Iranian Journal of Mathematical Sciences and Informatics
Vol. 16, No. 2 (2021), pp 179-195
DOI: 10.29252/ijmsi.16.2.163

# $\mathcal{N}$-Subalgebras of $BCK/BCI$-Algebras which are Induced from Hyperfuzzy Structures

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Abstract. In the paper [J. Ghosh and T.K. Samanta, Hyperfuzzy sets and hyperfuzzy group, Int. J. Advanced Sci Tech. 41 (2012), 27–37], Ghosh and Samanta introduced the concept of hyperfuzzy sets as a generalization of fuzzy sets and interval-valued fuzzy sets, and applied it to group theory. The aim of this manuscript is to study $\mathcal{N}$-structures in $BCK/BCI$-algebras induced from hyperfuzzy structures.

**Keywords:** Hyperfuzzy set, Hyperfuzzy structure, Hyperfuzzy subalgebra, $\mathcal{N}$-Subalgebra, Induced $\mathcal{N}$-Function.
**2000 Mathematics subject classification:** 06F35, 03G25, 03B52

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Received 08 April 2018; Accepted 05 January 2021

## 1. Introduction

Fuzzy set theory is firstly introduced by Zadeh [15] and opened a new path of thinking to mathematicians, physicists, chemists, engineers and many others due to its diverse applications in various fields. Algebraic hyperstructure, which is introduced by the French mathematician Marty [13], represent a natural extension of classical algebraic structures. Since then, many papers and several books have been written in this area. Nowadays, hyperstructures have a lot of applications in several domains of mathematics and computer sciences. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The study of fuzzy hyperstructures is an interesting research area of fuzzy sets. As a generalization of fuzzy sets and interval-valued fuzzy sets, Ghosh and Samanta [9] introduced the notion of hyperfuzzy sets, and applied it to group theory. Jun et al. [11] applied the hyperfuzzy sets to $BCK/BCI$-algebras, and introduced the notion of $k$-fuzzy substructure for $k \in \{1, 2, 3, 4\}$. They introduced the concepts of hyperfuzzy substructures of several types by using $k$-fuzzy substructures, and investigated their basic properties. They also introduced the notion of hyperfuzzy subalgebras of type $(i, j)$ for $i, j \in \{1, 2, 3, 4\}$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [12] introduced and used a new function which is called negative-valued function. The important achievement of the paper [12] was that one can deal with positive and negative information simultaneously by combining ideas in [12] and already well known positive information.

In this paper, we study $\mathcal{N}$-structures in $BCK/BCI$-algebras induced from hyperfuzzy structures. We introduce the notions of $\mathcal{N}_k$-subalgebras in $BCK/BCI$-algebras for $k \in \{1, 2, 3, 4\}$, and investigate several properties. We investigate relations between $\mathcal{N}_k$-subalgebras induced from hyperfuzzy sets and $(i, j)$-hyperfuzzy subalgebras in $BCK/BCI$-algebras for $i, j, k \in \{1, 2, 3, 4\}$.

## 2. Preliminaries

By a *$BCI$-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a $BCI$-algebra $X$ satisfies $0 * x = 0$ for all $x \in X$, then we say that $X$ is a *$BCK$-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0).$$

In a $BCK/BCI$-algebra $X$, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{2.1}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2.2}$$

A non-empty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [10] and [14] for further information regarding $BCK/BCI$-algebras.

By a *fuzzy structure* over a nonempty set $X$ we mean an ordered pair $(X, \rho)$ of $X$ and a fuzzy set $\rho$ on $X$.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set $X$ to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from $X$ to $[-1, 0]$ (briefly, $\mathcal{N}$-*function* on $X$.) By an $\mathcal{N}$-*structure* we mean an ordered pair $(X, \lambda)$ of $X$ and an $\mathcal{N}$-function $\lambda$ on $X$.

Let $X$ be a nonempty set. A mapping $\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0, 1])$ is called a *hyperfuzzy set* over $X$ (see [9]), where $\tilde{\mathcal{P}}([0, 1])$ is the family of all nonempty subsets of $[0, 1]$. An ordered pair $(X, \tilde{\lambda})$ is called a *hyper structure* over $X$.

Given a hyper structure $(X, \tilde{\lambda})$ over a nonempty set $X$, we consider two fuzzy structures $(X, \tilde{\lambda}_{\inf})$ and $(X, \tilde{\lambda}_{\sup})$ over $X$ in which

$$\tilde{\lambda}_{\inf} : X \to [0, 1],\ x \mapsto \inf\{\tilde{\lambda}(x)\},$$

$$\tilde{\lambda}_{\sup} : X \to [0, 1],\ x \mapsto \sup\{\tilde{\lambda}(x)\}.$$

Given a nonempty set $X$, let $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$ denote the collection of all $BCK$-algebras and all $BCI$-algebras, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

**Definition 2.1** ([11])**.** For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure $(X, \lambda)$ over $(X, *, 0)$ is called a

- *fuzzy subalgebra* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\})\,, \tag{2.3}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\lambda(x * y) \leq \min\{\lambda(x), \lambda(y)\})\,, \tag{2.4}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\lambda(x * y) \geq \max\{\lambda(x), \lambda(y)\})\,, \tag{2.5}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\})\,. \tag{2.6}$$

**Definition 2.2** ([11])**.** For any $(X, *, 0) \in \mathcal{B}(X)$ and $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ is called an $(i, j)$-*hyperfuzzy subalgebra* of $(X, *, 0)$ if $(X, \tilde{\lambda}_{\inf})$ is an $i$-fuzzy subalgebra of $(X, *, 0)$ and $(X, \tilde{\lambda}_{\sup})$ is a $j$-fuzzy subalgebra of $(X, *, 0)$.

