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Surfaces Generated by Translation Surfaces of Type 1 in I_3^1

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Abstract. In this paper, we classify surfaces at a constant distance from the edge of regression on a translation surface of Type 1 in the three dimensional simply isotropic space $\mathbb{I}_3^1$ satisfying some algebraic equations in terms of the coordinate functions and the Laplacian operators with respect to the first, the second and the third fundamental forms of the surface. We also give explicit forms of these surfaces.

Keywords: Simply isotropic space, Translation surfaces, Surface at a constant distance from the edge of regression on a surface.

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1. Introduction

Let $\mathbf{x} : \mathbf{M} \to \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$ induced from that of $\mathbb{E}^m$, respectively. Takahashi proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta\mathbf{x} = \lambda\mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$, are either the minimal submanifolds ($\mathbf{H} = 0$) of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$ [18].

As an extension of Takahashi theorem, Garay studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue in [10]. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{1.1}$$

where $\mathbf{A} \in Mat\ (m, \mathbb{R})$ is an $m \times m$- diagonal matrix, and proved that such hypersurfaces are minimal in $\mathbb{E}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders. Related to this, Dillen, Pas and Verstraelen investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$\Delta\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}, \tag{1.2}$$

where $\mathbf{A} \in Mat\ (3, \mathbb{R})$ is a 3×3-real matrix and $\mathbf{B} \in \mathbb{R}^3$ [8]. In other words, each coordinate function is of 1-type in the sense of Chen [7]. The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $\mathbf{G}$ satisfies the condition

$$\Delta\mathbf{G} = \mathbf{A}\mathbf{G} \tag{1.3}$$

where $\mathbf{A} \in Mat\ (3, \mathbb{R})$ [9]. Tarakci and Hacisalihoglu defined the surface $\mathbf{M}^f$ at a constant distance from the edge of regression on a surface $\mathbf{M}$ and investigated some properties of $\mathbf{M}^f$ [19]. Cakmak and Tarakci investigated the surface at a constant distance from the edge of regression on a surface of revolution indicated by $\mathbf{M}^f$, condition that $\mathbf{M}$ is denoted by a surface of revolution in $\mathbb{E}^3$ [5]. Saglam and Kalkan defined the surfaces $\mathbf{M}^f$ at a constant distance from the edge of regression on a surfaceM in $\mathbb{E}^3_1$ [14]. Yurttancikmaz and Tarakci investigated the relationship between focal surfaces and surfaces at a constant distance from the edge of regression on a surface [22].

Yoon studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map $\mathbf{G}$ satisfying the condition (1.3) and also translation surfaces in a 3-dimensional Galilean space $\mathbb{G}_3$ satisfies the condition

$$\Delta \mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{1.4}$$

where $\lambda_i \in \mathbb{R}$ and provided some examples of new classes of translation surfaces [20, 21]. Baba-Hamed, Bekkar and Zoubir classified all translation surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{R}_1^3$ under the condition (1.4) [2]. Bekkar and Senoussi studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

$$\Delta^{\mathbf{III}} \mathbf{r}_i = \mu_i \mathbf{r}_i,$$

where $\mu_i \in \mathbb{R}$ and $\Delta^{\mathbf{III}}$ denotes the Laplacian of the surface with respect to the third fundamental form **III** [3]. They showed that in both spaces a translation surface satisfying the preceding relation is a surface of Scherk. Aydin and Sipus studied constant curvatures of translation surfaces in the three dimensional simply isotropic space [1, 16]. Karacan, Yoon and Bukcu classified translation surfaces of Type 1 satisfying $\Delta^J \mathbf{x}_i = \lambda_i\ \mathbf{x}_i$, $j = 1, 2$ and $\Delta^{\mathbf{III}} \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $\lambda_i \in \mathbb{R}$ [4, 12].

In this paper, we classify the surfaces at a constant distance from the edge of regression on a translation surface of Type 1 in the three dimensional simply isotropic space under the condition $\Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $J = \mathbf{I}, \mathbf{II}, \mathbf{III}$, where $\lambda_i \in \mathbb{R}$. Δ^J denotes the Laplace operator with respect to the fundamental forms **I**, **II** and **III**.

