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Relative Non-normal Graphs of a Subgroup of Finite Groups

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Abstract. Let G be a finite group and H,K be two subgroups of G. We introduce the relative non-normal graph of K with respect to H , denoted by $\mathfrak{N}_{H,K}$, which is a bipartite graph with vertex sets $H\setminus H_K$ and $K\setminus N_K(H)$ and two vertices $x \in H\setminus H_K$ and $y \in K\setminus N_K(H)$ are adjacent if $x^y \notin H$, where $H_K = \bigcap_{k\in K} H^k$ and $N_K(H) = \{k \in K : H^k = H\}$. We determined some numerical invariants and state that when this graph is planar or outerplanar.

Keywords: Non-normal graph, Relative non-normal graph, Normality degree, Outer planar.

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1. Introduction

There are many ways to assign a graph to groups and many graphs have been associated to a group, such as non-cyclic graph, Engel graph and non-commuting graph (see [3, 1, 2]). Saeedi, Farrokhi and Jafari [8] introduced the subgroup normality degree of finite groups as the ratio of the number of pairs

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$(h,g) \in H \times G$ such that $h^g \in H$ by $|H||G|$, where G is a finite group and H is a subgroup of G. Erfanian, Farrokhi and Tolue [7] defined non-normal graph of finite groups as follows: Let H be a subgroup of a group G. Then non-normal graph of G with respect to H, denoted by $\mathfrak{N}_{H,G}$, is defined as a bipartite graph with vertex sets $H \setminus H_G$ and $G \setminus N_G(H)$ as its parts in such a way that two vertices $h \in H \setminus H_G$ and $g \in G \setminus N_G(H)$ are adjacent if $h^g \notin H$. Also they gave some properties of $\mathfrak{N}_{H,G}$ such as girth, diameter and planarity.

In this paper, we aim to give a generalization of non-normal graph. We note that the idea of non-normal graph comes from the probability of a subgroup H is normal in G. Now, we may replace group G by another subgroup K of G. In other words, we can consider normality of H with respect to the subgroup K i.e. H is normal with respect to K whenever $h^k \in H$ for all $k \in K$ and all $h \in H$. Thus, we state the related graph namely relative non-normal graph as the following. For any two subgroups H and K of G, we remind that $H_K = \bigcap_{k\in K} H^k$ and $N_K(H) = \{k \in K : H^k = H\} = N_G(H) \bigcap K$. So for all $h \in H$ and $k \in K$, if $h \in H_K$ or $k \in N_K(H)$ then $h^k \in H$. Assume that $|H| \leq |N_K(H)|$, the relative non-normal graph of K with respect to H, denoted by $\mathfrak{N}_{H,K}$, is defined as a bipartite graph with vertex sets $H \setminus H_K$ and $K \setminus N_K(H)$ as its parts in such a way that two vertices $h \in H \setminus H_K$ and $k \in K \setminus N_K(H)$ are adjacent if $h^k \notin H$.

Clearly, if H is normal with respect to K, then $\mathfrak{N}_{H,K}$ is a null graph. Moreover, if $K = G$, $\mathfrak{N}_{H,K}$ and $\mathfrak{N}_{H,G}$ are concide. As it is mentioned before, the subgroup normality degree of H in G is defined as the following :

$$P_N(H,G) = \frac{|\{(h,g) \in H \times G : h^g \in H\}|}{|H||G|}.$$

So the relative normality degree of H in K can be similarly defined. It is easy to see that, the graph $\mathfrak{N}_{H,K}$ and the relative normality degree of H in K are associated through the equality

$$|E(\mathfrak{N}_{H,K})| = |H||K|(1 - P_N(H,K)),$$

where $E(\mathfrak{N}_{H,K})$ denotes the set of all edges of $\mathfrak{N}_{H,K}$.

In this paper, we state some results which are mostly new or an improvement of results given in [7]. In the next section, we give some basic properties of this graph. Section 3 deals with diameter and girth of the graph and classify all cases that diameter is 2, 3 or 4. In section 4, planarity and outer planarity are investigated. Given a graph $\Gamma = (V,E)$, a dominating set for Γ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(\Gamma)$ is the number of vertices in a smallest dominating set for Γ. An independent set or stable set is a set of vertices in a graph , no two of which are adjacent. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. An undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. That is, no vertex is totally surrounded by edges. Alternatively, a graph Γ is outerplanar if the graph formed from Γ by adding a new vertex, with edges connecting it to all the other vertices, is a planar graph. For set X, we assume $X^2 = \{x^2 : x \in X\}$.