## 3. $\mathcal{N}$-subalgebras based on hyperfuzzy structures

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ unless otherwise specified.

**Definition 3.1.** Given a hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$, we define an $\mathcal{N}$-function on $(X, *, 0)$ as follows:

$$\tilde{\lambda}_{\mathcal{N}} : X \to [-1, 0],\ x \mapsto \tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x),$$

which is called an *induced* $\mathcal{N}$*-function* from $(X, \tilde{\lambda})$ on $(X, *, 0)$.

**Definition 3.2.** A hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ is called an

- $\mathcal{N}_1$-*subalgebra* of $(X, *, 0)$ if

$$(\forall x, y \in X) \left( \tilde{\lambda}_{\mathcal{N}}(x * y) \geq \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \right), \tag{3.1}$$

- $\mathcal{N}_2$-*subalgebra* of $(X, *, 0)$ if

$$(\forall x, y \in X) \left( \tilde{\lambda}_{\mathcal{N}}(x * y) \leq \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \right), \tag{3.2}$$

- $\mathcal{N}_3$-*subalgebra* of $(X, *, 0)$ if

$$(\forall x, y \in X) \left( \tilde{\lambda}_{\mathcal{N}}(x * y) \geq \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \right), \tag{3.3}$$

- $\mathcal{N}_4$-*subalgebra* of $(X, *, 0)$ if

$$(\forall x, y \in X) \left( \tilde{\lambda}_{\mathcal{N}}(x * y) \leq \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \right). \tag{3.4}$$

Example 3.3. Consider a $BCK$-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [14]).

Table 1. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 | 0 |
| 4 | 4 | 3 | 4 | 1 | 0 |

Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\lambda}$ is given as follows:

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.2, 0.4) & \text{if } x = 0, \\ (0.1, 0.3] \cup [0.5, 0.9) & \text{if } x = 1, \\ [0.1, 0.3] & \text{if } x = 2, \\ [0.3, 0.4) \cup [0.5, 0.6] & \text{if } x = 3, \\ [0.3, 0.8] & \text{if } x = 4. \end{cases}$$

Then the induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$ is given by Table 2.

TABLE 2. Induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\tilde{\lambda}_{\mathcal{N}}$ | $-0.2$ | $-0.8$ | $-0.2$ | $-0.3$ | $-0.5$ |

EXAMPLE 3.4. Consider a $BCI$-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ which is given in Table 3 (see [14]).

TABLE 3. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | $a$ | $b$ | $c$ |
| 1 | 1 | 0 | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | 0 | $c$ | $b$ |
| $b$ | $b$ | $b$ | $c$ | 0 | $a$ |
| $c$ | $c$ | $c$ | $b$ | $a$ | 0 |

Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\lambda}$ is given as follows:

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.2, 1.0) & \text{if } x = 0, \\ (0.1, 0.4] \cup [0.5, 0.8] & \text{if } x = 1, \\ [0.4, 0.9] & \text{if } x = a, \\ [0.3, 0.6) & \text{if } x \in \{b, c\} \end{cases}$$

The induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$ is given by Table 4.

TABLE 4. Induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$

| $X$ | 0 | 1 | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|
| $\tilde{\lambda}_{\mathcal{N}}$ | $-0.8$ | $-0.7$ | $-0.5$ | $-0.3$ | $-0.3$ |

It is routine to verify that $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$.

Given a hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ and $t \in [-1, 0]$, consider the following sets:

$$U_{\mathcal{N}}(\tilde{\lambda}; t) := \{x \in X \mid \tilde{\lambda}_{\mathcal{N}}(x) \geq t\}, \tag{3.5}$$

$$L_{\mathcal{N}}(\tilde{\lambda}; t) := \{x \in X \mid \tilde{\lambda}_{\mathcal{N}}(x) \leq t\}. \tag{3.6}$$

**Theorem 3.5.** *A hyper structure* $(X, \tilde{\lambda})$ *over* $(X, *, 0)$ *is an* $\mathcal{N}_1$*-subalgebra of* $(X, *, 0)$ *if and only if the following assertion is valid.*

$$(\forall t \in [-1, 0]) \Big( U_{\mathcal{N}}(\tilde{\lambda}; t) \neq \emptyset \;\Rightarrow\; U_{\mathcal{N}}(\tilde{\lambda}; t) \text{ is a subalgebra of } (X, *, 0) \Big). \tag{3.7}$$

*Proof.* Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$ and let $t \in [-1, 0]$ be such that $U_{\mathcal{N}}(\tilde{\lambda}; t)$ is nonempty. If $x, y \in U_{\mathcal{N}}(\tilde{\lambda}; t)$, then $\tilde{\lambda}_{\mathcal{N}}(x) \geq t$ and $\tilde{\lambda}_{\mathcal{N}}(y) \geq t$. It follows from (3.1) that

$$\tilde{\lambda}_{\mathcal{N}}(x * y) \geq \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \geq t$$

and so that $x * y \in U_{\mathcal{N}}(\tilde{\lambda}; t)$. Hence $U_{\mathcal{N}}(\tilde{\lambda}; t)$ is a subalgebra of $(X, *, 0)$.