2. Preliminaries

A simply isotropic space $\mathbb{I}_3^1$ is a Cayley–Klein space defined from the three dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure which is an ordered triple (w, f_1, f_2), where w is a plane in $\mathcal{P}(\mathbb{R}^3)$ and f_1, f_2 are two complex-conjugate straight lines in w. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane w is given by $x_0 = 0$ and the absolute lines f_1, f_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. The intersection point $\mathbb{F}(0 : 0 : 0 : 1)$ of these two lines is called the absolute point. The group of motions of the simply isotropic space is a six-parameter group given in the affine coordinates $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ by

$$\begin{aligned} \overline{x} &= a + x\cos\theta - y\sin\theta \\ \overline{y} &= b + x\sin\theta + y\cos\theta \\ \overline{z} &= c + +c_1 x + c_2 y + z, \end{aligned} \tag{2.1}$$

where $a, b, c, c_1, c_2, \theta \in \mathbb{R}$. Such affine transformations are called isotropic congruence transformations or i-motions.

Isotropic geometry has different types of lines and planes with respect to the absolute figure. A line is called non-isotropic (resp. completely isotropic) if its point at infinity does not coincide (coincides) with the point $\mathbb{F}$. A plane is called non-isotropic (resp. isotropic) if its line at infinity does not contain $\mathbb{F}$ (resp.otherwise). Completely isotropic lines and isotropic planes in this affine model appear as vertical, i.e., parallel to the z-axis. Finally, the metric of the simply isotropic space $\mathbb{I}_3^1$ is given by

$$ds^2 = dx^2 + dy^2.$$

A surface $\mathbf{M}$ immersed in $\mathbb{I}_3^1$ is called admissible if it has no isotropic tangent planes. For such a surface, the coefficients E, F, G of its first fundamental form are calculated with respect to the induced metric and the coefficients L, M, N of the second fundamental form, with respect to the normal vector field of a surface which is always completely isotropic. The (isotropic) Gaussian and (isotropic) mean curvature are defined by

$$\mathbf{K} = k_1k_2 = \frac{LN - M^2}{EG - F^2}, \quad 2\mathbf{H} = k_1 + k_2 = \frac{EN - 2FM + GL}{EG - F^2}, \tag{2.2}$$

where k_1, k_2 are principal curvatures, i.e., extrema of the normal curvature determined by the normal section (in completely isotropic direction) of a surface. Since $EG - F^2 > 0$, for the function in the denominator we often put $W^2 = EG - F^2$. The surface $\mathbf{M}$ is said to be isotropic flat (resp. isotropic minimal) if $\mathbf{K}$ (resp.$\mathbf{H}$) vanishes. The unit normal vector field of $\mathbf{M}$ is the isotropic vector $\mathbf{N} = (0, 0, 1)$ since it is perpendicular to all non-isotropic vectors [1, 13, 16, 17].

Definition 2.1. Let $\mathbf{M}$ and $\mathbf{M}^h$ be two admissible surfaces in $\mathbb{I}_3^1$ and $\mathbf{N}_P$ be a isotropic unit normal vector at a point P of the surface $\mathbf{M}$. Take a unit vector at a point P

$$\mathbf{Z}_P = d_1\mathbf{x}_u + d_2\mathbf{x}_v + d_3\mathbf{N}_P, \tag{2.3}$$

where $\mathbf{x}_u$, $\mathbf{x}_v$ are tangent vectors at P and $d_1^2 + d_2^2 = 1$. If there is a function h defined by

$$\begin{aligned} h \quad &: \quad \mathbf{M} \to \mathbf{M}^h, \\ h(P) \quad &= \quad P + r\mathbf{Z}_P, \end{aligned}$$

where r is constant, then the surface $\mathbf{M}^h$ is called the surface at a constant distance from the edge of regression on $\mathbf{M}$. $\mathbf{M}$ and $\mathbf{M}^h$ are shown by the pair $\left(\mathbf{M}, \mathbf{M}^h\right)$. If $d_1 = d_2 = 0$, then we have $\mathbf{Z}_P = r\mathbf{N}_P$ and so $\mathbf{M}$ and $\mathbf{M}^h$ are parallel surfaces.

