t)\mathcal{W}(t) > 0$) $\forall\ t \in {T_S}^z$. The set of all right–dense continuous regressive functions (resp. right–dense continuous positively regressive functions) will be denoted by $\mathcal{R}_{\mathcal{G}}(T_S)$ (resp. $\mathcal{R}_{\mathcal{G}}(T_S)^+$). The delta derivative of the function $W : T_S \to \mathbb{R}$ on $t \in T_S^z$, is given by

$$W^{\Delta}(t) = \lim_{s \to t,\ s \neq \Theta(t)} \frac{W(\Theta(t)) - W(s)}{\Theta(t) - s}.$$

For a rd–continuous function $W : T_S \to \mathbb{R}$, the $\Delta-$integral is defined to be

$$\int_a^b W(t)\Delta t = w(b) - w(a), \text{ for all } a, b \in T_S,$$

where w is the anti–derivative of W, i.e., $w^{\Delta} = W$ on ${T_S}^z$.

For $p \in \mathcal{R}_{\mathcal{G}}(T_S)$, the generalized exponential function is defined by

$$e_p(a, b) = \exp\left(\int_a^b \alpha_{\mu(s)} p(s) \Delta s \right) \text{ for all } a, b \in T_S,$$

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while,

$$\alpha_{\mu(t)}p(t) = \begin{cases} \dfrac{Log(1+\mu(t)p(t))}{\mu(t)}, & \text{if } \mu(t) \neq 0, \\ p(t), & \text{if } \mu(t) = 0, \end{cases}$$

is the cylindrical transformation.

The fundamental matrix $\Psi_M(t, t_0)$ is the unique solution of the dynamic equation $\omega^{\Delta}(t) = M(t)\omega(t),\ \omega(t_0) = \omega_0,\ t \in {T_S}^0$.

3. Basic concepts and remarks

Let $C(J, \mathbb{R}^n)$ (resp. $PC(J, \mathbb{R}^n)$) be the Banach space of all continuous functions (resp. the Banach space of piecewise continuous functions) with the norm $||\omega||_{\infty} = \sup_{t \in J} ||\omega(t)||,\ J = [s_0 - \lambda, t_f] \cap T_S$ and $\mathbb{R}$ represents the set of real numbers. Finally, we denote by $PC^1(J, \mathbb{R}^n) = \{\omega \in PC(J, \mathbb{R}^n) : \omega^{\Delta} \in PC(J, \mathbb{R}^n)\}$, the Banach space with norm $\|\omega\|_1 = \max\{\|\omega\|_{\infty}, \|\omega^{\Delta}\|_{\infty}\}$. Here, as usual we denote by $\|x\| = \sum_{i=1}^{n} |x_i|$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Consider the following inequalities,

$$\begin{cases} \left|\left|\phi^{\Delta}(t) - M(t)\phi(t) - \displaystyle\int_{t_0}^{t} \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s\right|\right| \leq \epsilon, \\ \quad t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \ldots, m, \\ \left|\left|\phi(t) - I^{\alpha}_{t_i,t} g_i(t, \phi(t), \phi(h(t)))\right|\right| \leq \epsilon,\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \ldots, m, \end{cases} \tag{3.1}$$

$$\begin{cases} \left|\left|\phi^{\Delta}(t) - M(t)\phi(t) - \displaystyle\int_{t_0}^{t} \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s\right|\right| \leq \varphi(t), \\ \quad t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \ldots, m, \\ \left|\left|\phi(t) - I^{\alpha}_{t_i,t} g_i(t, \phi(t), \phi(h(t)))\right|\right| \leq \kappa,\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \ldots, m, \end{cases} \tag{3.2}$$

where $\epsilon > 0,\ \kappa \geq 0$ and $\varphi \in PC(J, \mathbb{R}^+)$ is an increasing function.

Definition 3.1. Eq. (1.1) is said to be stable in the sense of Hyers–Ulam, if for every $\epsilon > 0$ there exists a positive number K such that for every $\phi \in PC^1(J, \mathbb{R}^n)$ satisfying (3.1), there exists a solution $\phi_0 \in PC^1(J, \mathbb{R}^n)$ of (1.1) such that $\|\phi_0(t) - \phi(t)\| \leq K\epsilon$ for all $t \in J$. Here K is a positive number that depends on ϵ and do not depend on f_i.