Conversely, assume that $U_{\mathcal{N}}(\tilde{\lambda}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [-1, 0]$ with $U_{\mathcal{N}}(\tilde{\lambda}; t) \neq \emptyset$. If there exist $a, b \in X$ such that

$$\tilde{\lambda}_{\mathcal{N}}(a * b) < \min\{\tilde{\lambda}_{\mathcal{N}}(a), \tilde{\lambda}_{\mathcal{N}}(b)\},$$

then $a, b \in U_{\mathcal{N}}(\tilde{\lambda}; t)$ and so $a * b \in U_{\mathcal{N}}(\tilde{\lambda}; t)$ by taking $t := \min\{\tilde{\lambda}_{\mathcal{N}}(a), \tilde{\lambda}_{\mathcal{N}}(b)\}$. Thus $\tilde{\lambda}_{\mathcal{N}}(a * b) \geq t$, which is a contradiction. Hence

$$\tilde{\lambda}_{\mathcal{N}}(x * y) \geq \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}$$

for all $x, y \in X$. Therefore $(X, \tilde{\lambda})$ is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$. □

**Corollary 3.6.** *If a hyper structure* $(X, \tilde{\lambda})$ *over* $(X, *, 0)$ *is an* $\mathcal{N}_3$*-subalgebra of* $(X, *, 0)$*, then the assertion* (3.7) *is valid.*

The converse of Corollary 3.6 may not be true as seen in the following example.

Example 3.7. Consider a $BCI$-algebra $X = \{0, 1, 2, a, b\}$ with the binary operation $*$ which is given in Table 5 (see [14]).

Table 5. Cayley table for the binary operation "$*$"

| $*$ | 0 | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $a$ | $a$ |
| 1 | 1 | 0 | 1 | $b$ | $a$ |
| 2 | 2 | 2 | 0 | $a$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | 0 | 0 |
| $b$ | $b$ | $a$ | $b$ | 1 | 0 |

Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\lambda}$ is given as follows:

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.2, 0.4) & \text{if } x = 0, \\ (0.1, 0.4] \cup [0.5, 0.7] & \text{if } x = 1, \\ [0.5, 0.8] & \text{if } x = 2, \\ [0.4, 0.5) \cup (0.6, 0.8] & \text{if } x = a, \\ [0.3, 0.9) & \text{if } x = b \end{cases}$$

The induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$ is given by Table 6.

Table 6. Induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$

| $X$ | 0 | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| $\tilde{\lambda}_{\mathcal{N}}$ | $-0.2$ | $-0.6$ | $-0.3$ | $-0.4$ | $-0.6$ |

Hence we have

$$U_{\mathcal{N}}(\tilde{\lambda}; t) = \begin{cases} \emptyset & \text{if } t \in (-0.2, 0], \\ \{0\} & \text{if } t \in (-0.3, -0.2], \\ \{0, 2\} & \text{if } t \in (-0.4, -0.3], \\ \{0, 2, a\} & \text{if } t \in (-0.6, -0.4], \\ X & \text{if } t \in [-1, -0.6], \end{cases}$$

and so $U_{\mathcal{N}}(\tilde{\lambda}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [-1, 0]$ with $U_{\mathcal{N}}(\tilde{\lambda}; t) \neq \emptyset$. But $(X, \tilde{\lambda})$ is not an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$ since

$$\tilde{\lambda}_{\mathcal{N}}(b * a) = \tilde{\lambda}_{\mathcal{N}}(1) = -0.6 < -0.4 = \max\{\tilde{\lambda}_{\mathcal{N}}(b), \tilde{\lambda}_{\mathcal{N}}(a)\}.$$

**Theorem 3.8.** *A hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ if and only if the following assertion is valid.*

$$(\forall t \in [-1, 0]) \Big( L_{\mathcal{N}}(\tilde{\lambda}; t) \neq \emptyset \ \Rightarrow\ L_{\mathcal{N}}(\tilde{\lambda}; t) \text{ is a subalgebra of } (X, *, 0) \Big). \quad (3.8)$$

*Proof.* Assume that $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ and let $t \in [-1, 0]$ be such that $L_{\mathcal{N}}(\tilde{\lambda}; t)$ is nonempty. If $x, y \in L_{\mathcal{N}}(\tilde{\lambda}; t)$, then $\tilde{\lambda}_{\mathcal{N}}(x) \leq t$ and $\tilde{\lambda}_{\mathcal{N}}(y) \leq t$. It follows from (3.4) that

$$\tilde{\lambda}_{\mathcal{N}}(x * y) \leq \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \leq t$$

and so that $x * y \in L_{\mathcal{N}}(\tilde{\lambda}; t)$. Hence $L_{\mathcal{N}}(\tilde{\lambda}; t)$ is a subalgebra of $(X, *, 0)$.