Now, we represent parametrization of surfaces at a constant distance from the edge of regression on $\mathbf{M}$. Let $\mathbf{x}(u, v)$ be a parametrization of $\mathbf{M}$. In this case, $\{\mathbf{x}_u, \mathbf{x}_v\}$ is non-isotropic orthonormal bases the surface $\mathbf{M}$. Let $\mathbf{N}_P$ be a isotropic unit normal vector at a point P and $d_1, d_2, d_3 \in \mathbb{R}$ be constant numbers. Then we can write a parametric representation of $\mathbf{M}^h$ is

$$\mathbf{x}^h(u,v) = \mathbf{x}(u,v) + r\mathbf{Z}(u,v).$$

Thus we obtain

$$\mathbf{x}^h(u,v) = \mathbf{x}(u,v) + r\left(d_1\mathbf{x}_u + d_2\mathbf{x}_v + d_3\mathbf{N}(0,0,1)\right). \tag{2.4}$$

If we take $rd_1 = \eta,\ rd_2 = \mu,\ rd_3 = \gamma$, where $\eta^2 + \mu^2 = r^2$. Thus we get

$$\mathbf{x}^h(u,v) = \mathbf{x}(u,v) + \eta\mathbf{x}_u + \mu\mathbf{x}_v + \gamma\mathbf{N}(0,0,1). \tag{2.5}$$

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$, $\Delta^{\mathbf{III}}$ of the first, the second and the third fundamental forms on $\mathbf{M}$ are defined by

$$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\left(\frac{G\mathbf{x}_u - F\mathbf{x}_v}{\sqrt{EG-F^2}}\right) - \frac{\partial}{\partial v}\left(\frac{F\mathbf{x}_u - E\mathbf{x}_v}{\sqrt{EG-F^2}}\right)\right], \tag{2.6}$$

$$\Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{LN-M^2}}\begin{bmatrix} \dfrac{\partial}{\partial u}\left(\dfrac{N\mathbf{x}_u - M\mathbf{x}_v}{\sqrt{LN-M^2}}\right) \\ -\dfrac{\partial}{\partial v}\left(\dfrac{M\mathbf{x}_u - L\mathbf{x}_v}{\sqrt{LN-M^2}}\right)\end{bmatrix}, \tag{2.7}$$

$$\Delta^{\mathbf{III}}\mathbf{x} = -\frac{\sqrt{EG-F^2}}{LN-M^2}\begin{bmatrix} \dfrac{\partial}{\partial u}\left(\dfrac{Z\mathbf{x}_u - Y\mathbf{x}_v}{(LN-M^2)\sqrt{EG-F^2}}\right) \\ -\dfrac{\partial}{\partial v}\left(\dfrac{Y\mathbf{x}_u - X\mathbf{x}_v}{(LN-M^2)\sqrt{EG-F^2}}\right)\end{bmatrix}, \tag{2.8}$$

where

$$\begin{aligned} X &= EM^2 - 2FLM + GL^2, \\ Y &= EMN - FLN + GLM - FM^2, \\ Z &= GM^2 - 2FNM + EN^2 \end{aligned}$$

[2, 3, 4, 9, 11, 12].

3. Translation Surfaces in $\mathbb{I}_3^1$

In order to describe the isotropic analogues of translation surfaces of constant curvatures, we consider translation surfaces obtained by translating two planar curves. The local surface parametrization is given by

$$\mathbf{x}(u,v) = \alpha(u) + \beta(v). \tag{3.1}$$

Since there are, with respect to the absolute figure, different types of planes in $\mathbb{I}_3^1$, there are in total three different possibilities for planes that contain translated curves: the translated curves can be curves in isotropic planes (which can be chosen, by means of isotropic motions, as $y = 0$, resp. $x = 0$); or one curve is in a non-isotropic plane ($z = 0$) and one curve in an isotropic plane ($y = 0$); or both curves are curves in non-isotropic perpendicular planes ($y - z = \pi$, resp.

$y+z=\pi$). Therefore, the obtained translation surfaces allow the following parametrizations:

Type 1: The surface $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=(u,v,f(u)+g(v)), \tag{3.2}$$

and the translated curves are $\alpha(u)=(u,0,f(u))$, $\beta(v)=(0,v,g(v))$. Thus, surfaces at a constant distance from the edge of regression on a translation surface of Type 1 is given by

$$\mathbf{x}^h(u,v)=(u+\eta,\ v+\mu,\ (f(u)+\eta f'(u))+(g(v)+\mu g'(v))+\gamma), \tag{3.3}$$