Definition 3.2. Eq. (1.1) is said to be stable in the sense of Hyers–Ulam–Rassias, provided for all $(\varphi, \kappa) \in PC(J, \mathbb{R}^+) \times \mathbb{R}^+$ there exists $M > 0$ such that for all $\phi \in PC^1(J, \mathbb{R}^n)$ satisfying (3.2), there exists a solution $\phi_0 \in PC^1(J, \mathbb{R}^n)$ of (1.1) such that the inequality $\|\phi_0(t) - \phi(t)\| \leq M\varphi(t)$ is true for all $t \in J$. Here $M > 0$ depends on (φ, κ).

Definition 3.3. In a metric space $(X; d)$, a mapping $\Lambda : X \to X$ is said to be Picard operator if it has precisely a unique fixed point $x^* \in X$, so that for every $x \in X$, the sequence $\{\Lambda^{(n)}(x)\}_{n \in \mathbb{N}}$ converges to x^*.

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Lemma 3.4. [16] *Suppose* $\tau \in T_S^+$, $y, b \in \mathcal{R}_{\mathcal{G}}(T_S{}^+)$, $p \in \mathcal{R}_{\mathcal{G}}(T_S{}^+)^+$ *and* $c,\ b_k \in \mathbb{R}^+$, $k = 1, 2, \dots,$ *so*

$$y(t) \leq c + \int_\tau^t p(s)y(s)\Delta s + \sum_{\tau < t_k < t} b_k y(t_k),$$

implies

$$y(t) \leq c \prod_{\tau < t_k < t} (1 + b_k) e_p(t, \tau),\ t \geq \tau.$$

Lemma 3.5. ***(Abstract Grönwall Lemma*** [22]***):*** *Let* $(X, d, \leq)$ *be an ordered metric space and let* x^* *be a fixed point for the increasing mapping* $\Lambda : X \to X$*. So, being arbitrary* $x \in X$, $x \leq \Lambda(x)$ *entails* $x \leq x^*$ *and* $x \geq \Lambda(x)$ *entails* $x \geq x^*$*, where* x^* *denotes the fixed point in* Λ*.*

Remark 3.6. A function $\phi \in PC^1(J, \mathbb{R}^n)$ satisfies inequality (3.1) (resp. inequality (3.2)) if and only if there exist a function $f \in PC^1(J, \mathbb{R}^n)$ and a finite sequence $\{f_k : k = 1, \dots, m\} \subset \mathbb{R}^n$ (dependent on ϕ) such that $\|f(t)\| \leq \epsilon$ for all $t \in J$ and $\|f_i\| \leq \epsilon$ (resp. $\|f_i\| \leq \kappa$) for every $i = 1, 2, \dots, m$ and

$$\begin{cases} \phi^\Delta(t) = M(t)\phi(t) + \displaystyle\int_{t_0}^t \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s + f(t), \\ t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \dots, m, \\ \phi(t) = I_{t_i,t}^\alpha g_i(t, \phi(t), \phi(h(t))) + f_i,\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \dots, m. \end{cases}$$

Lemma 3.7. *If* $\phi \in PC^1(J, \mathbb{R}^n)$ *satisfies inequality* (3.1) (*resp. inequality* (3.2))*, then the following inequalities*

$$\begin{cases} \Bigg\|\phi(t) - \phi_0 - \Psi_M(t, t_0)\phi_0 - \displaystyle\int_{s_i}^t \Psi_M(t, \Theta(s)) \int_{s_0}^s \mathcal{K}(s, u, \phi(u), \phi(h(u)))\Delta u \Delta s \\ - I_{t_i,t}^\alpha g_i(t, \phi(t), \phi(h(t)))\Bigg\| \leq (Ct_f - Cs_i + m)\epsilon, \\ t \in (s_i, t_{i+1}] \cap T_S,\ i = 1, 2, \dots, m, \\ \Big\|\phi(t) - I_{t_i,t}^\alpha g_i(t, \phi(t), \phi(h(t)))\Big\| \leq m\epsilon,\ (\text{resp. } m\kappa), \\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \dots, m, \end{cases}$$

are true. Here C *is the bound of fundamental matrix* $\Psi_M(t, \Theta(s))$*.*

Proof. If $\phi \in PC^1(J, \mathbb{R}^n)$ satisfies (3.1), then by Remark 3.6, we have

$$\begin{cases} \phi^\Delta(t) = M(t)\phi(t) + \displaystyle\int_{t_0}^t \mathcal{K}(t, s, \phi(s), \phi(h(s)))\Delta s + f(t), \\ t \in (s_i, t_{i+1}] \cap T_S,\ i = 0, 1, \dots, m, \\ \phi(t) = I_{t_i,t}^\alpha g_i(t, \phi(t), \phi(h(t))) + f_i,\ t \in (t_i, s_i] \cap T_S,\ i = 1, 2, \dots, m. \end{cases} \tag{3.3}$$