Conversely, suppose that $L_{\mathcal{N}}(\tilde{\lambda}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [-1, 0]$ with $L_{\mathcal{N}}(\tilde{\lambda}; t) \neq \emptyset$. Assume that there exist $a, b \in X$ such that

$$\tilde{\lambda}_{\mathcal{N}}(a * b) > \max\{\tilde{\lambda}_{\mathcal{N}}(a), \tilde{\lambda}_{\mathcal{N}}(b)\}.$$

If we take $t := \max\{\tilde{\lambda}_{\mathcal{N}}(a), \tilde{\lambda}_{\mathcal{N}}(b)\}$, then $a, b \in L_{\mathcal{N}}(\tilde{\lambda}; t)$ and so $a * b \in L_{\mathcal{N}}(\tilde{\lambda}; t)$. Thus $\tilde{\lambda}_{\mathcal{N}}(a * b) \leq t$, which is a contradiction. Hence

$$\tilde{\lambda}_{\mathcal{N}}(x * y) \leq \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}$$

for all $x, y \in X$. Therefore $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$. □

**Corollary 3.9.** *If a hyper structure* $(X, \tilde{\lambda})$ *over* $(X, *, 0)$ *is an* $\mathcal{N}_2$*-subalgebra of* $(X, *, 0)$*, then the assertion* (3.8) *is valid.*

The converse of Corollary 3.9 may not be true as seen in the following example.

Example 3.10. Let $X = \{0, 1, 2, a, b\}$ be the $BCI$-algebra in Example 3.7. Consider a hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ in which $\tilde{\lambda}$ is given as follows:

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.1, 0.3] \cup (0.4, 0.7) & \text{if } x = 0, \\ (0.2, 0.5] & \text{if } x = 1, \\ [0.3, 0.7] & \text{if } x = 2, \\ [0.4, 0.5) \cup (0.5, 0.6] & \text{if } x = a, \\ [0.5, 0.7) & \text{if } x = b \end{cases}$$

Then $(X, \tilde{\lambda})$ induces the $\mathcal{N}$-function given by Table 7,

Table 7. Induced $\mathcal{N}$-function from $(X, \tilde{\lambda})$

| $X$ | 0 | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| $\tilde{\lambda}_{\mathcal{N}}$ | $-0.6$ | $-0.3$ | $-0.4$ | $-0.2$ | $-0.2$ |

and so

$$L_{\mathcal{N}}(\tilde{\lambda}; t) = \begin{cases} \emptyset & \text{if } t \in [-1, -0.6), \\ \{0\} & \text{if } t \in [-0.6, -0.4), \\ \{0, 2\} & \text{if } t \in [-0.4, -0.3), \\ \{0, 1, 2\} & \text{if } t \in [-0.3, -0.2), \\ X & \text{if } t \in [-0.2, 0]. \end{cases}$$

Thus $L_{\mathcal{N}}(\tilde{\lambda}; t)$ is a subalgebra of $(X, *, 0)$ for all $t \in [-1, 0]$ with $L_{\mathcal{N}}(\tilde{\lambda}; t) \neq \emptyset$. Since

$$\tilde{\lambda}_{\mathcal{N}}(b * 1) = \tilde{\lambda}_{\mathcal{N}}(a) = -0.2 > -0.3 = \min\{\tilde{\lambda}_{\mathcal{N}}(b), \tilde{\lambda}(1)\},$$

$(X, \tilde{\lambda})$ is not an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$.

**Theorem 3.11.** *Given a subalgebra* $A$ *of* $(X, *, 0)$ *and* $B_1, B_2 \in \tilde{\mathcal{P}}([0,1])$ *with* $B_1 \subsetneq B_2$*, the hyper structure* $(X, \tilde{\lambda})$ *over* $(X, *, 0)$ *given by*

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} B_2 & \text{if } x \in A, \\ B_1 & \text{otherwise} \end{cases} \tag{3.9}$$

*is an* $\mathcal{N}_4$*-subalgebra of* $(X, *, 0)$.

*Proof.* From (3.9), we have

$$(\forall x \in X) \left( \tilde{\lambda}_{\mathcal{N}}(x) = \begin{cases} \inf\{B_2\} - \sup\{B_2\} & \text{if } x \in A, \\ \inf\{B_1\} - \sup\{B_1\} & \text{otherwise.} \end{cases} \right). \tag{3.10}$$

Since $B_1 \subsetneq B_2$, we have $\inf\{B_2\} - \sup\{B_2\} \leq \inf\{B_1\} - \sup\{B_1\}$. For any $x, y \in X$, if $x, y \in A$, then $x * y \in A$ and so

$$\tilde{\lambda}_{\mathcal{N}}(x * y) = \inf\{B_2\} - \sup\{B_2\} = \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}.$$

If $x, y \notin A$, then $\tilde{\lambda}_{\mathcal{N}}(x * y) \leq \inf\{B_1\} - \sup\{B_1\} = \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}$. Assume that $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$). Then

$$\tilde{\lambda}_{\mathcal{N}}(x * y) \leq \inf B_1 - \sup B_1 = \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}.$$

Therefore $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$. □

The hyper structure $(X, \tilde{\lambda})$ in Theorem 3.11 is not an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$ as seen in the following example.

