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Uniform Number of a Graph

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Abstract. We introduce the notion of uniform number of a graph. The uniform number of a connected graph G is the least cardinality of a nonempty subset M of the vertex set of G for which the function $f_M : M^c \to \mathcal{P}(X) - \{\emptyset\}$ defined as $f_M(x) = \{D(x, y) : y \in M\}$ is a constant function, where $D(x, y)$ is the detour distance between x and y in G and $\mathcal{P}(X)$ is power set of $X = \{D(x_i, x_j) : x_i \neq x_j\}$. We obtain some basic results and compute the newly introduced graph parameter for some specific graphs.

Keywords: Graphs, Detour distance, Uniform number, Hamiltonian connected graphs.

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1. Introduction

All the graphs unless otherwise specified are finite, undirected, connected and simple. For standard graph theory terminology and notations not defined here, we refer Buckley and Harary [3]. We also refer [7] and [9] for the notions of theory of algorithms and their complexity.

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For an arbitrary pair of vertices $x, y \in V(G)$, the distance $d(x, y)$ is the length

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of a shortest path and the detour distance $D(x, y)$ is the length of a longest path between x and y in G. The study on detour distances was initiated by Chartrand *et al.* [5]. The detour eccentricity $e_D(G)$ of a vertex x is the detour distance from x to a vertex farthest from x. The detour radius $rad_D(G)$ of a connected graph G is the minimum detour eccentricity among the vertices of G and the detour diameter $diam_D(G)$ is the maximum detour eccentricity among the vertices of G.

Before we proceed any further, we recall few facts about detour distances.

Theorem 1.1. [5]

(1) *For any graph G, $D(x, y) = 1$ if and only if xy is a bridge in G.*
(2) *For any graph G, $d(x, y) = D(x, y)$ for every pair of vertices x and y of G if and only if G is a tree.*

As with distance, detour distance is also known to be a metric on the vertex set of any connected graph [6]. Finding the longest path between two vertices of a graph is known to be NP-complete [7]. However, when we consider the longest path problem (LPP) which involves finding a path of maximum length in a given graph where the length of a path may either be measured by its number of edges, or for weighted graphs by the sum of the weights of its edges, is NP-hard [9].

The notion of detour distance appears in various real world problems. Wong *et al.* [23] mentioned LPP on graphs in the context of information retrieval on peer-to-peer networks where the weights are associated with vertices. LPP has also been addressed in [18] for evaluating the worst-case packet delay of Switched Ethernet. LPP also appears in the domain of high-performance printed circuit board design in which one needs to find the longest path between two specified vertices on a rectangle grid routing [22]. In [17], the authors described LPP in the context of multi-robot patrolling.

In view of the above mentioned real-life applications and the algorithmic complexity of LPP, the notion of uniform sets of graph and uniform number of graph are introduced and investigated in this article. Our investigation of uniform number of a graph is also motivated from the work of Slater where he introduced the notion of locating sets [20] which are also referred as metric dimension [11] or resolving sets [4] in the current literature, the notion of distinct distance sets [21] and homometric sets [2] in a graph. An analogous but complementary notion of detour distance pattern distinguishing set of G was introduced and explored in [1].

2. Uniform Sets and Uniform Number of a Graph

For any connected graph G, let M^c be the complement of $M \subseteq V(G)$, $\mathcal{P}(X)$ be the powerset of the set $X = \{D(x_i, x_j) : x_i \neq x_j\}$ and $Y(X) = \mathcal{P}(X) - \{\emptyset\}$. For a nonempty subset $M \subseteq V(G)$, we define a function $f_M : M^c \to Y(X)$

as $f_M(x) = \{D(x,y) : y \in M\}$. We call the assignment $f_M(x)$ as the detour pattern of x with respect to M. A set M is said to be a uniform detour pattern set of the graph G if f_M is a constant function. For any $w \in V(G)$ the set $\{w\}^c$ is trivially a uniform detour pattern set of G. The trivial uniform detour pattern sets of G being analogous to trivial solutions, would be excluded from our discussion in this article. Henceforth a non-trivial uniform detour pattern set of a graph would be referred as a uniform set.

To begin with, we examine the behavior of a pendant vertex with respect to the uniform sets of a graph.

Proposition 2.1. *Let G be a graph of order $n \geq 3$ and v be a pendant vertex of G, then the set $\{v\}$ cannot be a uniform set of G.*

Proof. Let v be a pendant vertex in G and $M = \{v\}$. We want to show that M is not a uniform set of G. Since $n = |V(G)| \geq 3$, there exist at least one pair of vertices $u, w \in V(G)$ such that $vw, wu \in E(G)$. The edge vw being a bridge in G, $D(v,w) = 1$ and therefore

$$\begin{aligned} f_M(w) &= \{D(w,v)\} = \{1\}, \text{ and} \\ f_M(u) &= \{D(u,v)\} = \{D(u,w) + D(w,v)\} = \{D(u,w)+1\}. \end{aligned}$$

Detour distance being a metric, $f_M(w) \neq f_M(u)$ in G. Hence the proof. □

Let S be the set of all pendant vertices of a graph G. The above result simply asserts that: if $S' \subseteq S$ and $|S'| = 1$, then S' cannot be a uniform set of G. This raises a natural question: Whether or not any proper subset S' of S forms a uniform set of G? The foregoing question is answered by our next result.

Proposition 2.2. *Let S be the set of all pendant vertices of a graph G and $S' \subset S$, then S' alone cannot be a uniform set of G.*

Proof. Let $S = \{u_i : 1 \leq i \leq n_0 < n\}$ be the set of all pendant vertices of a graph G of order n. We want to show that any proper subset S' of S cannot be a uniform set of G.

Let $W = \{w_j : 1 \leq j \leq k\}$ be the set of vertices with $deg_G(w_j) \geq 2$ which partitions the set S into k-classes $S_j = \{v_{ji} : 1 \leq i \leq n_j\}$.

Case 1: If $S' \bigcap S_j = S'$ for some j, say $j = 1$.
Then $S' = \{v_{1i} : 1 \leq i \leq n_1\}$, and for all $2 \leq j \leq k$ we have

$$\begin{aligned} f_{S'}(w_1) &= \{D(w_1, v_{1i})\} = \{1\}, \text{ and} \\ f_{S'}(v_{ji}) &= \{D(v_{ji}, v_{1i})\} = \{D(v_{ji}, w_1) + D(w_1, v_{1i})\} = \{D(v_{ji}, w_1) + 1\}. \end{aligned}$$

Detour distance being a metric it follows that $f_{S'}(w_1) \neq f_{S'}(v_{ji})$ and hence S' is not a uniform set of G.

Case 2: If $S' \bigcap S_j = \emptyset$ for at least one j, say $j = 1$. Let $S' = \bigcup_{2}^{k} S_j$. Then for $j \neq 1$ and $w_1 \in W$ we have

$$f_{S'}(w_1) = \bigcup_{j=2}^{k}\{D(w_1, v_{ji})\} = \bigcup_{j=2}^{k}\{D(w_1, w_j)+D(w_j, v_{ji})\} = \bigcup_{j=2}^{k}\{D(w_1+w_j)+1\}.$$

Similarly, for $j \neq 1$ and each $v_{1i} \in S_1$

$$\begin{aligned} f_{S'}(v_{1i}) &= \bigcup_{j=2}^{k}\{D(v_{1i}, v_{ji})\} = \bigcup_{j=2}^{k}\{D(v_{1i}, w_k) + D(w_k, w_j) + D(w_j, v_{ji})\} \\ &= \bigcup_{j=2}^{k}\{1 + D(w_k, w_j) + 1\} = \bigcup_{j=2}^{k}\{D(w_1, w_j) + 2\}. \end{aligned}$$

Clearly, $f_{S'}(w_k) \neq f_{S'}(v_{ki})$ and S' is not a uniform set of G.