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Clearly the solution of (3.3) is given as

$$\phi(t)=\begin{cases}\phi_0+\Psi_M(t,t_0)\phi_0+\displaystyle\int_{s_i}^{t}\Psi_M(t,\Theta(s))\Big(\int_{s_0}^{s}\mathcal{K}(s,u,\phi(u),\phi(h(u)))\Delta u+f(s)\Big)\Delta s\\ \quad +I^{\alpha}_{t_i,t}g_i(t,\phi(t),\phi(h(t))),\ t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\dots,m,\\ I^{\alpha}_{t_i,t}g_i(t,\phi(t),\phi(h(t)))+f_i,\ t\in(t_i,s_i]\cap T_S,\ i=1,2,\dots,m.\end{cases}$$

For $t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\dots,m$, we get

$$\begin{aligned}\Big\|\phi(t)-\phi_0-\Psi_M(t,t_0)\phi_0 &- \int_{s_i}^{t}\Psi_M(t,\Theta(s))\int_{s_0}^{s}\mathcal{K}(s,u,\phi(u),\phi(h(u)))\Delta u\Delta s\\ &\quad -I^{\alpha}_{t_i,t}g_i(t,\phi(t),\phi(h(t)))\Big\|\\ &\le \int_{s_i}^{t}||\Psi_M(t,\Theta(s))||\|f(s)\|ds+\sum_{i=1}^{m}\|f_i\|\\ &\le (Ct-Cs_i+m)\epsilon\\ &\le (Ct_f-Cs_i+m)\epsilon.\end{aligned}$$

Proceeding as above we derive

$$\Big\|\phi(t)-I^{\alpha}_{t_i,t}g_i(t,\phi(t),\phi(h(t)))\Big\| \le m\epsilon,\ t\in(t_i,s_i]\cap T_S,\ i=1,2,\dots,m.$$

We have similar processions for (3.2). □

4. Main Results

Onward we will state our major results. The first solution to be establish is Hyers–Ulam stablity. First we assume the following conditions:
($\mathbf{A_1}$) The function $\mathcal{K}$ is piecewise continuous with the Lipschitz condition $||\mathcal{K}(t,s,x_1,x_2)-\mathcal{K}(t,s,y_1,y_2)||\le\sum_{k=1}^{2}L||x_k-y_k||$, $L>0$, for all $t\in(s_i,t_{i+1}]\cap T_S,\ i=0,1,\dots,m$ and $x_k,y_k\in\mathbb{R}^n$, $k\in\{1,2\}$;

($\mathbf{A_2}$) $g_i:(t_i,s_i]\cap T_S\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the Lipschitz condition $||g_i(t,u_1,u_2)-g_i(t,v_1,v_2)||\le\sum_{k=1}^{2}L_{g_i}||u_k-v_k||$, $L_{g_i}>0$, for all $t\in(t_i,s_i]\cap T_S,\ i=1,2,\dots,m$ and $u_1,u_2,v_1,v_2\in\mathbb{R}^n$;

($\mathbf{A_3}$) $\left(\frac{2L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\Delta s+2CL\int_{s_i}^{t}\int_{s_0}^{s}\Delta u\Delta s\right)<1,\ i=1,2,\dots,m;$

($\mathbf{A_4}$) $\varphi\in PC(J,\mathbb{R}^+)$ is increasing so that for some $\rho>0$,

$$\int_{t_0}^{t}\varphi(r)\Delta r\le\rho\varphi(t).$$

Theorem 4.1. *If conditions* ($\mathbf{A_1}$) − ($\mathbf{A_3}$) *hold, then Eq. (1.1) has precisely a unique solution in* $PC^1(J,\mathbb{R}^n)$.