Example 3.12. Consider the $BCK$-algebra $(X, *, 0)$ in Example 3.3, and take a subalgebra $A = \{0, 1, 2\}$ of $(X, *, 0)$. Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ given by

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} (0.2, 0.7) & \text{if } x \in A, \\ [0.3, 0.6) & \text{otherwise.} \end{cases}$$

Then $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ by Theorem 3.11. But it is not an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$ since

$$\tilde{\lambda}_{\mathcal{N}}(3 * 1) = \tilde{\lambda}_{\mathcal{N}}(3) = -0.3 > -0.5 = \min\{\tilde{\lambda}_{\mathcal{N}}(3), \tilde{\lambda}_{\mathcal{N}}(1)\}.$$

**Theorem 3.13.** *If $B_2 \subsetneq B_1$ in Theorem 3.11, then $(X, \tilde{\lambda})$ is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$.*

*Proof.* If $B_2 \subsetneq B_1$, then $\inf\{B_2\} - \sup\{B_2\} \geq \inf\{B_1\} - \sup\{B_1\}$. For any $x, y \in X$, the following assertion is clear.

$$x, y \in A \ \Rightarrow\ \tilde{\lambda}(x * y) = \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}.$$

If $x \notin A$ or $y \notin A$, then $\tilde{\lambda}_{\mathcal{N}}(x) = \inf\{B_1\} - \sup\{B_1\}$ or $\tilde{\lambda}_{\mathcal{N}}(y) = \inf\{B_1\} - \sup\{B_1\}$. It follows that

$$\tilde{\lambda}_{\mathcal{N}}(x * y) \geq \inf\{B_1\} - \sup\{B_1\} = \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}.$$

Therefore $(X, \tilde{\lambda})$ is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$. □

The hyper structure $(X, \tilde{\lambda})$ in Theorem 3.13 is not an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$ as seen in the following example.

Example 3.14. Consider the $BCK$-algebra $(X, *, 0)$ in Example 3.3. Given a subalgebra $A = \{0, 1, 2\}$ of $(X, *, 0)$, let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ given by

$$\tilde{\lambda} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} \{0.3n \mid n \in (0.4, 0.7]\} & \text{if } x \in A, \\ \{0.3n \mid n \in [0.2, 0.9)\} & \text{otherwise.} \end{cases}$$

Then $(X, \tilde{\lambda})$ is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$ by Theorem 3.13. But it is not an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$ since

$$\tilde{\lambda}_{\mathcal{N}}(3 * 1) = \tilde{\lambda}_{\mathcal{N}}(3) = -0.21 < -0.09 = \max\{\tilde{\lambda}_{\mathcal{N}}(3), \tilde{\lambda}(1)\}.$$

**Theorem 3.15.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.4). If $(X, \tilde{\lambda})$ is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$.*

*Proof.* Assume that $(X, \tilde{\lambda})$ is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.4). Then $\tilde{\lambda}_{\inf}(x*y) \leq \tilde{\lambda}_{\inf}(x)$ and $\tilde{\lambda}_{\inf}(x * y) \leq \tilde{\lambda}_{\inf}(y)$ for all $x, y \in X$, and $(X, \tilde{\lambda}_{\sup})$ is a 1-fuzzy subalgebra of $X$. It follows from (2.3) that

$$\begin{aligned}
\tilde{\lambda}_{\mathcal{N}}(x * y) &= \tilde{\lambda}_{\inf}(x * y) - \tilde{\lambda}_{\sup}(x * y) \\
&\leq \tilde{\lambda}_{\inf}(x * y) - \min\{\tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\sup}(y)\} \\
&= \max\{\tilde{\lambda}_{\inf}(x * y) - \tilde{\lambda}_{\sup}(x),\ \tilde{\lambda}_{\inf}(x * y) - \tilde{\lambda}_{\sup}(y)\} \\
&\leq \max\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x),\ \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(y)\} \\
&= \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}
\end{aligned}$$

for all $x, y \in X$. Therefore $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$. □

**Corollary 3.16.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.4). If $(X, \tilde{\lambda})$ is a $(k, 3)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$.*

In general, any $\mathcal{N}_4$-subalgebra may not be a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ as seen in the following example.

Example 3.17. In Example 3.4, the $\mathcal{N}_4$-subalgebra $(X, \tilde{\lambda})$ of $(X, *, 0)$ is not a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ since

$$\begin{aligned}
&\tilde{\lambda}_{\inf}(b * b) = \tilde{\lambda}_{\inf}(0) = 0.2 < 0.3 = \min\{\tilde{\lambda}_{\inf}(b), \tilde{\lambda}_{\inf}(b)\}, \\
&\tilde{\lambda}_{\inf}(b * c) = \tilde{\lambda}_{\inf}(a) = 0.4 > 0.3 = \min\{\tilde{\lambda}_{\inf}(b), \tilde{\lambda}_{\inf}(c)\}, \\
&\tilde{\lambda}_{\inf}(a * a) = \tilde{\lambda}_{\inf}(0) = 0.2 < 0.4 = \max\{\tilde{\lambda}_{\inf}(a), \tilde{\lambda}_{\inf}(a)\}, \\
&\tilde{\lambda}_{\inf}(b * c) = \tilde{\lambda}_{\inf}(a) = 0.4 > 0.3 = \max\{\tilde{\lambda}_{\inf}(b), \tilde{\lambda}_{\inf}(c)\}.
\end{aligned}$$

We consider a condition for an $\mathcal{N}_4$-subalgebra to be a $(k, 1)$-hyperfuzzy subalgebra for $k \in \{1, 2, 3, 4\}$.