Case 3: If $S' \bigcap S_j \neq \emptyset$ for all j, then $S' \subset \bigcup_{j=1}^{k} S_j = S$. Since $S' \subset S$, there exist at least one pair (j, i), say $(1, 1)$ for which $v_{11} \notin S'$. For $2 \leq i \leq n_1$, and $w_1 \in W$ and $v_{11} \in S_1$, we have

$$\begin{aligned} f_{S'}(w_1) &= \bigcup_{j=2}^{k}\{D(w_1, v_{ji})\} \bigcup \{D(w_1, v_{1i})\} \\ &= \bigcup_{j=2}^{k}\{D(w_1, w_j) + D(w_j, v_{ji})\} \bigcup \{1\} \\ &= \bigcup_{j=2}^{k}\{D(w_1, w_j) + 1\} \bigcup \{1\}. \end{aligned}$$

Again, $2 \leq i \leq n_1$ and $v_{11} \in S_1$, we have

$$\begin{aligned} f_{S'}(v_{11}) &= \bigcup_{j=2}^{k}\{D(v_{11}, v_{ji})\} \bigcup \{D(v_{11}, v_{1i})\} \\ &= \bigcup_{j=2}^{k}\{D(v_{11}, w_1) + D(w_1, w_j) + D(w_j, v_{ji})\} \bigcup \{2\} \\ &= \bigcup_{j=2}^{k}\{1 + D(w_1, w_j) + 1\} \bigcup \{2\} = \bigcup_{j=2}^{k}\{D(w_1, w_j) + 2\} \bigcup \{2\}. \end{aligned}$$

Clearly, $f_{S'}(w_1) \neq f_{S'}(v_{11})$. S' yet again is not a uniform set of G.

Hence in all the possible cases S' is not a uniform set of G. $\square$

In view of Proposition 2.1 and 2.2 it is evident that any proper subset of S cannot be a uniform set of G. Hence it would be meaningful to explore, whether or not the set of all pendant vertices S of a graph G forms a uniform set of

G? And, in the event of S alone being a uniform set of G, what is the least cardinality of S? Also, it would be meaningful to characterize the graphs for which the set of all pendant vertices is a uniform set.

The following result not only determines the least cardinality of the set S for which it is a uniform set, but it also characterizes the class of graphs for which it is a uniform set.

Proposition 2.3. *If G is a graph of order $n \geq 4$ with exactly two pendant vertices, say u, v, then the set $\{u, v\}$ is a uniform set of G if and only if $G \cong P_4$.*

Proof. Let $G \cong P_4$ and $\{v_1, v_2, v_3, v_4\}$ be the vertices of P_4 taken in a sequential order. Let $M = \{v_1, v_4\}$, then M is a uniform set of P_4 as $f_M(v_2) = \{1, 2\} = f_M(v_3)$.

Conversely, let G be a graph of order $n \geq 4$ with exactly two pendant vertices, say u, v such that $M = \{u, v\}$ is a uniform set of G. We shall show $G \cong P_4$.

If the pendant vertices of G are both adjacent to a vertex w in G, then

$$f_M(w) \quad = \quad \{D(w, u), D(w, v)\} = \{1\}.$$

Since $|V(G)| \geq 4$, there exists at least one vertex x different from u, v, w in G. For such a vertex $x \in V(G)$, we have

$$\begin{aligned} f_M(x) &= \{D(x, u), D(x, v)\} = \{D(x, w) + D(w, u), D(x, w) + D(w, v)\} \\ &= \{D(x, w) + 1, D(x, w) + 1\} = \{D(x, w) + 1\}. \end{aligned}$$

Since $x \neq w \in V(G)$, $D(w, x) > 1$. Hence $f_M(w) \neq f_M(x)$, a contradiction to the fact that M is a uniform set of G. It follows that $uw, vw \notin E(G)$.

Let there exist a pair of vertices, say $u', v' \in V(G)$ such that $u'u, v'v \in E(G)$. To show that G is a path, it is enough to show that the degree of every vertex different from u, v in G is 2. Assuming to the contrary, let $deg(u') \geq 3$. Then there exists a $y \in V(G)$ such that $u'y \in E(G)$. For such $y, u' \in V(G)$, we have

$$f_M(u') \quad = \quad \{D(u', u), D(u', v)\} = \{1, D(u', v') + D(v', v)\} = \{1, 1 + D(u', v')\}$$

and

$$\begin{aligned} f_M(y) &= \{D(y, u), D(y, v)\} = \{D(y, u') + D(u', u), D(y, v') + D(v', v)\} \\ &= \{1 + D(y, u'), 1 + D(y, v')\}. \end{aligned}$$

Since $D(u', y), D(v', y) \geq 1$, $f_M(u') \neq f_M(y)$. Again a contradiction to the fact that M is a uniform set of G, which implies that $deg(u') < 3$. As $u' \neq u, v$ in G we have $1 < deg(u') < 3$. With similar arguments, $deg_G(v') < 3$. Therefore $deg(u') = 2 = deg(v')$.

Finally, to show that $G \cong P_4$ it is enough to show that $u'v' \in E(G)$. Suppose not, then there exists at least one $z \neq v' \in V(G)$ such that $u'z \in E(G)$. For such $z \in V(G)$, we have

$$\begin{aligned} f_M(z) &= \{D(z,u), D(z,y)\} = \{D(z,u') + D(u',u), D(z,v') + D(v',v)\} \\ &= \{1 + D(z,u'), 1 + D(z,v')\}. \end{aligned}$$

Also,

$$f_M(u') = \{1, 1 + D(u',v')\}.$$

Since $D(u',z), D(v',z) \geq 1$, $f_M(u') \neq f_M(z)$. Yet another contradiction to the fact that M is a uniform set of G, consequently $u'v' \in E(G)$. Hence, $G \cong P_4$. $\square$

Remark 2.4. The center of P_4 is also a uniform set.

Uniform sets of the following graphs are given below.

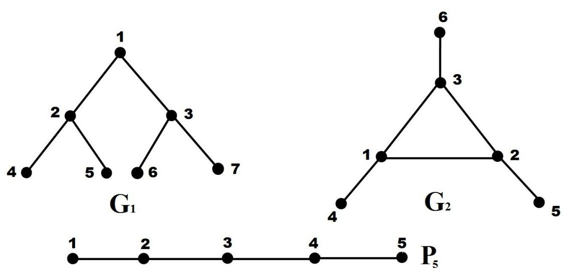

FIGURE 1

For the graph G_1 in Fig. 1, $M_1 = \{1,2,3\}$ and $M_2 = \{1,4,5,6,7\}$ are two uniform sets of different cardinalities. Next, for the graph G_2 in Fig. 1, $M_3 = \{1,2,3\}$ and $M_4 = \{4,5,6\}$ are two uniform set of same cardinality. Finally, in the Fig. 1, $M_5 = \{2,3,4\}$ is the only uniform set of the path P_5. One can easily verify that for the uniform sets, exhibited for the graphs G_1, G_2 and P_5 in Fig. 1, none of their proper subsets are uniform sets of the respective graphs.

This observation prompts us to define minimal uniform and minimum uniform sets of G. A uniform set M of G is minimal if it contains no proper uniform set. And, a uniform set M of G is minimum if it is of the least cardinality. The cardinality of the minimum uniform set of G will be referred as the uniform number of G, denoted as $\varsigma(G)$.
As an immediate consequence of the definition of $\varsigma(G)$, we have the following bounds for $\varsigma(G)$.

Proposition 2.5. *For any connected graph G of order n, $1 \leq \varsigma(G) \leq n-2$.*

The following result is analogous to classical theorem on domination in locally finite graphs due to Ore [15].