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Proof. i) Determine an operator $\Lambda : PC(J,\mathbb{R}^n) \to PC(J,\mathbb{R}^n)$ as

$$(\Lambda\omega)(t) = \begin{cases} \alpha(t),\ t \in [s_0-\lambda, s_0]\cap T_S, \\ I^{\alpha}_{t_i,s_i} g_i(t,\omega(t),\omega(h(t))),\ t\in(t_i,s_i]\cap T_S,\ i=1,2,\ldots,m,\ \alpha\in(0,1), \\ \alpha(t_0)+\Psi_M(t,t_0)\omega_0 + I^{\alpha}_{t_i,s_i} g_i(s_i,\omega(s_i),\omega(h(s_i))) \\ +\displaystyle\int_{s_i}^{t}\Psi_M(t,\Theta(s))\int_{s_0}^{s}\mathcal{K}(s,u,\omega(u),\omega(h(u)))\Delta u\Delta s, \\ t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\ldots,m,\ \alpha\in(0,1). \end{cases} \tag{4.1}$$

For any $\omega_1,\, \omega_2 \in PC(J,\mathbb{R}^n),\ t \in (s_i,t_{i+1}]\cap T_S,\ i=1,2,\ldots,m$, we have

$$\begin{aligned} \big|\big|(\Lambda\omega_1)(t)-(\Lambda\omega_2)(t)\big|\big| &\leq \big|\big|I^{\alpha}_{t_i,s_i} g_i(s_i,\omega_1(s_i),\omega_1(h(s_i))) - I^{\alpha}_{t_i,s_i} g_i(s_i,\omega_2(s_i),\omega_2(h(s_i)))\big|\big| \\ &\quad + \int_{s_i}^{t}\big|\big|\Psi_M(t,\Theta(s))\big|\big|\int_{s_0}^{s}\Big|\Big|\mathcal{K}(s,u,\omega_1(u),\omega_1(h(u))) \\ &\quad -\mathcal{K}(s,u,\omega_2(u),\omega_2(h(u)))\Big|\Big|\Delta u\Delta s \\ &\leq \frac{1}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|g_i(s,\omega_1(s),\omega_1(h(s))) \\ &\quad -g_i(s,\omega_2(s),\omega_2(h(s)))\big|\big|\Delta s \\ &\quad +L\int_{s_i}^{t}\big|\big|\Psi_M(t,\Theta(s))\big|\big|\int_{s_0}^{s}\big|\big|\omega_1(u)-\omega_2(u)\big|\big|\Delta u\Delta s \\ &\quad +L\int_{s_i}^{t}\big|\big|\Psi_M(t,\Theta(s))\big|\big|\int_{s_0}^{s}\big|\big|\omega_1(h(u))-\omega_2(h(u))\big|\big|\Delta u\Delta s \\ &\leq \frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|\omega_1(s)-\omega_2(s)\big|\big|\Delta s \\ &\quad +\frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|\omega_1(h(s))-\omega_2(h(s))\big|\big|\Delta s \\ &\quad +2CL\int_{s_i}^{t}\int_{s_0}^{s}\sup_{s_i\leq s\leq t_{i+1}}\big|\big|\omega_1(u)-\omega_2(u)\big|\big|\Delta u\Delta s \\ &\leq \frac{2L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\sup_{t_i\leq s\leq s_i}\big|\big|\omega_1(s)-\omega_2(s)\big|\big|\Delta s + \\ &\quad 2CL\int_{s_i}^{t}\int_{s_0}^{s}\sup_{s_i\leq s\leq t_{i+1}}\big|\big|\omega_1(s)-\omega_2(s)\big|\big|\Delta u\Delta s \\ &\leq \left(\frac{2L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\Delta s + 2CL\int_{s_i}^{t}\int_{s_0}^{s}\Delta u\Delta s\right)\|\omega_1-\omega_2\|_\infty. \end{aligned}$$

According to (c), we are dealing here with the strictly contractive operator on $(s_i,t_{i+1}]\cap T_S,\ i=1,2,\ldots,m$, and hence a Picard operator on $PC(J,\mathbb{R}^n)$. Regarding to (4.1), it shows that the unique solution of Eq. (1.1) in $PC^1(J,\mathbb{R}^n)$

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is in fact the unique fixed point of this operator.

□

Theorem 4.2. *If conditions* $(\mathbf{A_1})-(\mathbf{A_3})$ *hold, then Eq. (1.1) has Hyers–Ulam stability on* J.