2. Preliminary Results

Let H and K be two subgroups of a finite group G and $\mathfrak{N}_{H,K}$ be the relative non-normal graph of K with respect to H. Remind that $\mathfrak{N}_{H,K}$ is a bipartite graph with bipartition $H \setminus H_K$ and $K \setminus N_K(H)$. As $K \setminus N_K(H)$ is a union of right cosets of $N_K(H)$, we have

$$|H \setminus H_K| < |H| \leq |N_K(H)| \leq |K \setminus N_K(H)|.$$

Now let $h \in H \setminus H_K$ and $k \in K \setminus N_K(H)$. Then the neighbor of h in $\mathfrak{N}_{H,K}$, denoted by $N_{\mathfrak{N}_{H,K}}(h)$ is the set of all elements $x \in K \setminus N_K(H)$ such that $h^x \notin H$ that is $N_{\mathfrak{N}_{H,K}}(h) = K \setminus A(K,H,h)$, where $A(K,H,h) = \{x \in K : h^x \in H\}$. Similarly the neighbor of k in $\mathfrak{N}_{H,K}$ equals $H \setminus B(K,H,k)$, where $B(K,H,k) = \{y \in H : y^k \in H\}$. It is evident that $B(K,H,k) = H \cap H^{k^{-1}}$ hence $N_{\mathfrak{N}_{H,K}}(k) = H \setminus H \cap H^{k^{-1}}$. As $A(K,H,h)$ is a union of right cosets of $N_K(H)$ we observe that $N_{\mathfrak{N}(H,K)}(h)$ is a non-empty union of right cosets of $N_K(H)$ and hence

$$\deg_{\mathfrak{N}_{H,K}}(h) = |N_{\mathfrak{N}_{H,K}}(h)| \geq |N_K(H)| \geq |H| > |H \setminus H \cap H^{k^{-1}}| = \deg_{\mathfrak{N}_{H,K}}(k),$$

where $\deg_{\mathfrak{N}_{H,K}}(h)$ and $\deg_{\mathfrak{N}_{H,K}}(k)$ denote the degree of h and k in $\mathfrak{N}_{H,K}$, respectively. In particular, $\mathfrak{N}_{H,K}$ is never a regular graph.

Lemma 2.1. *If H and K are two subgroups of a finite group G, then $\mathfrak{N}_{H,K}$ is an induced subgraph of $\mathfrak{N}_{H,G}$.*

Proof. The proof follows from the fact that $H \setminus H_K \subseteq H \setminus H_G$ and $K \setminus N_K(H) \subseteq G \setminus N_G(H)$ directly. □

Theorem 2.2. *We have*

(i) *$K \setminus N_K(H)$ is a maximal independent set of $\mathfrak{N}_{H,K}$,*
(ii) *the size of maximal dominating sets of $\mathfrak{N}_{H,K}$ are at most $d(H) + [K : N_K(H)] - 1$.*

Proof. (*i*) Clearly $H \setminus H_K$ and $K \setminus N_K(H)$ are independent sets of $\mathfrak{N}_{H,K}$. If X is a maximal independent set of $\mathfrak{N}_{H,K}$, then $X = A \cup B$, where $A \subseteq H \setminus H_K$ and $B \subseteq K \setminus N_K(H)$. Since $|X|$ is maximum, B is a union of right cosets of

$N_K(H)$. Now if $X \neq K \setminus N_K(H)$, then $|B| \leq |K \setminus N_K(H)| - |N_K(H)|$, from which it follows that

$$|A| + |B| < |H| + |K \setminus N_K(H)| - |N_K(H)| \leq |K \setminus N_K(H)|,$$

which is a contradiction. Therefore $K \setminus N_K(H)$ is a maximal independent set of $\mathfrak{N}_{H,K}$ and the proof of (i) is completed.