Type 2: The surface $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=(u,f(u)+g(v),v) \tag{3.4}$$

and the translated curves are $\alpha(u)=(u,f(u),0)$, $\beta(v)=(0,g(v),v)$. In order to obtain admissible surfaces, $g'(v)\neq 0$ is assumed (i.e. $g(v)\neq$const.).The surfaces at a constant distance from the edge of regression on translation surface of type 2 is given by

$$\mathbf{x}^h(u,v)=(u+\eta,(f(u)+\eta f'(u))+(g(v)+\mu g'(v)),v+\mu+\gamma), \tag{3.5}$$

Type 3: The surface $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\frac{1}{2}\left(f(u)+g(v),\ u-v+\pi,\ u+v\right) \tag{3.6}$$

and the translated curves are

$$\alpha(u)=\frac{1}{2}\Big(f(u),\ u+\frac{\pi}{2},\ u-\frac{\pi}{2}\Big),\beta(v)=\Big(g(v),\ \frac{\pi}{2}-v,\ \frac{\pi}{2}+v\Big). \tag{3.7}$$

In order to obtain admissible surfaces, $f'(u)+g'(v)\neq 0$ is assumed (i.e. $f'(u)\neq -g'(v)=a=$ constant)[16]. The surfaces at a constant distance from the edge of regression on translation surface of type 3 is given by

$$\mathbf{x}^h(u,v)=\frac{1}{2}\begin{pmatrix}(f(u)+\eta f'(u))+(g(v)+\mu g'(v)),\\ u-v+\pi+\eta-\mu,\\ u+v+\eta+\mu+\gamma\end{pmatrix}. \tag{3.8}$$

In this paper, we will investigate surfaces at a constant distance from the edge of regression on a translation surface of type 1.

4. Surfaces at a Constant Distance from the Edge of Regression on a Translation Surface of Type 1 Satisfying $\Delta^{\mathbf{I}}\mathbf{x}_i^h=\lambda_i\mathbf{x}_i^h$

In this section, we classify surfaces at a constant distance from the edge of regression on a translation surface of Type 1 in $\mathbb{I}_3^1$ satisfying the equation

$$\Delta^{\mathbf{I}}\mathbf{x}_i^h=\lambda_i\mathbf{x}_i^h, \tag{4.1}$$

where $\lambda_i\in\mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{I}}\mathbf{x}^h=\left(\Delta^{\mathbf{I}}\mathbf{x}_1^h,\ \Delta^{\mathbf{I}}\mathbf{x}_2^h,\ \Delta^{\mathbf{I}}\mathbf{x}_3^h\right),$$

where

$$\mathbf{x}_1^h = u+\eta,\ \mathbf{x}_2^h = v+\mu,\ \mathbf{x}_3^h = (f(u)+\eta f'(u)) + (g(v)+\mu g'(v)) + \gamma.$$

For the surface given by (3.3), the coefficients of the first and second fundamental forms are

$$E=1,\ F=0,\ G=1, \tag{4.2}$$

$$L = f''+\eta f''',\ M=0,\ N = g''+\mu g''', \tag{4.3}$$

respectively. The Gaussian curvature $\mathbf{K}^h$ and the mean curvature $\mathbf{H}^h$ are

$$\mathbf{K}^h = (f''+\eta f''')(g''+\mu g'''),\quad \mathbf{H}^h = \frac{(f''+\eta f''')+(g''+\mu g''')}{2}, \tag{4.4}$$

respectively.

Suppose that $\mathbf{K}^h$ satisfies the condition $\mathbf{K}^h=0$. In this case, we define as a surface satisfying that condition isotropic flat. Then, from (4.4) we can write

$$(f''+\eta f''')(g''+\mu g''')=0 \tag{4.5}$$

In above differential equation, for the best case, i.e. $(f''+\eta f''')=0$ and $(g''+\mu g''')=0$. So, the solutions of (4.5) are given by

$$\begin{aligned} f(u) &= c_1\eta^2 e^{-\frac{u}{\eta}} + c_2 u + c_3, \\ g(v) &= c_4\mu^2 e^{-\frac{v}{\mu}} + c_5 v + c_6, \end{aligned}$$

where $c_i\in\mathbb{R}$. If $\mathbf{H}^h=0$, then

$$\begin{aligned} f(u) &= a\frac{u^2}{2} + c_1\eta^2 e^{-\frac{u}{\eta}} + c_2 u + c_3, \\ g(v) &= -a\frac{v^2}{2} + c_4\mu^2 e^{-\frac{v}{\mu}} + c_5 v + c_6, \end{aligned}$$

where $c_i\in\mathbb{R}$. By a straightforward computation, the Laplacian operator on $\mathbf{M}^h$ with the help of (4.2) and (2.6) turns out to be

$$\Delta^{\mathbf{I}}\mathbf{x}_i^h = (0,\ 0,\ -(f''+\eta f''')-(g''+\mu g''')). \tag{4.6}$$