Proposition 2.6. *For any graph G, every uniform set contains a minimal one.*

The following definition due to Lee, Schmeichel and Shee [14] will be useful for the remainder of this paper. For any two graphs G and H, let u and v be fixed vertices of G and H, respectively. Then the vertex amalgamation of G and H is the graph obtained from G and H by identifying G and H at the vertices u and v. The next result is analogous to the results due to Buckley and Chartrand respectively.

Theorem 2.7. *For every graph G, $V(G)$ is a uniform set of some graph.*

Proof. Let $V = \{u_i : 1 \leq i \leq n\}$ be the vertex set of G and, u_1 and u_n be any pair of antipodal vertices of G. For $m \geq 2$, let H be a graph obtained by amalgamating the center of a star $K_{1,m}$ either with u_1 or u_n in G. Without loss of generality, let the center of the star $K_{1,m}$ be amalgamated with u_1 in G. For the graph H, let $M = V(G)$. We claim that M is a uniform set of H. Let $S = \{w_j : 1 \leq j \leq m\}$ be the set of pendant vertices in H. For all $w_j \in S$

$$D(w_j, u_i) \quad = \quad \begin{cases} 1, & i = 1 \\ D(w_j, u_1) + D(u_1, u_i), & i \neq 1, \end{cases}$$

Since $f_M(w_j) = f_M(w_{j'})$, $V(G)$ is a uniform set of H. □

In view of Theorem 2.7, it is quite natural to seek for an analogous result for the case of a disconnected graph. We shall now show that Theorem 2.7 can be extended to the case of disconnected graph in the following manner as follows: Let $S = \{w_j : 1 \leq j \leq m\}$ be the set of isolated vertices, and G_i be connected graphs of order n_i, where $1 \leq i \leq k$. Let $u_i \in V(G_i)$ be one of the antipodal vertices of graph G_i and G be the graph obtained by making $u_i \in V(G_i)$ and w_j adjacent for all i and j. Then by using arguments similar to that of Theorem 2.7 the following result can be established.

Theorem 2.8. *If G is a disconnected graph with k components, then $V(G)$ is a uniform set of some graph.*

Remark 2.9. For the embedding scheme proposed in Theorem 2.7, if $G \cong K_2$ or P_3, then G is neither a minimum nor a minimal uniform set of H.

Even before we attempt to recognize the graphs which are minimum or minimal uniform sets of some graphs, the existence of such a class of graphs has to be shown. Our first result in the next section demonstrates a class of graphs which are minimum uniform sets of some graphs.

3. New Results on Uniform Numbers

In this section we investigate the newly introduced graph parameter, uniform number, $\varsigma(G)$. To begin with, we prove the following result.

Theorem 3.1. *For every positive integer $t > 3$, there exists a graph H such that $\varsigma(H) = t$.*

Proof. For $m \geq t \geq 3$, let H be a graph obtained by amalgamating the center of a star $K_{1,m}$ with any vertex of a complete graph K_t. Let $V(K_t) = \{u_i : 1 \leq i \leq t\}$ and, the center of a star $K_{1,m}$ be amalgamated with u_1 in K_t. We claim that $M = V(K_t)$ is a uniform set of H. Let $S = \{w_j : 1 \leq j \leq m\}$ denote the set of pendant vertices in H. Since for all $w_j \in S$ and $u_i \in M$

$$D(w_j, u_i) = \begin{cases} 1, & i = 1 \\ t, & i \neq 1, \end{cases}$$

$f_M(w_j) = \{1, t\}$ for all $w_j \in S$. Hence, M is a uniform set of H. We shall now show that M is minimum too.

Suppose not, then there exists a uniform set M' of H such that $|M'| < |M| = t$. And, we have the following exhaustive cases: either $M' \subseteq M^c$ or $M' \subseteq M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Let $M' \subseteq M^c$. Since $|M'| < t \leq m$, there exists at least one element in M^c, say w_1, such that $w_1 \notin M'$, hence

$$f_{M'}(w_1) = \{D(w_1, w_j) : 2 \leq j \leq m\} = \{2\}.$$

Also, for $u_1 \in M$,

$$f_{M'}(u_1) = \{D(u_1, w_j) : 1 \leq j \leq m\} = \{1\}.$$

Consequently, $f_{M'}(u_1) \neq f_{M'}(w_1)$. A contradiction to our assumption that M' is a uniform set of H, hence $M' \nsubseteq M^c$.
So, let $M' \subseteq M$. Since $|M'| < t$, there exists at least one $u_k \in M$ such that $u_k \notin M'$, hence

$$f_{M'}(u_k) = \{D(u_k, u_i) : 1 \leq i \neq k \leq t\} = \{t - 1\}.$$

Also, for all $w_j \in M^c$,

$$f_{M'}(w_j) = \{D(w_j, u_i)\} = \{1, t\}.$$

Since $f_{M'}(u_k) \neq f_{M'}(w_j)$, a contradiction to our assumption that M' is a uniform set of H. Hence, $M' \nsubseteq M$.
Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. In this case M' contains some but not all elements from M and some but not all elements from M^c. Since $|M'| < t \leq m$, there exists at least one vertex $u_{i'} \in M$ such that $u_{i'} \notin M'$ and at least one vertex $w_{j'} \in M^c$ such that $w_{j'} \notin M'$. Thus we have

$$\begin{aligned} f_{M'}(u_{i'}) &= \begin{cases} \{t-1, t\}, & i' \neq 1 \\ \{t-1, 1\}, & i' = 1 \end{cases} \quad \text{and} \\ f_{M'}(w_{j'}) &= \{t, 2\}. \end{aligned}$$

Hence, $f_{M'}(u_{i'}) \neq f_{M'}(w_{j'})$. Yet another contradiction to our assumption that M' is a uniform set of H. Hence, $\varsigma(H) = t \geq 3$. $\square$

The embedding scheme proposed in Theorem 3.1 asserts the existence of class of graph which are minimum uniform set of some graph. With the preceding Theorem 3.1 one may ponder “Does there exists a graph G for which $\varsigma(G) = 1$ or 2?” We settle this question in affirmative and also provide a new characterization of Hamiltonian connected graphs.

In 1963, Ore [16] defined a graph G to be Hamilton-connected if it has a spanning path for all pairs of vertices x and y in G. Examples of Hamilton-connected graphs include $K_n \times P_m$, antiprism graphs, complete graphs K_n, Möbius ladders, prism graphs of odd order, wheel graphs, the truncated prism graph, truncated cubical graph, truncated tetrahedral graph, Grötzsch graph, Frucht graph, and Hoffman-Singleton graph.

Theorem 3.2. *A graph G of order $n \geq 3$ is Hamiltonian connected if and only if every vertex of G is a uniform set of G.*

Proof. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ be the vertex set of a Hamiltonian connected graph G. Since for any pair of vertices $v_i, v_j \in V(G)$ there exists a spanning path between them,

$$D(v_i, v_j) \quad = \quad n - 1.$$

The set $M = \{v_k\}$ is a uniform set of G, since for all v_i different from v_k in G

$$f_M(v_i) \quad = \quad \{n - 1\}.$$

The choice of $v_k \in V(G)$ was arbitrary, hence every vertex of G is a uniform set of G and $\varsigma(G) = 1$.

To see that the converse is also true, let G be a graph in which every vertex is a uniform set. In view of Proposition 2.1, $deg(u) \geq 2$ for all $u \in V(G)$. We need to show that G is Hamiltonian connected.