(ii) If X is a minimal generating set for H, then it is easy to see that every element of $K \setminus N_K(H)$ is adjacent to some elements of X. Since the neighbor of every element of $H \setminus H_K$ is a union of right cosets of $N_K(H)$, every element of $H \setminus H_K$ is adjacent to some element of Y, where Y is a set of representatives of non-trivial right cosets of $N_K(H)$ in K. Hence the size of every dominating set of $\mathfrak{N}_{H,K}$ is bounded above by $|X| + |Y| = d(H) + [K : N_K(H)] - 1$ and the proof is complete. □

In the sequel, G stands for a finite group and H and K denote two non-normal subgroups of G.

3. Diameter and Girth

In the previous section, we gave some elementary properties of $\mathfrak{N}_{H,K}$. Now we shall determine some more properties of $\mathfrak{N}_{H,K}$. We start with the following simple lemma which is necessary to find an upper bound for the diameter of $\mathfrak{N}_{H,K}$.

Lemma 3.1. *$\mathfrak{N}_{H,K}$ has a pendant vertex if and only if $|H| = 2$ and $\mathfrak{N}_{H,K}$ is a star graph.*

Proof. Let $x \in V(\mathfrak{N}_{H,K})$ be a pendant vertex. If $x \in H \setminus H_K$, then $|K \setminus A(K, H, x)| = \deg x = 1$. But $A(K, H, x)$ is a union of right cosets of $N_K(H)$ and so $|N_K(H)|$ divides $|K \setminus A(K, H, x)|$, which is impossible. Thus $x \in K \setminus N_K(H)$. Then $|H \setminus H \cap H^{x^{-1}}| = \deg x = 1$. Now since $H \cap H^{x^{-1}}$ is a subgroup of H, $|H \cap H^{x^{-1}}|$ divides $|H \setminus H \cap H^{x^{-1}}|$ and so $|H \cap H^{x^{-1}}| = 1$. Hence $|H| = 2$ and the result follows. The converse is obvious. □

Theorem 3.2. $\operatorname{diam}(\mathfrak{N}_{H,K}) \leq 4$.

Proof. Let x and y be two non-adjacent vertices of $\mathfrak{N}_{H,K}$. First assume that $x, y \in K \setminus N_K(H)$. Then there exists $h_1, h_2 \in H \setminus H_K$ such that $h_1^x, h_2^y \notin H$. If either x and h_2 are adjacent, or y and h_1 are adjacent, then $d(x, y) = 2$ and we are done. Thus we may assume that $h_2^x, h_1^y \in H$. But then $h_1h_2 \in H \setminus H_K$ is adjacent to both x, y and $d(x, y) = 2$. Now assume that x, y belong to different parts of $\mathfrak{N}_{H,K}$, say $x \in H \setminus H_K$ and $y \in K \setminus N_K(H)$. Let $k \in K \setminus N_K(H)$ be a vertex adjacent to x. Then $d(x, y) \leq d(y, k) + 1$ =3. Finally suppose that $x, y \in H \setminus H_K$ and x, y be adjacent to vertices $u, v \in K \setminus N_K(H)$, respectively. Then $d(x, y) \leq d(u, v) + 2 \leq 4$ and the proof is complete. □

By the above lemma the relative non-normal graph is connected. It is easy to see that $\mathrm{diam}(\mathfrak{N}_{H,K}) = 4$ if and only if there exist two vertices x, y in a same part of $\mathfrak{N}_{H,K}$, which have no common neighbor. Let $\mathrm{diam}(\mathfrak{N}_{H,K}) = 4$ and h_1, h_2 be two vertices in a same part such that have no common neighbor. By the proof of Theorem 3.2, h_1 and h_2 must be in part $H \setminus H_K$. Then $(K \setminus A(K,H,h_1)) \cap (K \setminus A(K,H,h_2)) = \emptyset$. Hence $K = A(K,H,h_1) \cup A(K,H,h_2)$, as required. The converse is clear. Therefore $\mathrm{diam}(\mathfrak{N}_{H,K}) = 4$ if and only if $K = A(K,H,h_1) \cup A(K,H,h_2)$ for some $h_1, h_2 \in H \setminus H_K$.