**Theorem 3.18.** *If $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}$ is constant on $X$, then it is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

*Proof.* Assume that $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}$ is constant on $X$. It is clear that $(X, \tilde{\lambda}_{\inf})$ is a $k$-fuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. Let $\tilde{\lambda}_{\inf}(x) = t$ for all $x \in X$. Then

$$\begin{aligned}\tilde{\lambda}_{\sup}(x * y) &= \tilde{\lambda}_{\inf}(x * y) - \tilde{\lambda}_{\mathcal{N}}(x * y)\\ &= t - \tilde{\lambda}_{\mathcal{N}}(x * y)\\ &\geq t - \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\\ &= \min\{t - \tilde{\lambda}_{\mathcal{N}}(x), t - \tilde{\lambda}_{\mathcal{N}}(y)\}\\ &= \min\{\tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\sup}(y)\}\end{aligned}$$

for all $x, y \in X$. Thus $(X, \tilde{\lambda}_{\sup})$ is a 1-fuzzy subalgebra of $X$. Therefore $(X, \tilde{\lambda})$ is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

**Corollary 3.19.** *If $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}$ is constant on $X$, then it is a $(k, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

**Theorem 3.20.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.5). If $(X, \tilde{\lambda})$ is a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$.*

*Proof.* Let $(X, \tilde{\lambda})$ be a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.5). Then $\tilde{\lambda}_{\sup}(x * y) \geq \tilde{\lambda}_{\sup}(x)$ and $\tilde{\lambda}_{\sup}(x * y) \geq \tilde{\lambda}_{\sup}(y)$ for all $x, y \in X$. Since $(X, \tilde{\lambda}_{\inf})$ is a 4-fuzzy subalgebra of $(X, *, 0)$, we have

$$\begin{aligned}\tilde{\lambda}_{\mathcal{N}}(x * y) &= \tilde{\lambda}_{\inf}(x * y) - \tilde{\lambda}_{\sup}(x * y)\\ &\leq \max\{\tilde{\lambda}_{\inf}(x), \tilde{\lambda}_{\inf}(y)\} - \tilde{\lambda}_{\sup}(x * y)\\ &= \max\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x * y), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(x * y)\}\\ &\leq \max\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(y)\}\\ &= \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$. □

**Corollary 3.21.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.5). If $(X, \tilde{\lambda})$ is a $(2, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$.*

**Theorem 3.22.** *If $(X, \tilde{\lambda})$ is an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\sup}$ is constant on $X$, then it is a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

*Proof.* Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_4$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\sup}$ is constant on $X$. It is clear that $(X, \tilde{\lambda}_{\sup})$ is a $k$-fuzzy subalgebra of $X$ for $k \in \{1, 2, 3, 4\}$.

Let $\tilde{\lambda}_{\sup}(x) = t$ for all $x \in X$. Then

$$\begin{aligned}\tilde{\lambda}_{\inf}(x*y) &= \tilde{\lambda}_{\mathcal{N}}(x*y) + \tilde{\lambda}_{\sup}(x*y)\\ &\leq \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} + t\\ &= \max\{\tilde{\lambda}_{\mathcal{N}}(x) + t, \tilde{\lambda}_{\mathcal{N}}(y) + t\}\\ &= \max\{\tilde{\lambda}_{\inf}(x), \tilde{\lambda}_{\inf}(y)\}\end{aligned}$$

for all $x, y \in X$, that is, $(X, \tilde{\lambda}_{\inf})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Hence $(X, \tilde{\lambda})$ is a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

**Corollary 3.23.** *If $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\sup}$ is constant on $X$, then it is a $(4, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

**Theorem 3.24.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.5). If $(X, \tilde{\lambda})$ is a $(k, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$.*

*Proof.* Assume that $(X, \tilde{\lambda})$ is a $(k, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.5). Then $\tilde{\lambda}_{\inf}(x*y) \geq \tilde{\lambda}_{\inf}(x)$ and $\tilde{\lambda}_{\inf}(x*y) \geq \tilde{\lambda}_{\inf}(y)$ for all $x, y \in X$, and $(X, \tilde{\lambda}_{\sup})$ is a 4-fuzzy subalgebra of $X$. Hence

$$\begin{aligned}\tilde{\lambda}_{\mathcal{N}}(x*y) &= \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(x*y)\\ &\geq \tilde{\lambda}_{\inf}(x*y) - \max\{\tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\sup}(y)\}\\ &= \min\{\tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(x),\ \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(y)\}\\ &\geq \min\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x),\ \tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(y)\}\\ &= \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\end{aligned}$$

for all $x, y \in X$, and so $(X, \tilde{\lambda})$ is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$. □

**Corollary 3.25.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.5). If $(X, \tilde{\lambda})$ is a $(k, 2)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$.*

**Theorem 3.26.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}$ is constant. Then every $\mathcal{N}_1$-subalgebra is a $(k, 4)$-hyperfuzzy subalgebra for $k \in \{1, 2, 3, 4\}$.*

*Proof.* Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}(x) = t$ for all $x \in X$. It is obvious that $(X, \tilde{\lambda}_{\inf})$ is a $k$-fuzzy subalgebra of $(X, *, 0)$ for

$k \in \{1,2,3,4\}$. Also we have

$$\begin{aligned}\tilde{\lambda}_{\sup}(x*y) &= \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\mathcal{N}}(x*y) = t - \tilde{\lambda}_{\mathcal{N}}(x*y)\\ &\leq t - \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\\ &= \max\{t - \tilde{\lambda}_{\mathcal{N}}(x), t - \tilde{\lambda}_{\mathcal{N}}(y)\}\\ &= \max\{\tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\sup}(y)\}\end{aligned}$$

for all $x, y \in X$, and hence $(X, \tilde{\lambda}_{\sup})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Therefore $(X, \tilde{\lambda})$ is a $(k, 4)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1,2,3,4\}$. □