Suppose that $\mathbf{M}^h$ satisfies (4.1). Then from (4.6), we have

$$-(f''+\eta f''')-(g''+\mu g''') = \lambda(f+g+\eta f'+\mu g'+\gamma), \tag{4.7}$$

where $\lambda\in\mathbb{R}$. This means that $\mathbf{M}^h$ is at most of 1-type. First of all, we assume that $\mathbf{M}^h$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x}_i^h=0$. We call a surface satisfying that condition is a harmonic surface or isotropic minimal. In this case, we get from (4.7)

$$-(f''+\eta f''')-(g''+\mu g''')=0. \tag{4.8}$$

Here u and v are independent variables, so each side of (4.8) must be equal to a constant, call it a. Hence, the two equations

$$-(f''+\eta f''') = a = (g''+\mu g'''). \tag{4.9}$$

Its general solutions are

$$
\begin{aligned}
f(u) &= -a\frac{u^2}{2} + \eta^2 c_1 e^{-\frac{u}{\eta}} + c_2 u + c_3,\\
f'(u) &= -au + \eta c_1 e^{-\frac{u}{\eta}} + c_2,\\
g(v) &= a\frac{v^2}{2} + \mu^2 c_4 e^{-\frac{v}{\mu}} + c_5 v + c_6,\\
g'(v) &= av - \mu c_4 e^{-\frac{v}{\mu}} + c_5,
\end{aligned}
\tag{4.10}
$$

where $a, c_i \in \mathbb{R}$. In this case, $\mathbf{M}^h$ is parametrized by (3.3) with (4.10).

Theorem 4.1. *Let $\mathbf{M}^h$ be a surface at a constant distance from the edge of regression on a translation surface of Type 1 given by (3.3) in $\mathbb{I}_3^1$. If $\mathbf{M}^h$ is harmonic or isotropic minimal, then it is congruent to an open part of the surface (3.3) with (4.10), where f and g are given in (4.10).*

If $\lambda \neq 0$, from (4.7), we have

$$
-\left[\eta f''' + f'' + \eta\lambda f' + \lambda f\right] - \left[\mu g''' + g'' + \lambda\mu g' + \lambda g + \lambda\gamma\right] = 0. \tag{4.11}
$$

Here u and v are independent variables, so each side of (4.11) is equal to a constant, call it a. Hence, the two equations

$$
-\left[\eta f''' + f'' + \eta\lambda f' + \lambda f\right] = a = \left[\mu g''' + g'' + \lambda\mu g' + \lambda g + \lambda\gamma\right]. \tag{4.12}
$$

Their general solutions are given by

$$
\begin{aligned}
f(u) &= -\frac{a}{\lambda} + c_1 e^{-\frac{u}{\eta}} + c_2 \cos u\sqrt{\lambda} + c_3 \sin u\sqrt{\lambda},\\
g(v) &= -\frac{a-\gamma\lambda}{\lambda} + c_4 e^{-\frac{v}{\mu}} + c_5 \cos v\sqrt{\lambda} + c_6 \sin v\sqrt{\lambda},
\end{aligned}
\tag{4.13}
$$

where $a, c_i \in \mathbb{R}$. So $\mathbf{M}^h$ is parametrized by

$$
\mathbf{x}^h(u,v) = \left(\begin{array}{c}
u+\eta,\\
v+\mu,\\
\left(c_2 + c_3\eta\sqrt{\lambda}\right)\cos u\sqrt{\lambda} + \left(c_5 + c_6\mu\sqrt{\lambda}\right)\cos v\sqrt{\lambda}\\
+\left(c_3 - c_2\eta\sqrt{\lambda}\right)\sin u\sqrt{\lambda} + \left(c_6 - c_5\mu\sqrt{\lambda}\right)\sin v\sqrt{\lambda} + \gamma
\end{array}\right)
\tag{4.14}
$$