Theorem 3.3. *If $|H| > 2$, then the girth of $\mathfrak{N}_{H,K}$ is 4.*

Proof. Since $\mathfrak{N}_{H,K}$ is a bipartite graph and by Lemma 3.1, $\mathfrak{N}_{H,K}$ has a cycle we have that $gr(\mathfrak{N}_{H,K}) \geq 4$. Hence we have to show that $\mathfrak{N}_{H,K}$ indeed has a cycle of length four. If $(H \setminus H_K)^2 \neq 1$ such that $(H \setminus H_K)^2 = \{a^2 : a \in H \setminus H_K\}$, then there exist $a \in H \setminus H_K$ such that $a \neq a^{-1}$. By Lemma 3.1, a is not pendant then there exist $x, y \in K \setminus N_K(H)$ such that a is adjacent to x and y. Then the elements a, a^{-1}, x, y induce a cycle of length 4 and hence the girth of $\mathfrak{N}_{H,K}$ is 4. Suppose $(H \setminus H_K)^2 = 1$. By Lemma 3.2, $\mathrm{diam}(\mathfrak{N}_{H,K}) \leq 4$. If $\mathrm{diam}(\mathfrak{N}_{H,K}) = 2$, then $\mathfrak{N}_{H,K}$ is complete bipartite graph and girth of $\mathfrak{N}_{H,K}$ is 4. If $\mathrm{diam}(\mathfrak{N}_{H,K}){=}3$, then for every $a, b \in H \setminus H_K$, $d(a,b) {=}2$. Let $x, y \in K \setminus N_K(H)$ and $a^x \notin H$, $a^y \in H$, $b^y \notin H$ and $b^x \in H$, in this case since $a \neq b = b^{-1}$, then $ab \neq a$ and $ab \neq b$, hence $d(b, ab) = 2$ then there exist $z \in K \setminus N_K(H)$ such that z is adjacent to b and ab, also ab and y are adjacent and the elements b, ab, y and z induce a cycle of length 4. Finally if $\mathrm{diam}(\mathfrak{N}_{H,K}) {=}4$, in this case $a, b \in H \setminus H_K$ such that $d(a,b) = 4$. Let $x, y \in K \setminus N_K(H)$ and $c \in H \setminus H_K$ such that $a^x \notin H$, $b^y \notin H$, $a^y \in H$, $b^x \in H$, $c^x \notin H$ and $c^y \notin H$. In this case ab is adjacent to x and y, then the elements c, x, ab and y induce a cycle of length 4 and hence the girth of $\mathfrak{N}_{H,K}$ is 4. □

Let H and K be two subgroups of G. H is called a TI-subgroup with respect to K if $H \cap H^k = 1$ for all $k \in K \setminus N_K(H)$. For the following theorem and two corollaries, we assumed that H_K is a normal subgroup of K.

Theorem 3.4. $\mathrm{diam}(\mathfrak{N}_{H,K}) = 2$ *if and only if $\mathfrak{N}_{H,K}$ is a complete bipartite graph if and only if H/H_K is a TI-subgroup with respec to K/H_K.*

Proof. It is obvious that $\mathrm{diam}(\mathfrak{N}_{H,K}) = 2$ if and only if $\mathfrak{N}_{H,K}$ is a complete bipartite graph. Let $\overline{H} = H/H_K$ and $\overline{K} = K/H_K$. If $\overline{H}$ is a TI-subgroup with respec to $\overline{K}$ and $\overline{k} \in \overline{K} \setminus N_{\overline{K}}(\overline{H})$, then $\overline{H} \cap \overline{H}^{\overline{k}} = \overline{1}$. So $\overline{k}^{-1}$ is adjacent to $\overline{h}$ for all $\overline{h} \in \overline{H} \setminus \{\overline{1}\}$ that is $\overline{h}^{\overline{k}^{-1}} \notin \overline{H}$ for all $\overline{k} \in \overline{K} \setminus N_{\overline{K}}(\overline{H})$ and $\overline{h} \in \overline{H} \setminus \{\overline{1}\}$. Then $h^{k^{-1}} \notin H$ for all $k \in K \setminus N_K(H)$ and $h \in H \setminus H_K$. So $\mathfrak{N}_{H,K}$ is a complete bipartite graph. The converse is similar. □

A subgroup K of G is called a *Krutik* group if $A(K,H,h)$ is a subgroup of K for each subgroup H of G and element $h \in H$. For instance, take $G = S_4$,

$K = S_3$ and $H = \langle(1234)\rangle$. Then $N_K(H) = \{1, (13)\}$, $H_K = \{1\}$ and $\mathfrak{N}_{H,K}$ is isomorphic to $K_{3,4}$, so H is a TI-subgroup with respect to K also K is a *Krutik* group.
In the following two corollaries we consider the case where the diameter is 3.