Proof. Assume that (3.1) has a solution $\phi \in PC^1(J,\mathbb{R}^n)$. Then for dynamic equation

$$\begin{cases} \omega^{\Delta}(t) = M(t)\omega(t) + \int_{t_0}^{t} \mathcal{K}(t,s,\omega(s),\omega(h(s)))\Delta s,\ t\in(s_i,t_{i+1}]\cap T_S,\ i=0,1,\dots,m, \\ \omega(t) = I^{\alpha}_{t_i,t} g_i(t,\omega(t),\omega(h(t))),\ t\in(t_i,s_i]\cap T_S,\ i=1,2,\dots,m, \\ \omega(t)=\phi(t),\ t\in[s_0-\lambda,s_0]\cap T_S, \\ \omega(t_0)=\phi(t_0)=\omega_0, \end{cases}$$

we have the unique solution

$$\omega(t) = \begin{cases} \phi(t),\ t\in[s_0-\lambda,s_0]\cap T_S, \\ I^{\alpha}_{t_i,s_i} g_i(t,\omega(t),\omega(h(t))),\ t\in(t_i,s_i]\cap T_S,\ i=1,2,\dots,m,\ \alpha\in(0,1), \\ \phi(t_0)+\Psi_M(t,t_0)\omega_0 + I^{\alpha}_{t_i,s_i} g_i(s_i,\omega(s_i),\omega(h(s_i))) \\ +\int_{s_i}^{t}\Psi_M(t,\Theta(s))\int_{s_0}^{s}\mathcal{K}(s,u,\omega(u),\omega(h(u)))\Delta u\Delta s, \\ t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\dots,m. \end{cases}$$

We observe that for all $t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\dots,m$, using Lemma 3.7, we have

$$\begin{aligned} \big|\big|\phi(t)-\omega(t)\big|\big| \ \leq\ & \big|\big|\phi(t)-\phi_0-\Psi_M(t,t_0)\phi_0-\int_{s_i}^{t}\Psi_M(t,\Theta(s))\int_{s_0}^{s}\mathcal{K}(s,u,\phi(u),\phi(h(u)))\Delta u\Delta s \\ & -I^{\alpha}_{t_i,t} g_i(t,\phi(t),\phi(h(t)))\big|\big| + \big|\big|I^{\alpha}_{t_i,s_i} g_i(s_i,\phi(s_i),\phi(h(s_i))) \\ & -I^{\alpha}_{t_i,s_i} g_i(s_i,\omega(s_i),\omega(h(s_i)))\big|\big| \\ & +\int_{s_i}^{t}\big|\big|\Psi_M(t,\Theta(s))\big|\big|\int_{s_0}^{s}\big|\big|\mathcal{K}(s,u,\phi(u),\phi(h(u))) \\ & -\int_{s_0}^{s}\mathcal{K}(s,u,\omega(u),\omega(h(u)))\big|\big|\Delta u\Delta s \\ \leq\ & (m+Ct_f-Cs_i)\epsilon + \frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|\phi(s)-\omega(s)\big|\big|\Delta s \\ & +\frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|\phi(h(s))-\omega(h(s))\big|\big|\Delta s \\ & +CL\int_{s_i}^{t}\int_{s_0}^{s}\big|\big|\phi(u)-\omega(u)\big|\big|\Delta u\Delta s \\ & +CL\int_{s_i}^{t}\int_{s_0}^{s}\big|\big|\phi(h(u))-\omega(h(u))\big|\big|\Delta u\Delta s. \end{aligned}$$

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Next, we show that the operator $T : PC(J,\mathbb{R}^+) \to PC(J,\mathbb{R}^+)$ given below is an increasing Picard operator:

$$\begin{aligned}(Tg)(t) &= (m+Ct_f-Cs_i)\epsilon + \frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}g(s)\Delta s\\ &+\frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}g(h(s))\Delta s + CL\int_{s_i}^{t}\int_{s_0}^{s}g(u)\Delta u\Delta s\\ &+CL\int_{s_i}^{t}\int_{s_0}^{s}g(h(u))\Delta u\Delta s.\end{aligned}$$