In particular, for the case $\left(\eta = \frac{1}{2}, \lambda = 2\right)$, the solution of the differential equation (4.12) respect to $f(u)$ is given by

$$
\begin{aligned}
f(u) &= -\frac{a}{2} + c_1 e^{-2u} + c_2 \cos u\sqrt{2} + c_3 \sin u\sqrt{2},\\
f'(u) &= -2c_1 e^{-2u} - \sqrt{2}c_2 \sin u\sqrt{2} + \sqrt{2}c_3 \cos u\sqrt{2}
\end{aligned}
\tag{4.15}
$$

For the function $g(v)$, we have

$$
\begin{aligned}
g(v) &= -1 + \frac{a}{2} + c_4 e^{-2v} + c_5 \cos v\sqrt{2} + c_6 \sin v\sqrt{2},\\
g'(v) &= -2c_4 e^{-2v} - \sqrt{2}c_5 \sin u\sqrt{2} + \sqrt{2}c_6 \cos u\sqrt{2}
\end{aligned}
\tag{4.16}
$$

where $\left(\mu = \frac{1}{2}, \lambda = 2, \gamma = 1\right)$.

Theorem 4.2. *Let $\mathbf{M}^h$ be a non harmonic surface at a constant distance from the edge of regression on a translation surface of type 1 given by (3.3) in $\mathbb{I}_3^1$. If the surface $\mathbf{M}^h$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x}_i{=}\lambda_i\mathbf{x}_i$, where $\lambda_i{\in}\mathbb{R}$, $i{=}1,2,3$, then it is congruent to an open part of the surface (4.14).*

5. Surfaces at a constant distance from the edge of regression on a translation surface of Type 1 satisfying $\Delta^{\mathbf{II}}\mathbf{x}_i^h = \lambda_i\mathbf{x}_i^h$

In this section, we classify surfaces at a constant distance from the edge of regression on a translation surface of Type 1 with non-degenerate second fundamental form in $\mathbb{I}_3^1$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i^h = \lambda_i\mathbf{x}_i^h, \tag{5.1}$$

where $\lambda_i{\in}\mathbb{R}$, $i{=}1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x}^h = \left(\Delta^{\mathbf{II}}\mathbf{x}_1^h, \Delta^{\mathbf{II}}\mathbf{x}_2^h, \Delta^{\mathbf{II}}\mathbf{x}_3^h\right),$$

where

$$\mathbf{x}_1^h = u + \eta,\ \mathbf{x}_2^h = v + \mu,\ \mathbf{x}_3^h = (f(u) + \eta f'(u)) + (g(v) + \mu g'(v)) + \gamma.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}^h$ with the help of (4.3) and (2.7) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x}^h = \begin{pmatrix} \frac{f'''+\eta f^{(4)}}{2(f''+\eta f''')^2}, \\ \frac{g'''+\mu g^{(4)}}{2(g''+\mu g''')^2}, \\ -\frac{4\left(f''+\eta f'''\right)^2\left(g''+\mu g'''\right)^2}{2(f''+\eta f''')^2(g''+\mu g''')^2} \\ +\frac{\left(f'+\eta f''\right)\left(f'''+\eta f^{(4)}\right)\left(g''+\mu g'''\right)^2}{2(f''+\eta f''')^2(g''+\mu g''')^2} \\ +\frac{\left(f''+\eta f'''\right)^2\left(g'+\mu g''\right)\left(g'''+\mu g^{(4)}\right)}{2(f''+\eta f''')^2(g''+\mu g''')^2} \end{pmatrix}. \tag{5.2}$$

The equation (5.1) by means of (5.2) gives rise to the following system of ordinary differential equations

$$\frac{f''' + \eta f^{(4)}}{2\left(f'' + \eta f'''\right)^2} = \lambda_1 (u + \eta) \tag{5.3}$$

$$\frac{g''' + \mu g^{(4)}}{2\left(g'' + \mu g'''\right)^2} = \lambda_2 (v + \mu), \tag{5.4}$$

$$\begin{aligned} &-\frac{4\left(f'' + \eta f'''\right)^2\left(g'' + \mu g'''\right)^2}{2\left(f'' + \eta f'''\right)^2\left(g'' + \mu g'''\right)^2} + \frac{\left(f' + \eta f''\right)\left(f''' + \eta f^{(4)}\right)\left(g'' + \mu g'''\right)^2}{2\left(f'' + \eta f'''\right)^2\left(g'' + \mu g'''\right)^2} \\ &+\frac{\left(f'' + \eta f'''\right)^2\left(g' + \mu g''\right)\left(g''' + \mu g^{(4)}\right)}{2\left(f'' + \eta f'''\right)^2\left(g'' + \mu g'''\right)^2} = \lambda_3 \left(f + g + \eta f' + \mu g' + \gamma\right) \end{aligned} \tag{5.5}$$