Corollary 3.5. *If K is a Krutik subgroup of G, then* $\mathrm{diam}(\mathfrak{N}_{H,K}) = 3$ *for all non-normal subgroup H of G such that H/H_K is not a TI-subgroup with respect to K/H_K.*

Corollary 3.6. *If H is a cyclic subgroup of G such that H/H_K is not a TI-subgroup with respect to K/H_K, then* $\mathrm{diam}(\mathfrak{N}_{H,K}) = 3$.

Proof. It is straightforward to see that $A(K, H, h) = N_K(\langle h\rangle)$ is a subgroup of K for each $h \in H \setminus H_K$. Hence by Lemma 3.4, we have $\mathrm{diam}(\mathfrak{N}_{H,K}) = 3$. □

4. Planarity and Outer Planarity

This section is devoted to a determination of planarity of relative non-normal graphs. Except for few possible cases, we show that the relative non-normal graphs are not planar. We begin with some elementary lemmas.

Lemma 4.1. *If H is a cyclic subgroup of G, then $\mathfrak{N}_{H,K}$ has a subgraph isomorphic to $K_{\varphi(|H|),|K|-|N_K(H)|}$, where φ is the Euler's totient function. In particular if H is a cyclic group of order p, then $\mathfrak{N}_{H,K}$ is isomorphic to $K_{p-1,|K|-|N_K(H)|}$.*

Proof. The result follows from the fact that the generators of H are adjacent to all elements of $K \setminus N_K(H)$. □

Lemma 4.2. *If H_K is a maximal subgroup of H, then $\mathfrak{N}_{H,K}$ is isomorphic to $K_{|H|-|H_K|,|K|-|N_K(H)|}$.*

Proof. Every element of $H \setminus H_K$ is adjacent to all elements of $K \setminus N_K(H)$. Suppose on the contrary that there exist $h \in H \setminus H_K$ such that h is not adjacent to some element $k \in K \setminus N_K(H)$. Let $N = \langle H_K \cup \langle h\rangle\rangle$. Then we show that $N \neq H$. Since $k \in K \setminus N_K(H)$ there exist $h_0 \in H \setminus H_K$ such that $h_0^k \notin H$. If $h_0 \in \langle h\rangle$ or $h_0 \in N$ so $h_0^k \in H$, which is a contradiction. So $h_0 \in H \setminus N$ and $N \neq H$ which contradicts maximality of H_K in H. □

Lemma 4.3. *If $|H| > 2$, $a \in H \setminus H_K$, $a^2 \neq 1$ and $b \in H \setminus H_K$ not adjacent to at least three vertices adjacent to a, then $\mathfrak{N}_{H,K}$ is not planar.*

Proof. Lemma 3.1 implies that the degree of every vertex is at least 2. Also for every $h \in H \setminus H_K$ and $k \in K \setminus N_K(H)$, $deg(h) > deg(k)$. Let x, y, z be neighbors of a but not b, then the subgraph of $\mathfrak{N}_{H,K}$ induced by a, a^{-1}, ab, x, y, z is isomorphic to $K_{3,3}$, which contradicts planarity of $\mathfrak{N}_{H,K}$ by Kuratowski theorem, (see [6]). □

Lemma 4.4. *If* $|H\backslash H_K| > 2$*, where* H *is non-cyclic, and* $(H \setminus H_K \cap N_K(H))^2 \neq \{1\}$*, then* $\mathfrak{N}_{H,K}$ *is not planar.*