Suppose not, then there exist at least one pair of vertices, say, $x, y \in V(G)$ such that there is no spanning path $P(x, y)$ in G and therefore $D(x, y) = r < n - 1$. Let the vertices of $P(x, y)$ be given $x = v_1, v_2, v_3, \ldots, v_{r+1} = y$. Hence, $V(G)$ is partitioned into two classes *viz*, V_1 and V_2 such that V_1 contains all the vertices of the detour path $P(x, y)$ and V_2 containing all the vertices of G which are avoided by $P(x, y)$ in G. Since the length of $P(x, y)$ is r, $|V_2| = n - 1 - r$. Let $V_2 = \{w_1, w_2, w_3, \ldots, w_{n-1-r}\}$. Since G is connected and $deg(u) \geq 2$ for all $u \in V(G)$, there exists at least one vertex in each partition, say v_j, w_i, such that $v_j w_i \in E(G)$. Since $D(x, y) = r$ and every vertex in G is a uniform set, in particular $\{x\}$ is also a uniform set of G and therefore $D(x, z) = r$ for all $z \in V(G)$. Let us consider the detour path $P(x, v_j)$ for which we have the following possibilities:

Case 1: If $P(x, v_j)$ avoids all the vertices of V_1, then such a path must be of the form $x, w_t, w_{t+1}, \ldots, w_{t-2-r}, v_j$, where $w_t, w_{t+1}, \ldots, w_{t-2-r} \in V_2$ such that $P(x, v_j)$ of length r. Hence, $D(x, y) = D(x, v_j) + D(v_j, y) \geq r + 1$ a contradiction to the fact that $D(x,y)=r$. Thus, there exists no detour $P(x,v_j)$ avoiding all the vertices of V_1.

Case 2: The detour path $P(x,v_j)$ avoids all the vertices of V_2. If $P(x,v_j)$ avoids all the vertices of V_2, then there exists a relabeling of the vertices of V_1 such that $x=u_1,u_2,u_3,\ldots,u_{r+1}=v_j$. Also, $v_jw_i\in E(G)$ and therefore $D(x,w_i)=D(x,v_j)+D(v_j,w_i)\geq r+1$, again a contradiction to the fact that $D(x,w_i)=r$. Hence there exists no detour $P(x,v_j)$ avoiding all the vertices of V_2.

Case 3: The detour path between x and v_j avoids some, but not all the vertices of V_1 and V_2. Let $P_1(x,v_j)$ be such a path of length r in G. In such a case either $D(x,v_{j-1})\geq r+1$ or $D(x,v_{j+1})\geq r+1$ as $v_{j-1}v_j, v_jv_{j+1}\in E(G)$. In either of the case, we have a contradiction to the fact that $D(x,z)=r$ for all $z\in V(G)$.

Which implies that there exists no such partition V_1 and V_2 of $V(G)$. Hence, $r=n-1$, and the proof is seen to be complete. □

Hamilton connected graphs provides us with an instance where every singleton subset of the vertex set is minimum uniform set. We propose the following problem.

Problem 3.3. *Characterize the graphs for which every k-subset of the vertex set is a minimum uniform set.*

Theorem 3.2 settles the Problem 3.3 for $k=1$. Hamiltonian connectedness of graph G is a necessary condition for $\varsigma(G)=1$, and the converse is not true, as can be seen from the following results.

Theorem 3.4. *If T is a tree of order $n\geq 3$, then $\varsigma(T)=1$ if and only if $T\cong K_{1,n-1}$.*

Proof. Let u be the center and $S=\{v_i: 1\leq i\leq n-1\}$ be the set of pendent vertices of $K_{1,n-1}$. Since $D(v_i,u)=1$ for all $v_i\in S$, the set $M=\{u\}$ is a uniform set of $K_{1,n-1}$. Consequently, $\varsigma(K_{1,n-1})=1$ for $n\geq 3$.

Conversely, let T be a tree of order n such that $\varsigma(T)=1$. Let $V(T)=\{v_i: 1\leq i\leq n\}$ be the vertex set of T. Since $\varsigma(T)=1$, any uniform set must be of the form $M=\{v_k\}$ for some $v_k\in V(T)$. In view of Proposition 2.1, $deg(v_k)\neq 1$. To show that T is a star it is enough to show that every vertex in T different from v_k is of degree one. Suppose not, then there exist at least one pair of vertices $v_j,v_{j+1}\in V(T)$ such that $deg(v_j)\geq 2$ and $v_jv_{j+1}\in E(T)$. In view of the fact that any two vertices of a tree is connected by a unique path, $f_M(v_j)\neq f_M(v_{j+1})$. A contradiction to the fact that M' is a uniform set of T. Hence, there exists no such vertex $v_j\in V(T)$ such that $deg(v_j)\geq 2$ with $v_{j+1}\in V(T)$ as the successor of v_j in T. Hence, $T\cong K_{1,n-1}$. □

Theorem 3.5. *If $G=P_n+K_1$ where $n\geq 3$, then $\varsigma(G)=1$.*

Proof. Let $\{u_1, u_2, u_3, \ldots, u_n\}$ be the vertices of P_n taken in a sequential order and v be the vertex of K_1. Since $D(v, u_1) = n$ and $D(u_j, u_1) = n$ for all $2 \leq j \leq n$, the set $M = \{u_1\}$ is seen to be a uniform set of G as $f_M(v) = \{n\} = f_M(u_j)$. Hence, $\varsigma(G) = 1$. □

Remark 3.6. In view of the geometric symmetry of $P_n + K_1$ about the vertex v, the set $\{u_n\}$ is also a minimum uniform set of $P_n + K_1$.

The friendship graphs [8] F_n are graphs obtained by vertex amalgamation of n-copies of K_3. Figure 2 illustrates F_4. K_3 being a complete graph is Hamilton connected. We extend the notion of friendship graph to generalized friendship graphs by replacing K_3 by a Hamilton connected graph of order m. Let H be a Hamilton connected graph of order m and u be a fixed vertex of H. Generalized friendship graph denoted as F_n^* is a graph obtained by amalgamating the vertex u in n-copies of H.

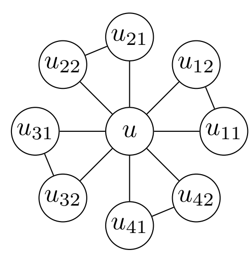

Figure 2. Friendship graph, F_4

Theorem 3.7. *If F_n^* is a generalized friendship graph of order $n \geq 2$, then* $\varsigma(F_n^*) = 1$.

Proof. Let $u \in V(H)$ be a fixed vertex of a Hamilton connected graph H of order m and F_n^* be a graph obtained by amalgamating the vertex u in n-copies of H. Let $V(F_n^*) = V_1 \bigcup V_2$, where $V_1 = \{u\}$ and $V_2 = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$. Since $D(u, u_{ij}) = m - 1$ for all $1 \leq i \leq n$ and $1 \leq j \leq m-1$, the set $M = \{u\}$ is a uniform set of F_n^*. Consequently, $\varsigma(F_n^*) = 1$. □

Corollary 3.8. *If F_n is a friendship graph of order $n \geq 2$, then $\varsigma(F_n) = 1$.*

Having discussed the graphs for which $\varsigma(G) = 1$, we now exhibit some classes of graphs for which $\varsigma(G) = 2$.

A bipartite graph G with partitions V_1 and V_2 is Hamilton-laceable [19], if for any $u \in V_1$ and $v \in V_2$, there is a Hamilton path whose terminal vertices are u and v. The notion of Hamilton-laceable graphs was extended as strongly Hamiltonian-laceable graphs [12]. A bipartite graph G of order n with its partite sets of equal size is said to be strongly Hamiltonian-laceable if there is a Hamiltonian path between every two vertices that belong to different partite sets and there is a path of (maximal) length $n-2$ between every two vertices that belong to the same partite set.