**Corollary 3.27.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}$ is constant. Then every $\mathcal{N}_3$-subalgebra is a $(k, 4)$-hyperfuzzy subalgebra for $k \in \{1,2,3,4\}$.*

**Theorem 3.28.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.4). For every $k \in \{1,2,3,4\}$, every $(1, k)$-hyperfuzzy subalgebra is an $\mathcal{N}_1$-subalgebra.*

*Proof.* For every $k \in \{1,2,3,4\}$, let $(X, \tilde{\lambda})$ be a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.4). Then $\tilde{\lambda}_{\sup}(x*y) \leq \tilde{\lambda}_{\sup}(x)$ and $\tilde{\lambda}_{\sup}(x*y) \leq \tilde{\lambda}_{\sup}(y)$ for all $x, y \in X$. Since $(X, \tilde{\lambda}_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$, we have

$$\begin{aligned}\tilde{\lambda}_{\mathcal{N}}(x*y) &= \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(x*y)\\ &\geq \min\{\tilde{\lambda}_{\inf}(x), \tilde{\lambda}_{\inf}(y)\} - \tilde{\lambda}_{\sup}(x*y)\\ &= \min\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x*y), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(x*y)\}\\ &\geq \min\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(y)\}\\ &= \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\end{aligned}$$

for all $x, y \in X$. Thus $(X, \tilde{\lambda})$ is an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$. □

**Corollary 3.29.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.4). For every $k \in \{1,2,3,4\}$, every $(3, k)$-hyperfuzzy subalgebra is an $\mathcal{N}_1$-subalgebra.*

**Theorem 3.30.** *Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$. If $\tilde{\lambda}_{\sup}$ is constant on $X$, then $(X, \tilde{\lambda})$ is a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1,2,3,4\}$.*

*Proof.* Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_1$-subalgebra of $(X, *, 0)$ such that $\tilde{\lambda}_{\sup}(x) = t$ for all $x \in X$. Obviously, $(X, \tilde{\lambda}_{\sup})$ is a $k$-fuzzy subalgebra of $(X, *, 0)$ for $k \in$

$\{1, 2, 3, 4\}$, and

$$\begin{aligned}\tilde{\lambda}_{\inf}(x * y) &= \tilde{\lambda}_{\sup}(x * y) + \tilde{\lambda}_{\mathcal{N}}(x * y) \\ &\geq t + \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \\ &= \min\{t + \tilde{\lambda}_{\mathcal{N}}(x), t + \tilde{\lambda}_{\mathcal{N}}(y)\} \\ &= \min\{\tilde{\lambda}_{\inf}(x), \tilde{\lambda}_{\inf}(y)\}\end{aligned}$$

for all $x, y \in X$, and so $(X, \tilde{\lambda}_{\inf})$ is a 1-fuzzy subalgebra of $(X, *, 0)$. Therefore $(X, \tilde{\lambda})$ is a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

**Corollary 3.31.** *Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$ in which $\tilde{\lambda}_{\sup}$ is constant on $X$. Then $(X, \tilde{\lambda})$ is a $(1, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

**Theorem 3.32.** *Given a hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ in which $\tilde{\lambda}_{\inf}$ is constant on $X$, if $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra (resp., $\mathcal{N}_3$-subalgebra) of $(X, *, 0)$, then $(X, \tilde{\lambda})$ is a $(k, 3)$-hyperfuzzy (resp., $(k, 2)$-hyperfuzzy ) subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

*Proof.* Let $(X, \tilde{\lambda})$ be an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$ such that $\tilde{\lambda}_{\inf}(x) = t$ for all $x \in X$. Then $(X, \tilde{\lambda}_{\inf})$ is a $k$-fuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, and

$$\begin{aligned}\tilde{\lambda}_{\sup}(x * y) &= \tilde{\lambda}_{\inf}(x * y) - \tilde{\lambda}_{\mathcal{N}}(x * y) = t - \tilde{\lambda}_{\mathcal{N}}(x * y) \\ &\geq t - \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} \\ &= \max\{t - \tilde{\lambda}_{\mathcal{N}}(x), t - \tilde{\lambda}_{\mathcal{N}}(y)\} \\ &= \max\{\tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\sup}(y)\}\end{aligned}$$

for all $x, y \in X$, and so $(X, \tilde{\lambda}_{\sup})$ is a 3-fuzzy subalgebra of $(X, *, 0)$. Therefore $(X, \tilde{\lambda})$ is a $(k, 3)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. Similarly, if $(X, \tilde{\lambda})$ is an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$ such that $\tilde{\lambda}_{\inf}(x) = t$ for all $x \in X$, then $(X, \tilde{\lambda})$ is a $(k, 2)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

**Theorem 3.33.** *Let $(X, \tilde{\lambda})$ be a $(k, 2)$-hyperfuzzy (resp., $(k, 3)$-hyperfuzzy ) subalgebra of $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.5) (resp., (2.4)) for $k \in \{1, 2, 3, 4\}$. Then $(X, \tilde{\lambda})$ is an $\mathcal{N}_3$-subalgebra (resp., $\mathcal{N}_2$-subalgebra) of $(X, *, 0)$.*