where $\lambda_i \in \mathbb{R}$. This means that $\mathbf{M}^h$ is at most of 3- types. Combining equations (5.3), (5.4) and (5.5), we have

$$\begin{aligned} -2 + \lambda_1 (u+\eta)(f' + \eta f'') - \lambda_3 (f + \eta f') = -\lambda_2 (v+\mu)(g' + \mu g'') \\ + \lambda_3 (g + \mu g') + \lambda_3 \gamma \end{aligned} \tag{5.6}$$

The differential equation (5.6) cannot be solved analytically according to constants $\lambda_1, \lambda_2, \lambda_3$. Since the remained cases with respect to $\lambda_1, \lambda_2, \lambda_3$ there are no any solutions analytically, we discuss only one case respect to constants $\lambda_1, \lambda_2, \lambda_3$: that $\lambda_1 = \lambda_2 = 0, \lambda_3 \neq 0$, from (5.6), we obtain

$$-2 - \lambda_3 (f + \eta f') = \lambda_3 (g + \mu g') + \lambda_3 \gamma. \tag{5.7}$$

Here u and v are independent variables, so each side of (5.7) is equal to a constant, call it p. Hence, the two equations

$$-2 - \lambda_3 (f + \eta f') = p = \lambda_3 (g + \mu g') + \lambda_3 \gamma. \tag{5.8}$$

Their general solutions are

$$\begin{aligned} f(u) &= c_1 e^{-\frac{u}{\eta}} - \frac{2+p}{\lambda_3}, \\ g(v) &= c_2 e^{-\frac{v}{\mu}} + \frac{p - \gamma\lambda_3}{\lambda_3}, \end{aligned} \tag{5.9}$$

where $c_1, c_2 \neq 0$ are some constants. In this case, $\mathbf{M}^h$ is parametrized by (3.3) with (5.9).

Definition 5.1. A surface in the three dimensional simple isotropic space is said to be $\mathbf{II}-$harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x}^h = \mathbf{0}$.

Theorem 5.2. *Let $\mathbf{M}^h$ be a surface at a constant distance from the edge of regression on a translation surface of Type 1 given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. Then, there is no $\mathbf{II}-$harmonic surface.*

Theorem 5.3. *Let $\mathbf{M}^h$ be a non $\mathbf{II}$-harmonic surface at a constant distance from the edge of regression on a translation surface of Type 1 given by (3.3) in $\mathbb{I}_3^1$. If the surface $\mathbf{M}^h$ satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x}_i^h = \lambda_i \mathbf{x}_i^h$, where $\lambda_i \in \mathbb{R}$, $i=1,2,3$, then it is congruent to an open part of the surface (3.3) with (5.9).*

6. Surfaces at a constant distance from the edge of regression on a translation surface of Type 1 Satisfying $\Delta^{\mathbf{III}}\mathbf{x}_i^h = \lambda_i \mathbf{x}_i^h$

In this section, we classify surfaces at a constant distance from the edge of regression on a translation surface of Type 1 in $\mathbb{I}_3^1$ satisfying the equation

$$\Delta^{\mathbf{III}}\mathbf{x}_i^h = \lambda_i \mathbf{x}_i^h, \tag{6.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{III}}\mathbf{x}^h = \left(\Delta^{\mathbf{III}}\mathbf{x}_1^h, \Delta^{\mathbf{III}}\mathbf{x}_2^h, \Delta^{\mathbf{III}}\mathbf{x}_3^h\right),$$

where

$$\mathbf{x}_1^h = u+\eta,\ \mathbf{x}_2^h = v+\mu,\ \mathbf{x}_3^h = (f(u)+\eta f'(u)) + (g(v)+\mu g'(v)) + \gamma.$$