Theorem 3.9. *If G is a strongly Hamiltonian-laceable graph of order $n \geq 4$, then $\varsigma(G) = 2$.*

Proof. If G is a strongly Hamiltonian-laceable graph, then G is a bipartite graph with partitions of equal size. Let $V_1 = \{u_i : 1 \leq i \leq m\}$ and $V_2 = \{v_i : 1 \leq i \leq m\}$ be the partitions of G. Without loss of generality, let $M = \{v_1, u_1\}$. We claim that M is a uniform set of G. Since there is a Hamiltonian path between every two vertices that belongs to different partite sets and there is a path of length $2m-2$ between every two vertices that belongs to the same partite set in G, for all $u_i, v_j \in V(G)$ such that $u_i, v_j \notin M$

$$f_M(u_i) \quad = \quad f_M(v_j) = \{2m-1, 2m-2\}.$$

Next to see that M is a minimum uniform set, assume to the contrary. Then there exists a uniform set M' of G such that $|M'| < 2$. Let $M' = \{x\}$, where $x \in V_1$ or $x \in V_2$. Without loss of generality, let $x \in V_1$, then for all $u_i \in V_1$ different from x

$$f_{M'}(u_i) \quad = \quad \{D(u_i, x)\} = \{2m-2\}.$$

Also, for all $v_j \in V_2$

$$f_{M'}(v_j) \quad = \quad \{D(v_j, x)\} = \{2m-1\}.$$

A contradiction to our assumption that M' is a uniform set of G. Hence, $\varsigma(G) = 2$. □

In view of the preceding theorem and the fact that, n-dimensional hypercube is strongly Hamiltonian-laceable [13] the following is immediate consequence.

Corollary 3.10. *If Q_n is a n-dimensional hypercube, then $\varsigma(Q_n) = 2$.*

The converse of the statement in the Theorem 3.9 is not true as can be seen from our next result which characterizes trees for which $\varsigma(G) = 2$.

For arbitrary integers $r \geq 2$ and $s \geq 2$, a bistar [8] $B_{r,s}$ is a tree of diameter three and is obtained by taking two stars $K_{1,r-1}$ and $K_{1,s-1}$ on disjoint vertex sets and then by making their centers u and v adjacent to each other by introducing a new edge uv.

Theorem 3.11. *If T is a tree of order $n \geq 4$, then $\varsigma(T) = 2$ if and only if $T \cong B_{r,s}$.*

Proof. Let $\{u, v\}$ be the center of the bistar $B_{r,s}$. Let $S_1 = \{u_i : 1 \leq i \leq r\}$ and $S_2 = \{v_j : 1 \leq j \leq s\}$ be the set of pendant vertices adjacent to u and v respectively. Since $D(u_i, u) = 1 = D(v_j, v)$ and $D(u_i, v) = 2 = D(v_j, u)$, the set $M = \{u, v\}$ is a uniform set of $B_{r,s}$. We claim that M is a minimum uniform set.

Suppose not, then there exists a uniform set M' of $B_{r,s}$ such that $|M'| < |M| = 2$. Let $M' = \{x\}$, in view of proposition 2.1, $deg(x) \neq 1$. Since u and v are the only vertices in $B_{r,s}$ for which the degree is not equal to 1, without loss of generality let $x = u$. Then $f_{M'}(v) \neq f_{M'}(v_j)$.

Again a contradiction to the assumption that M' is a uniform set of $B_{r,s}$. Hence $M = \{u, v\}$ is a minimal uniform set of $B_{r,s}$.

Conversely, let T be a tree of order n with $\varsigma(T) = 2$. We need to show $T \cong B_{r,s}$. To show that $T \cong B_{r,s}$ it is enough to show that $diam(T) = 3$. Suppose not, then either $diam(T) < 3$ or $diam(T) > 3$. In view of Theorem 3.4, $diam(T) \nless 3$. Let $diam(T) > 3$. If T has exactly two pendant vertices, then $r = 1$ and $s = 1$ and $P_4 \cong B_{1,1}$, hence by Proposition 2.3 the result is seen to be true. So, let T contains at least three pendant vertices. Since $diam(T) > 3$ and T contains at least three pendant vertices, $n \geq 5$. Let $M = \{x, y\}$ be a minimum uniform set of T.

Case 1: If $deg(x) = 1$ or $deg(y) = 1$. Without loss of generality, let $deg(x) = 1$, then $deg(y) \geq 2$ by Proposition 2.1.

Subcase 1.a: When $xy \in E(T)$. Since $diam(T) > 3$ and $n \geq 5$, there exist at least one pair of vertices $w, z \in V(T)$ such that $yw, wz \in E(T)$. In such a case,

$$\begin{aligned} f_M(w) &= \{D(w,x), D(w,y)\}, \text{ and} \\ f_M(z) &= \{D(z,x), D(z,y)\} = \{D(z,w) + D(w,x), D(z,y) + D(w,y)\} \\ &= \{D(w,x) + 1, D(w,y) + 1\}. \end{aligned}$$

But $D(w,x), D(w,y) \geq 1$, hence $f_M(w) \neq f_M(z)$. A contradiction to the assumption that M is a uniform set of T. Hence, $xy \notin E(T)$.

Subcase 1.b: When $xy \notin E(T)$. Since $diam(T) > 3$ and $n \geq 5$, there exits at least one vertex $w \in V(T)$ such that $xw, yw \in E(T)$. For $w \in V(T)$,

$$f_M(w) = \{D(w,x), D(w,y)\} = \{1\}.$$

Since $deg(y) \geq 2$, there exists a $z \in V(T)$ such that $yz \in E(T)$ and

$$f_M(z) = \{D(z,x), D(z,y)\} = \{D(z,w) + D(w,x), D(z,y)\} = \{D(z,w) + 1, 1\}.$$

But $D(z,w) \geq 1$, hence $f_M(w) \neq f_M(z)$. Again a contradiction to the assumption that M is a uniform set of T. Hence, $deg(x) \neq 1$. Similarly, $deg(y) \neq 1$.

Case 2: When $deg(x) \neq 1$ and $deg(y) \neq 1$.

Let $a_1, a_2, \ldots, a_i, \ldots, x, \ldots, w, \ldots y, \ldots, a_j, \ldots a_{r-1}, a_r$ be any path in T containing x and y as its internal vertices such that a_i and a_j are not the pendant vertices. In such a case,

$$\begin{aligned} f_M(a_i) &= \{D(a_i,x), D(a_i,y)\}, \text{ and} \\ f_M(a_1) &= \{D(a_1,x), D(a_1,y)\} = \{D(a_1,a_i) + D(a_i,x), D(a_1,a_i) + D(a_i,y)\}. \end{aligned}$$

Since $a_1 \neq a_i$, $D(a_1, a_i) \geq 1$. Hence $f_M(a_i) \neq f_M(a_1)$, a contradiction to the assumption that M is a uniform set of T. Hence there exists no $a_i \in V(T)$ lying between a_1 and x in the $a_1 - a_r$ path in T. With a similar argument it can be proved that there exists no $a_j \in V(T)$ lying between y and a_{r-1} in the $a_1 - a_r$ path in T. Hence, $a_1 - a_r$ path is of the form $a_1, x, \ldots, w, \ldots y, a_r$. In such a case,

$$\begin{aligned} f_M(a_1) &= \{D(a_1,x), D(a_1,y)\} = \{D(a_1,x), D(a_1,x) + D(x,w) + D(w,y)\} \\ &= \{1, 1 + D(x,w) + D(w,y)\}, \text{ and} \\ f_M(w) &= \{D(x,w), D(w,y)\}. \end{aligned}$$

Since $D(x,w), D(w,y) \geq 1$, $f_M(a_1) \neq f_M(w)$. Again a contradiction to the assumption that M is a uniform set of T. Hence there exists no $w \in V(T)$ lying between x and y in the $a_1 - a_r$ path in T. Thus, $xy \in E(T)$ and $a_1 - a_r$ path is of the form a_1, x, y, a_r. Hence, $diam(T) = 3$. □

We now exhibit a few non-isomorphic classes of cyclic graphs for which $\varsigma(G) = 2$.