*Proof.* Assume that $(X, \tilde{\lambda})$ is a $(k, 2)$-hyperfuzzy subalgebra of $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.5). Then $\tilde{\lambda}_{\inf}(x*y) \geq \tilde{\lambda}_{\inf}(x)$ and $\tilde{\lambda}_{\inf}(x*y) \geq \tilde{\lambda}_{\inf}(y)$ for all $x, y \in X$. Since $(X, \tilde{\lambda}_{\sup})$ is a 2-fuzzy subalgebra of $(X, *, 0)$, it follows that

$$\begin{aligned}\tilde{\lambda}_{\mathcal{N}}(x*y) &= \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(x*y)\\ &\geq \tilde{\lambda}_{\inf}(x*y) - \min\{\tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\sup}(y)\}\\ &= \max\{\tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(y)\}\\ &\geq \max\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(y)\}\\ &= \max\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\end{aligned}$$

for all $x, y \in X$. Hence $(X, \tilde{\lambda})$ is an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$. Similarly, if $(X, \tilde{\lambda})$ is a $(k, 3)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\lambda}_{\inf})$ satisfies the condition (2.4), then it is an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$. □

**Theorem 3.34.** *Let $(X, \tilde{\lambda})$ be a hyper structure over $(X, *, 0)$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.5) (resp., (2.4)). If $(X, \tilde{\lambda})$ is a $(2, k)$-hyperfuzzy (resp., $(3, k)$-hyperfuzzy ) subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$, then $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra (resp., $\mathcal{N}_3$-subalgebra) of $(X, *, 0)$.*

*Proof.* Let $(X, \tilde{\lambda})$ be a $(2, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.5). Then $\tilde{\lambda}_{\sup}(x*y) \geq \tilde{\lambda}_{\sup}(x)$ and $\tilde{\lambda}_{\sup}(x*y) \geq \tilde{\lambda}_{\sup}(y)$ for all $x, y \in X$. Since $(X, \tilde{\lambda}_{\inf})$ is a 2-fuzzy subalgebra of $(X, *, 0)$, we have

$$\begin{aligned}\tilde{\lambda}_{\mathcal{N}}(x*y) &= \tilde{\lambda}_{\inf}(x*y) - \tilde{\lambda}_{\sup}(x*y)\\ &\leq \min\{\tilde{\lambda}_{\inf}(x), \tilde{\lambda}_{\inf}(y)\} - \tilde{\lambda}_{\sup}(x*y)\\ &= \min\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x*y), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(x*y)\}\\ &\leq \min\{\tilde{\lambda}_{\inf}(x) - \tilde{\lambda}_{\sup}(x), \tilde{\lambda}_{\inf}(y) - \tilde{\lambda}_{\sup}(y)\}\\ &= \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\}\end{aligned}$$

for all $x, y \in X$. Thus $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$. Similarly, we can verify that if $(X, \tilde{\lambda})$ is a $(3, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $(X, \tilde{\lambda}_{\sup})$ satisfies the condition (2.4), then $(X, \tilde{\lambda})$ is an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$. □

**Theorem 3.35.** *Given a hyper structure $(X, \tilde{\lambda})$ over $(X, *, 0)$ in which $\tilde{\lambda}_{\sup}$ is constant on $X$, if $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra (resp., $\mathcal{N}_3$-subalgebra) of $(X, *, 0)$, then $(X, \tilde{\lambda})$ is a $(2, k)$-hyperfuzzy (resp., $(3, k)$-hyperfuzzy ) subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$.*

*Proof.* Assume that $(X, \tilde{\lambda})$ is an $\mathcal{N}_2$-subalgebra of $(X, *, 0)$ such that $\tilde{\lambda}_{\sup}(x) = t$ for all $x \in X$. Obviously, $(X, \tilde{\lambda}_{\sup})$ is a $k$-fuzzy subalgebra of $(X, *, 0)$ for

$k \in \{1, 2, 3, 4\}$, and

$$\begin{aligned}\tilde{\lambda}_{\inf}(x*y) &= \tilde{\lambda}_{\mathcal{N}}(x*y) + \tilde{\lambda}_{\sup}(x*y) = \tilde{\lambda}_{\mathcal{N}}(x*y) + t \\ &\leq \min\{\tilde{\lambda}_{\mathcal{N}}(x), \tilde{\lambda}_{\mathcal{N}}(y)\} + t \\ &= \min\{\tilde{\lambda}_{\mathcal{N}}(x) + t, \tilde{\lambda}_{\mathcal{N}}(y) + t\} \\ &= \min\{\tilde{\lambda}_{\inf}(x), \tilde{\lambda}_{\inf}(y)\}\end{aligned}$$

for all $x, y \in X$. Hence $(X, \tilde{\lambda}_{\inf})$ is a 2-fuzzy subalgebra of $(X, *, 0)$, and therefore $(X, \tilde{\lambda})$ is a $(2, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. Similarly, we can prove that if $(X, \tilde{\lambda})$ is an $\mathcal{N}_3$-subalgebra of $(X, *, 0)$ such that $\tilde{\lambda}_{\sup}(x) = t$ for all $x \in X$, then $(X, \tilde{\lambda})$ is a $(3, k)$-hyperfuzzy subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. □

## Acknowledgments

The research is supported by the Iranian National Science Foundation (Grant No. 96008529).

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