Proof. Since $(H \setminus H_K \cap N_K(H))^2 \neq \{1\}$, there exists an element $a \in (H \setminus H_K \cap N_K(H))$ such that $a \neq a^{-1}$. By Lemma 3.1 $deg(a) > 2$ and there exists $x \in K \setminus N_K(H)$ such that a and x are adjacent. Also a^{-1} and x are adjacent. Since $a \in N_K(H) \leq K$ then $xa^{-1} \in K \setminus N_K(H)$. Suppose x is adjacent to all vertices of $H\backslash H_K$. As H is not cyclic and $|H\backslash H_K| \geq 3$, there exists $b \in H\backslash H_K$ such that $a^{-1} \neq b \neq a$ then it is adjacent to x, xa and xa^{-1}. But the subgraph of $\mathfrak{N}_{H,K}$ induced by elements a, a^{-1}, b, x, xa and xa^{-1} is isomorphic to $K_{3,3}$ and $\mathfrak{N}_{H,K}$ is not planar. If there exist $h \in H \setminus H_K$ such that $h^x \in H$, then x and ah are adjacent so in this case ah is adjacent to xa and xa^{-1}, hence the subgraph of $\mathfrak{N}_{H,K}$ induced by elements $a, a^{-1}, ah, x, xa, xa^{-1}$ is isomorphic to $K_{3,3}$ and again $\mathfrak{N}_{H,K}$ is not planar. □

Lemma 4.5. *Let* G *be a finite group and* H, K *be two subgroups of* G *such that* $\mathfrak{N}_{H,K}$ *is planar, then* $|H| \leq 11$.

Proof. First we observe that for every planar graph X with at least three vertices, we have $e \leq 3v - 6$, where e and v denote the number of edges and vertices of X, respectively, (see [5]). Hence $|E(\mathfrak{N}_{H,K})| \leq 3|V(\mathfrak{N}_{H,K})| - 6$. Also Corollary 2.6 of [8] can be generalized for the relative normality degree of H in K. Thus $P_N(H, K) \leq \frac{3}{4}$. Now we have

$$|E(\mathfrak{N}_{H,K})| = |H||K|(1 - P_N(H, K)) \geq |H||K|(1 - \frac{3}{4}) = \frac{1}{4}|H||K|.$$

Hence

$$\begin{aligned} \frac{1}{4}|H||K| &\leq 3(|H| - |H_K| + |K| - |N_K(H)|) - 6 \\ &\leq 3(|H| - 1 + |K| - |H|) - 6 = 3|K| - 9, \end{aligned}$$

which implies that

$$|H| \leq 12 - \frac{36}{|K|} < 12.$$

Therefore $|H| \leq 11$. □

Now by using the rigth coset H_K in H and $N_K(H)$ in K we show that the relative non-normal graphs are not planar in the following two cases.

Lemma 4.6. *Vertices in the same coset of part* $K \setminus N_K(H)$ *or* $H \setminus H_K$ *have the same neighbour.*

Proof. Suppose that $x, y \in kN_K(H)$ which $k \in K$ and $h \in H \setminus H_K$ is adjacent to x. We show that h is adjacent to y, too. Suppose that $x = kn_1$ and $y = kn_2$ that $n_1, n_2 \in N_K(H)$. As $h^x = h^{kn_1} \notin H$, we have $h^k \notin H$, so $h^y = h^{kn_2} \notin H$. Similarly, we can show that vertices in same coset of $H \setminus H_K$ have the same neighbours. □

Lemma 4.6 verifies that each right coset of $K \setminus N_K(H)$ and each rigth coset of $H \setminus H_K$ in $\mathfrak{N}_{H,K}$ form a complete bipartite subgraph or empty bipartite subgraph.

Lemma 4.7. *If $|H_K| \geq 3$, then $\mathfrak{N}_{H,K}$ is not planar.*

Proof. Since $H \setminus H_K$ is a union of rigth cosets of H_K, then $|H \setminus H_K| \geq 3$. Let $h \in H$. Since the coset hH_K has at least three elements, there exist $h_1, h_2, h_3 \in hH_K$. Let $x \in K$ and $x_1 \in xN_K(H) = \{x_1, x_2, ..., x_{|N_K(H)|}\}$ be a neighbor of h_1, where $|N_K(H)| \geq |H| \geq |H \setminus H_K| \geq 3$. So by Lemma 4.6, the elements $h_1, h_2, h_3, x_1, x_2, x_3$ induce a subgraph of $\mathfrak{N}_{H,K}$ that is isomorphic to $K_{3,3}$ and so $\mathfrak{N}_{H,K}$ is not planar. □

Lemma 4.8. *If $|H \setminus H_K| \geq 4$, then $\mathfrak{N}_{H,K}$ is not planar.*