Theorem 3.12. *If W_n^* is a graph obtained by attaching a pendant edge to the central vertex of a wheel W_n, then $\varsigma(W_n^*) = 2$.*

Proof. Let u be the central vertex, uv be the pendant edge and $R = \{w_i : 1 \leq i \leq n\}$ be the vertices on the rim of W_n^*. Since $D(w_i, u) = n$ and $D(w_i, v) = n+1$ for all $w_i \in R$, the set $M = \{u, v\}$ is a uniform set of W_n^*. We claim that M is minimum too. For if possible let M' be a uniform set of W_n^* such that $|M'| = 1$. The vertex v being a pendant vertex in W_n^*, in view of proposition 2.1, $v \notin M'$. So, either $M' = \{u\}$ or $M' = \{w_j\}$ for some j. Let $M' = \{u\}$. Since $D(w_i, u) = n$ for all $w_i \in R$, and $D(v, u) = 1$, $f_{M'}(w_i) \neq f_{M'}(v)$. A contradiction to the assumption that M' is a uniform set of W_n^*, hence $M' \neq \{u\}$. Finally, let $M' = \{w_j\}$ for some j. Since $D(u, w_i) = n$ and $D(v, w_i) = 1$ for all $w_i \in R$, $f_{M'}(u) \neq f_{M'}(v)$. Again a contradiction to the assumption that M' is a uniform set of W_n^*, hence $M' \neq \{w_j\}$. Hence, M is a minimum uniform set of W_n^*. □

The n-Barbell graph [10] is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge. K_n being a hamiltonian connected graph, a n-Barbell graph contains exactly two Hamiltonian connected graph, of order n as its induced subgraphs. We extend the notion of the n-Barbell graph by replacing K_n by a Hamilton connected graph of order n.

Theorem 3.13. *If G is a n-Barbell graph obtained by connecting two copies of a Hamiltonian connected graph of order n by a bridge, then $\varsigma(G) = 2$.*

Proof. Let H be a Hamiltonian connected graph of order n and G be the graph obtained as in the statement of the theorem.

Let H_1 and H_2 be the two copies of H in G such that $V(H_1) = \{u_i : 1 \leq i \leq n\}$ and $V(H_2) = \{v_i : 1 \leq i \leq n\}$, and $u_n v_n \in E(G)$ be the bridge in G. For any two vertices $x, y \in V(G)$ we have

$$\begin{array}{rcl} D(v_i, v_j) & = & n-1, \\ D(u_i, u_j) & = & n-1, \\ D(v_n, u_i) & = & n, \ i \neq n, \\ D(u_n, v_i) & = & n, \ i \neq n, \\ D(u_i, v_j) & = & \left\{ \begin{array}{ll} 2n-1, & 1 \leq i, j \leq n-1; \\ 1, & i = j = n. \end{array} \right. \end{array}$$

The set $M = \{u_n, v_n\}$ is a uniform set of G as

$$\begin{array}{rcl} f_M(u_i) & = & \{D(u_i, u_n) : 1 \leq i \leq n-1\} \bigcup \{D(u_i, v_n) : 1 \leq i \leq n\} \\ & = & \{n-1\} \bigcup \{2n-1\} = \{n-1, 2n-1\}, \text{and} \end{array}$$

$$\begin{array}{rcl} f_M(v_i) & = & \{D(v_i, u_n) : 1 \leq i \leq n\} \bigcup \{D(v_i, v_n) : 1 \leq i \leq n-1\} \\ & = & \{2n-1\} \bigcup \{n-1\} = \{n-1, 2n-1\}. \end{array}$$

And, therefore $f_M(u_i) = f_M(v_i)$ for all $1 \leq i \leq n-1$. We now claim that M is minimum too. Suppose not, then there exists a uniform set M' of G such that $|M'| = 1$. In such a case either $M' = \{u_l\}$ for some $u_l \in V(H_1)$ or $M' = \{v_m\}$ for some $v_m \in V(H_2)$ where $1 \leq l, m \leq n$. In view of the geometric symmetry of G about the edge $u_n v_n$, it is enough to consider the case $M' = \{u_l\}$ for some $u_l \in V(H_1)$.

Case 1: If $l \neq n$, then

$$\begin{array}{rcl} f_{M'}(u_i) & = & \{D(u_i, u_l) : i \neq l\} = \{n-1\}, \text{and} \\ f_{M'}(v_n) & = & \{D(v_n, u_l)\} = \{n\}. \end{array}$$

A contradiction to the assumption that M' is a uniform set of G.

Case 2: If $l = n$, then $M' = \{u_n\}$

$$\begin{array}{rcl} f_{M'}(u_i) & = & \{D(u_i, u_n)\} = \{n-1\}, \text{and} \\ f_{M'}(v_n) & = & \{D(v_n, u_n)\} = \{1\}. \end{array}$$

Again a contradiction to the assumption that M' is a uniform set of G.

In either of the cases, we arrive at a contradiction to the assumption that M' is a uniform set of G, hence the proof. □

The next result determines the uniform number of complete bipartite graph $K_{m,n}$.

Theorem 3.14. *If $K_{m,n}$ is a complete bipartite graph with $m, n \geq 2$, then* $\varsigma(K_{m,n}) = \left\{ \begin{array}{ll} 2, & \text{if } m = n \\ min\{m, n\}, & \text{if } m \neq n \end{array} \right.$.

Proof. Let the two color classes of $K_{m,n}$ be given by $V_1 = \{v_i : 1 \leq i \leq m\}$ and $V_2 = \{u_j : 1 \leq j \leq n\}$. When $m = n$, in view of Theorem 3.9 we have $\varsigma(K_{m,n}) = 2$. So, let $m \neq n$. Since the circumference of $K_{m,n}$ is equal to $2\min\{m,n\}$, for any adjacent pair of vertices u, v in $K_{m,n}$ we have $D(u,v) = 2\min\{m,n\} - 1$. If $m = \min\{m,n\}$, then V_1 is seen to be a uniform set of $K_{m,n}$. To see that $M = V_1$ is the minimum uniform set, assume to the contrary. Then there exists a uniform set M' of $K_{m,n}$, such that $|M'| < |M|$. For the sets $M', M \subseteq V(K_{m,n})$ we have the following exhaustive possibilities: either $M' \subseteq M^c$ or $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.
Let $M' \subseteq M^c$. Since $m = \min\{m,n\}$ and $|M'| < |M|$, $M' \subset M^c = V_2$ and, therefore there exists at least one u_j, say u_n such that $u_n \notin M'$. Hence, $M' \subseteq M - \{u_n\}$, and $f_{M'}(u_n) = \{2m-2\}$. Also, for all $v_i \in V_1$, $f_{M'}(v_i) = \{2m-1\}$. Clearly, $f_{M'}(v_i) \neq f_{M'}(u_n)$. A contradiction to the assumption that M' is a uniform set of $K_{m,n}$ and therefore $M' \nsubseteq M^c$.
So, let $M' \subset M$. There exists at least one vertex, say $v_1 \in V_1$, such that $v_1 \notin M'$. Hence, $M' \subseteq M - \{v_1\}$ and $f_{M'}(v_1) = \{2m-2\}$. Also, for all $u_j \in V_2$, $f_{M'}(u_j) = \{2m-1\}$. Clearly, $f_{M'}(v_1) \neq f_{M'}(u_j)$. A contradiction to the assumption that M' is a uniform set of $K_{m,n}$, hence our assumption $M' \subset M$ is not true.
Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. In this case M' contains some but not all elements from M and some but not all elements from M^c. Therefore, there exists a $v_{i'} \in M$ such that $v_{i'} \notin M'$ and there exists at least one $u_{j'} \in M^c$ such that $u_{j'} \in M'$. For such an M', we have

$$\begin{aligned} f_{M'}(v_{i'}) &= \{2m-2, 2m-1\}, \text{and} \\ f_{M'}(u_{j'}) &= \{2m-1, 2m\}. \end{aligned}$$

Clearly, $f_{M'}(v_1) \neq f_{M'}(u_j)$. Yet another contradiction to the assumption that M' is a uniform set of $K_{m,n}$. Hence, the proof is seen to be complete. □

In view of the Theorem 3.2 and 3.4, the lower bound prescribed for $\varsigma(G)$ in Proposition 2.5 is seen to be attained by infinite family of non-isomorphic graphs.