Suppose that the surface has non zero Gaussian curvature, so

$$(f''+\eta f''')\,(g''+\mu g''') \neq 0, \forall u, v \in I.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}^h$ with the help of (4.2), (4.3) and (2.8) turns out to be

$$\Delta^{\mathbf{III}}\mathbf{x}^h = \begin{pmatrix} \frac{f'''+\eta f^{(4)}}{(f''+\eta f''')^3}, \\ \frac{g'''+\mu g^{(4)}}{(g''+\mu g''')^3}, \\ -\frac{1}{(f''+\eta f''')} - \frac{1}{(g''+\mu g''')} \\ +\frac{\left(f'+\eta f''\right)\left(f'''+\eta f^{(4)}\right)}{(f''+\eta f''')^3} \\ +\frac{\left(g'+\mu g''\right)\left(g'''+\mu g^{(4)}\right)}{(g''+\mu g''')^3} \end{pmatrix} \tag{6.2}$$

Equation (6.1) by means of (6.2) gives rise to the following system of ordinary differential equations

$$\frac{f'''+\eta f^{(4)}}{\left(f''+\eta f'''\right)^3} = \lambda_1\left(u+\eta\right), \tag{6.3}$$

$$\frac{g'''+\mu g^{(4)}}{\left(g''+\mu g'''\right)^3} = \lambda_2(v+\mu), \tag{6.4}$$

$$\begin{aligned} &-\frac{1}{(f''+\eta f''')} - \frac{1}{(g''+\mu g''')} + \frac{\left(f'+\eta f''\right)\left(f'''+\eta f^{(4)}\right)}{\left(f''+\eta f'''\right)^3} \\ &+\frac{\left(g'+\mu g''\right)\left(g'''+\mu g^{(4)}\right)}{\left(g''+\mu g'''\right)^3} = \lambda_3\left(f+g+\eta f'+\mu g'\ +\gamma\right) \end{aligned} \tag{6.5}$$

where λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$. This means that $\mathbf{M}^h$ is at most of 3- types. Combining equations (6.3), (6.4) and (6.5), we have

$$\begin{aligned} -\frac{1}{(f''+\eta f''')} + \lambda_1\left(u+\eta\right)\left(f'+\eta f''\right) - \lambda_3\left(f+\eta f'\right) &= \frac{1}{(g''+\mu g''')} \\ &-\lambda_2\left(v+\mu\right)\left(g'+\mu g''\right) + \lambda_3\left(g+\mu g'\right) + \lambda_3\gamma. \end{aligned} \tag{6.6}$$

This nonlinear differential equation (6.6) cannot be solved analytically according to constants $\lambda_1,\ \lambda_2,\ \lambda_3$. Since the remained cases are not occur with respect to $\lambda_1,\ \lambda_2,\ \lambda_3$, we discuss only one case respect to constants $\lambda_1,\ \lambda_2,\ \lambda_3$: that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. From (6.6), we obtain

$$-\frac{1}{(f''+\eta f''')} = \frac{1}{(g''+\mu g''')}. \tag{6.7}$$

Here u and v are independent variables, so each side of (6.7) is equal to a constant, call it p. Hence, the two equations

$$-\frac{1}{(f''+\eta f''')} = p = \frac{1}{(g''+\mu g''')}. \tag{6.8}$$

Their general solutions are

$$
\begin{aligned}
f(u) &= -\frac{u^2}{2p} + c_1\frac{\eta^2}{p}e^{-\frac{u}{\eta}} + c_2u + c_3, \\
g(v) &= \frac{v^2}{2p} + c_4\frac{\mu^2}{p}e^{-\frac{v}{\mu}} + c_5v + c_6,
\end{aligned}
\tag{6.9}
$$

where for some constants $c_i, p \neq 0$. In this case, $\mathbf{M}^h$ is parametrized by (3.3) with (6.9).

Definition 6.1. A surface in the three dimensional simple isotropic space is said to be **III**-harmonic if it satisfies the condition $\Delta^{\mathbf{III}}\mathbf{x}^h = \mathbf{0}$.

Theorem 6.2. *Let $\mathbf{M}^h$ be a surface at a constant distance from the edge of regression on a translation surface of Type 1 given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. If $\mathbf{M}^h$ is* **III***-harmonic, then it is congruent to an open part of the surface (3.3) with (6.9).*

Theorem 6.3. *Let $\mathbf{M}^h$ be a non* **III***-harmonic surface surface at a constant distance from the edge of regression on a translation surface of Type 1 given by (3.3) in the three dimensional simply isotropic space $\mathbb{I}_3^1$. Then, there is no the surface $\mathbf{M}^h$ satisfies the condition $\Delta^{\mathbf{III}}\mathbf{x}_i^h = \lambda_i\mathbf{x}_i^h$, where $\lambda_i \in \mathbb{R}$.*

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