For any $g_1,\, g_2 \in PC(J,\mathbb{R}^+),\ t\in(s_i,t_{i+1}]\cap T_S,\ i=1,2,\ldots,m$, we have

$$\begin{aligned}\big|\big|(Tg_1)(t)-(Tg_2)(t)\big|\big| &\le \frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|g_1(s)-g_2(s)\big|\big|\Delta s\\ &+\frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\big|\big|g_1(h(s))-g_2(h(s))\big|\big|\Delta s\\ &+CL\int_{s_i}^{t}\int_{s_0}^{s}\big|\big|g_1(u)-g_2(u)\big|\big|\Delta u\Delta s\\ &+CL\int_{s_i}^{t}\int_{s_0}^{s}\big|\big|g_1(h(u))-g_2(h(u))\big|\big|\Delta u\Delta s\\ &\le \frac{2L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\sup_{t_i\le s\le s_i}\big|\big|g_1(s)-g_2(s)\big|\big|\Delta s\\ &+2CL\int_{s_i}^{t}\int_{s_0}^{s}\sup_{s_i\le s\le t_{i+1}}\big|\big|g_1(s)-g_2(s)\big|\big|\Delta u\Delta s\\ &\le \left(\frac{2L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\Delta s+2CL\int_{s_i}^{t}\int_{s_0}^{s}\Delta u\Delta s\right)\|g_1-g_2\|_\infty\\ &\le \left(\frac{2L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}\Delta s+2CL\int_{s_i}^{t}\int_{s_0}^{s}\Delta u\Delta s\right)\|g_1-g_2\|_\infty.\end{aligned}$$

Again according to (c), we are dealing here with the strictly contractive operator on $(s_i,t_{i+1}]\cap T_S,\ i=1,2,\ldots,m$ and hence a Picard operator on $PC(J,\mathbb{R}^+)$. Banach fixed point theorem imply, T is Picard operator having unique fixed point $g^*\in PC(J,\mathbb{R}^+)$ i.e.,

$$\begin{aligned}g^*(t) &= (m+Ct_f-Cs_i)\epsilon + \frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}g^*(s)\Delta s\\ &+\frac{L_{g_i}}{\Gamma(\alpha)}\int_{t_i}^{s_i}(s_i-s)^{\alpha-1}g^*(h(s))\Delta s + CL\int_{s_i}^{t}\int_{s_0}^{s}g^*(u)\Delta u\Delta s\\ &+CL\int_{s_i}^{t}\int_{s_0}^{s}g^*(h(u))\Delta u\Delta s.\end{aligned}$$

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As, g^* is increasing, therefore $g^*(h(t)) \leq g^*(t)$, $(m + Ct_f - Cs_i) \leq \delta$ for some $\delta > 0$ and for $i = 1, 2, \ldots, m$, we can write

$$g^*(t) \leq \delta\epsilon + \frac{2L_{g_i}}{m\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} g^*(s)\Delta s + 2CL \int_{s_0}^{t} \int_{s_0}^{s} g^*(u)\Delta u \Delta s.$$

Using Lemma 3.4, we have

$$g^*(t) \leq \delta\epsilon \prod_{s_i < s < t} \left(1 + \frac{2L_{g_i}}{m\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \Delta s\right) e_q(t, s_i).$$

where $q = 2CL \int_{s_0}^{s} \Delta u$. If we determine $g = ||\phi - \omega||$, then $g(t) \leq (Tg)(t)$, which follows by utilizing abstract Grönwall lemma that $g(t) \leq g^*$, hence

$$||\phi(t) - \omega(t)|| \leq \delta\epsilon \prod_{s_i < s < t} \left(1 + \frac{2L_{g_i}}{m\Gamma(\alpha)} \int_{t_i}^{s_i} (s_i - s)^{\alpha-1} \Delta s\right) e_q(t, s_i).$$

□

Similarly we can establish the Hyers–Ulam–Rassias stability of (1.1) on J. Its proof will be omitted.

Theorem 4.3. *If conditions* $(\mathbf{A_1}) - (\mathbf{A_4})$ *hold, then Eq.* (1.1) *has Hyers–Ulam–Rassias stability on* J.

5. Conclusion

This manuscript is about the establishment of Hyers–Ulam stability and Hyers–Ulam–Rassias stability of equation (1.1) with the utilization of fixed point approach. Also, the unique solution to (1.1) in $PC^1(J, \mathbb{R}^n)$ is obtained. Furthermore, abstract Grönwall lemma and Lemma 3.4 presented a fruitful outcome to our end. Our work assures the existence of an exact solution of (1.1) near to approximate solution. In fact, our results are significant when finding exact solution is quite difficult and hence are important to approximation theory etc.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Competing interest

The authors declare that they have no competing interest regarding this research work.

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Author's contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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