Our next result establishes the sharpness of the upper bound prescribed for $\varsigma(G)$.

Theorem 3.15. *If P_n is a path of order $n \geq 3$, then $\varsigma(P_n) = n - 2$.*

Proof. Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of P_n taken in a sequential order. For $n = 3$ it is easy to see that the set $M = \{u_3\}$ and for $n = 4$ the sets $M = \{u_2, u_3\}$ or $\{u_1, u_4\}$ are the minimum uniform sets of P_3 and P_4 respectively. Let $n \geq 4$. Since for any vertex $u_i, u_j \in V(P_n)$

$$D(u_i, u_j) = \begin{cases} i - j, & i > j, \\ j - i, & i < j, \end{cases}$$

the set $M = \{u_i : 2 \le i \le n-1\}$ is seen to be a uniform set of P_n as

$$f_M(u_1) \quad = \quad f_M(u_n) = \{i : 1 \le i \le n-2\}.$$

We now claim that M is minimum too. Suppose not, then there exists a uniform set M' of P_n such that $|M'| < |M|$. We have the following possibilities: either $M' \bigcap M = \emptyset$ or $M' \bigcap M \neq \emptyset$.
If $M' \bigcap M = \emptyset$, then $M' \subseteq M^c = \{u_1, u_n\}$. The vertices u_1 and u_2 being pendant vertices, in view of proposition 2.1 $M' \subsetneq M$. Hence, $M' = \{u_1, u_n\}$. Since any two vertices of a tree is connected by a unique path, for any pair of adjacent vertices $u_j, u_{j+1} \in V(P_n)$ we have $f_{M'}(u_j) \neq f_{M'}(u_{j+1})$. A contradiction to our assumption that M' is a uniform set of P_n, hence $M' \bigcap M \neq \emptyset$.
If $M' \bigcap M \neq \emptyset$, then either $M' \bigcap M^c = \emptyset$ or $M' \bigcap M^c \neq \emptyset$.

Case 1: Let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c = \emptyset$. Since $|M'| < |M|$ and path P_n is symmetrical about it's center, for $r \le \lceil \frac{n}{2} \rceil$ there exists $u_r \in M$ such that $u_r \notin M'$. Hence, $M' = \{u_2, u_3, \ldots, u_{r-1}, u_{r+1}, \ldots, u_{n-1}\}$. Note that for $u_1, u_r \notin M'$, $\max\{f_{M'}(u_1)\} = n-2$ and $\max\{f_{M'}(u_r)\} = n-r-1$. Also, $r \ge 2$ which implies that $n-r-1 < n-2$, hence $n-2 \notin f_{M'}(u_r)$ and $f_{M'}(u_1) \neq f_{M'}(u_r)$. A contradiction to our assumption that M' is a uniform set of P_n, hence $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c = \emptyset$ is not true.

Case 2: Let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. Since $|M'| \le n-3$, there exist at least two vertices, say, $u_r, u_s \in M$ such that $u_r, u_s \notin M'$ and either $u_1 \in M'$ or $u_n \in M'$. Without loss of generality, let $u_1 \in M'$, and $M' = \{u_1, u_2, u_3, \ldots, u_{r-2}, u_{r-3}, \ldots, u_{s-2}, u_{s-3}, \ldots, u_{n-1}\}$. For u_1, we have

$$\begin{aligned} f_{M'}(u_1) &= \{D(u_1, u_j) : 2 \le i \neq r, s \le n\} \\ &= \{1, 2, 3, \ldots, r-2, r-3, \ldots, s-2, s-3, \ldots, n-1\}, \end{aligned}$$

and using arguments similar to that of the preceding case, it can be seen that for r or $s \neq 1$ or n, $n-1 \notin f_{M'}(u_r)$. Hence, $f_{M'}(u_1) \neq f_{M'}(u_r)$, again a contradiction to our assumption that M' is a uniform set of P_n, hence $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$ is also not true. The proof is seen to be complete. $\square$

So far we have seen the graphs for which $\varsigma(G)$ is either close to the lower bound or close to the upper bound prescribed on $\varsigma(G)$. We now exhibit a class of graphs for which $\varsigma(G)$ lies almost in the middle of the prescribed bounds.

The next class of the graphs are obtained from a wheel graph by the subdivision of its edges. A gear graph [8], G_n, is obtained by subdividing the edges on the rim of a wheel graph W_n.

Theorem 3.16. *If G_n is a gear graph where $n \ge 3$, then $\varsigma(G_n) = n$.*

Proof. Let u be the central vertex, $\{v_i : 1 \le i \le n\}$ be the vertices on the rim and $\{w_i : 1 \le i \le n\}$ be the vertices obtained by the subdivision of edges $v_i v_{i+1}$ on the rim of W_n so that the resultant graph is a gear graph G_n. Let D

be the detour distance matrix of G_n, and for $1 \leq i, j \leq n$

$$\mathrm{D} = \begin{array}{c} \\ u \\ v_i \\ v_j \\ w_i \\ w_j \end{array} \begin{array}{c} \begin{array}{ccccc} u & v_i & v_j & w_i & w_j \end{array} \\ \begin{bmatrix} 0 & 2n-1 & 2n-1 & 2n & 2n \\ 2n-1 & 0 & 2n-2 & 2n-1 & 2n-1 \\ 2n-1 & 2n-2 & 0 & 2n-1 & 2n-1 \\ 2n & 2n-1 & 2n-1 & 0 & 2n \\ 2n & 2n-1 & 2n-1 & 2n & 0 \end{bmatrix} \end{array}$$

The set $M = \{v_i : 1 \leq i \leq n\}$ is a uniform set of G_n, since for all $w_j \in V(G_n)$ we have $f_M(w_j) = \{2n-1\} = f_M(u)$.

We claim that M is minimum. Suppose not, then there exists a set M' as a uniform set of G_n such that $|M'| < |M|$. We have the following exhaustive cases: either $M' \subseteq M^c$ or $M' \subseteq M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Case 1: Let $M' \subseteq M^c$. In view of our assumption that $|M'| < |M| = n$ it follows that $M' \subset M^c$. Since $|M'| < n$ and $|M^c| = n + 1$, there exist at least two elements $\alpha, \beta \in M^c$ such that $\alpha, \beta \notin M'$. Either $\alpha = u$ and $\beta = w_k$ for some $k \leq n$ or $\alpha = w_l$ and $\beta = w_m$ for some $l, m \leq n$. In either of the cases, $f_{M'}(w_k) \neq f_{M'}(v_i)$ for all $v_i \in V(G_n)$. A contradiction to the assumption that M' is a uniform set of G_n. Hence, $M' \nsubseteq M^c$.

Case 2: Let $M' \subseteq M$. Since $|M'| < |M|$, $M' \subset M$. Therefore $M' = M - \{v_t\}$ for some $t \leq n$. Since $f_{M'}(v_t) \neq f_{M'}(u)$, contrary to our assumption that M' is a uniform set of G_n. Hence $M' \nsubseteq M$.