Proof. By Lemma 3.1, degree of every vertex is at least 2, also $deg(h_i) > deg(k_i) \geq 2$ for all $h_i \in H \setminus H_K$ and $k_i \in K \setminus N_K(H)$, and $|N_K(H)| \geq |H| \geq |H \setminus H_K| \geq 4$. Let $h_1 \in H \setminus H_K$, there exist vertices $k_1, k_2, k_3 \in kN_K(H)$ such that they are adjacent to h_1. Since $deg(k_1) \geq 2$, then there exist $h_2 \in H \setminus H_K$ such that k_1 is adjacent to h_2. $|H \setminus H_K| \geq 4$, let $h_3, h_4 \in H \setminus H_K$. If h_3 (or similarly h_4) is adjacent to k_1, then by Lemma 4.6, the subgraph of $\mathfrak{N}_{H,K}$ induced by $h_1, h_2, h_3, k_1, k_2, k_3$ that is isomorphic to $K_{3,3}$ and $\mathfrak{N}_{H,K}$ is not planar. If h_3 and h_4 are not adjacent to k_1 and $h_1h_3 \neq h_2$, then h_1h_3 is adjacent to k_1 and in this case by Lemma 4.6, the elements of $h_1, h_2, h_1h_3, k_1, k_2, k_3$, induce a subgraph of $\mathfrak{N}_{H,K}$ that is isomorphic to $K_{3,3}$ and $\mathfrak{N}_{H,K}$ is not planar, otherwise we may replace h_1h_3 by h_1h_4 and the proof is complete. □

Now, using of the previous results will show that with exception of a few possible cases, the relative non-normal graphs are not outer planar.

Lemma 4.9. *If $|H| > 2$ and H is a cyclic group, then $\mathfrak{N}_{H,K}$ is not outer planar.*

Proof. By Lemma 4.1, $\mathfrak{N}_{H,K}$ has a subgraph isomorphic to $K_{\varphi(|H|),|K|-|N_K(H)|}$. As $|H| \geq 3$, we have $\varphi(|H|) \geq 2$ and $|H \setminus H_K| \geq 2$, $|K| - |N_K(H)| > |H \setminus H_K| \geq 2$. Then $|K| - |N_K(H)| \geq 3$ and $\mathfrak{N}_{H,K}$ has a subgraph isomorphic to $K_{2,3}$ and so $\mathfrak{N}_{H,K}$ is not outer planar, (see [4]). □

Lemma 4.10. *If $|H| > 2$ and H_K is a maximal subgroup of H, then $\mathfrak{N}_{H,K}$ is not outer planar.*

Proof. By Lemma 3.1, $\mathfrak{N}_{H,K}$ is not star graph, then $|H \setminus H_K| \geq 2$, also H_K is amaximal subgroup of H and by Lemma 4.2 and $\mathfrak{N}_{H,K}$ is isomorphic to $K_{|H|-|H_K|,|K|-|N_K(H)|}$. Also $|K| - |N_K(H)| > |H \setminus H_K| \geq 2$, deduce that $|K| - |N_K(H)| \geq 3$ and $\mathfrak{N}_{H,K}$ has a subgraph isomorphic to $K_{2,3}$ and therefore $\mathfrak{N}_{H,K}$ is not outer planar, (see [4]). □

Lemma 4.11. *If $|H| > 2$ and $|H \setminus H_K|^2 \neq 1$, then $\mathfrak{N}_{H,K}$ is not outer planar.*

Proof. Since $|H| > 2$ by Lemma 3.1, degree of every vertex is at least 2 and for every $h \in H \setminus H_K$ and $x \in K \setminus N_K(H)$, $deg(h) > deg(x)$, then every vertex in $H \setminus H_K$ has degree at least 3. Let $a \in H \setminus H_K$ and $a \neq a^{-1}$, then there exist $x, y, z \in K \setminus N_K(H)$ such that a adjacent to x, y, z. Thus the subgraph of $\mathfrak{N}_{H,K}$ induced by the elements a, a^{-1}, x, y, z is isomorphic to $K_{2,3}$ and $\mathfrak{N}_{H,K}$ is not outer planar (see [4]). □

Finally, one can also see that if $|H| \geq 2$ or $|H \setminus H_K| \geq 2$, then $\mathfrak{N}_{H,K}$ is not outer planar.

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