Case 3: Finally, if $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$, then $M' = X \cup Y$, where $X \subset M$ and $Y \subset M^c$. There exists $v_k, \gamma \in V(G_n)$ such that $v_k \notin X$ and $\gamma \in Y$. We now have the following possibilities:

If $\gamma = u$ and $M' = \{u\} \bigcup \{v_i : 1 \leq i \neq k \leq n\}$, then

$$f_{M'}(v_k) \quad = \quad \{D(v_k, u)\} \bigcup \{D(v_k, v_i) : 1 \leq i \neq k \leq n)\} = \{2n-1, 2n-2\}.$$

And, for all $w_i \in V(G_n)$

$$f_{M'}(w_i) \quad = \quad \{D(w_i, u)\} \bigcup \{D(w_i, v_i) : 1 \leq i \neq k \leq n\} = \{2n-1, 2n\}.$$

Contradicting the assumption that M' is a uniform set of G_n, hence $\gamma \neq \{u\}$.

So, let $\gamma = w_j$ for some j say $j = 1$ and $M' = \{w_1\} \bigcup \{v_i : 1 \leq i \neq k \leq n\}$. Then

$$\begin{aligned} f_{M'}(u) \quad &= \quad \{D(u, w_1)\} \bigcup \{D(u, v_i) : 1 \leq i \neq k \leq n\} = \{2n-1, 2n\}, \text{and} \\ f_{M'}(v_k) \quad &= \quad \{D(v_k, w_1)\} \bigcup \{D(v_k, v_i) : 1 \leq i \neq k \leq n)\} = \{2n-1, 2n-2\}. \end{aligned}$$

Again a contradicting our assumption that M' is a uniform set of G_n, hence $\alpha \neq \{w_j\}$.

Finally, let $M' = \{u, w_1\} \bigcup \{v_i : 1 \leq i \neq p, q, r \leq n\}$, then for all w_j different from w_1 in $V(G_n)$

$$\begin{aligned} f_{M'}(w_j) &= \{D(w_j, u)\} \bigcup \{D(w_j, w_1)\} \bigcup \{D(w_j, v_i) : 1 \leq i \neq p, q, r \leq n\} \\ &= \{2n-1, 2n\}, \text{and} \\ f_{M'}(v_p) &= \{D(u, v_p)\} \bigcup \{D(w_1, v_p)\} \bigcup \{D(v_p, v_i) : 1 \leq i \neq p, q, r \leq n\}\} \\ &= \{2n-1, 2n-2\}. \end{aligned}$$

Yet another contradiction to the assumption that M' is a uniform set of G_n, hence $M' \neq \{u, w_1\} \bigcup \{v_i : 1 \leq i \neq p, q, r \leq n\}$.

Since for all possibilities of $|M'| < |M|$, we have a contradiction to our basic assumption that M' is a uniform set of G_n, M is seen to be a minimum uniform set of G_n. $\square$

Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. The corona product $G_1 \circ G_2$ is defined as the graph obtained from G_1 and G_2 by taking one copy of G_1 and n_1 copies of G_2 and joining by an edge each vertex from the ith-copy of G_2 with the ith-vertex of G_1.

Theorem 3.17. *If G is a graph obtained as the corona product of Hamilton connected graph H of order n and K_1, then $\varsigma(G) = n$.*

Proof. Let $G \cong H \circ K_1$, where H is a Hamilton connected graph of order n. Let $\{u_i : 1 \leq i \leq n\}$ be the vertices of H and $\{v_i : 1 \leq i \leq n\}$ be the $n-$copies of K_1 adjacent to u_i in G. For any pair of vertices $x, y \in V(G)$, we have

$$\begin{aligned} D(u_i, u_j) &= n-1, \\ D(u_i, v_j) &= \begin{cases} 1, & i = j \\ n, & i \neq j, \end{cases} \\ D(v_i, v_j) &= n+1. \end{aligned}$$

The set $M = \{u_i : 1 \leq i \leq n\}$ is a uniform set of G, since $f_M(v_j) = \{1, n\}$ for all $v_j \in V(G)$. We now claim that M is minimum too. Suppose not then there exists a uniform set M' of G such that $|M'| < |M|$. Since $|M^c| = |M| = n$ and $|M'| < |M|$, we have the following exhaustive cases: either $M' \subset M^c$ or $M' \subset M$ or $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$.

Case 1: Let $M' \subset M^c$, then for some $k \leq n$ there exists $v_k \in M^c$ such that $M' = M^c - \{v_k\}$. Since

$$\begin{aligned} f_{M'}(v_k) &= \{D(v_k, v_j) : 1 \leq j \neq k \leq n\} = \{n+1\}, \text{and} \\ f_{M'}(u_1) &= \{D(u_1, v_j) : 1 \leq j \neq k \leq n\} = \{n\}, \end{aligned}$$

a contradiction to the assumption that M' is a uniform set of G. Hence our assumption that $M' \subset M^c$ is false.

Case 2: Let $M' \subset M$, then for some $l \leq n$ there exists at least one $u_l \in M$ such that $M' = M - \{u_l\}$. Since

$$\begin{aligned} f_{M'}(u_l) &= \{D(u_l, u_i) : 1 \leq i \neq l \leq n\} = \{n-1\}, \text{and} \\ f_{M'}(v_1) &= \{D(v_1, u_i) : 1 \leq i \neq l \leq n\} = \{n\}, \end{aligned}$$

contradiction to the assumption that M' is a uniform set of G. Hence our assumption that $M' \subset M$ is also false.

Case 3: Finally, let $M' \bigcap M \neq \emptyset$ and $M' \bigcap M^c \neq \emptyset$. Then there exists at least one $u_r \in M$ such that $u_r \notin M'$ and $v_s \in M^c$ such that $v_s \in M'$ for which, we have $M' = \{v_s\} \bigcup \{u_i : 1 \leq i \neq r \leq n\}$.

$$\begin{aligned} f_{M'}(u_r) &= \{D(u_r, u_i) : 1 \leq i \neq r \leq n\} \bigcup \{D(u_r, v_s)\} = \{n-1, n\}, \text{and} \\ f_{M'}(v_1) &= \{D(v_1, u_i) : 1 \leq i \neq r \leq n\} \bigcup \{D(v_1, v_s)\} = \{n, n+1\}. \end{aligned}$$

A contradiction to the assumption that M' is a uniform set of G. Hence the proof. $\square$

4. Conclusion and Scope for Further Research

In this article we have introduced the notion of uniform number of a graph $\varsigma(G)$. We have prescribed the bounds of $\varsigma(G)$ and established the sharpness of the prescribed bounds. We have also determined the uniform numbers of some standard classes of graphs. Although we have determined the uniform numbers of some classes of cyclic graphs, the determination of uniform number of cycles C_n remains an open problem. In the present article we have also initiated the determination of uniform numbers of graph products. A general investigation of uniform numbers of graph products is yet to be done. Further there are other open areas of investigation pertaining to the uniform number of graphs as discussed below.

Graphs G_1, G_2 and P_5 of Fig. 1 revisited: Recall that for the graph G_1 in Fig. 1, $M_1 = \{1, 2, 3\}$ and $M_2 = \{1, 4, 5, 6, 7\}$ are two uniform sets of different cardinalities. And, for the graph G_2 in Fig. 1, $M_3 = \{1, 2, 3\}$ and $M_4 = \{4, 5, 6\}$ are two uniform sets of same cardinality. Finally, $M = \{2, 3, 4\}$ is the only uniform set for the graph P_5 in Fig. 1.

From the preceding discussion it is evident that, a graph G may or may not have a unique uniform set. In case it is having more than one uniform sets, they may or may not have the same cardinalities. Hence the following problem is yet to be investigated.

Problem 4.1. Characterize the graphs having unique uniform set.

At this juncture one may suspect that a uniform set of a graph is either the center or the detour-center of the graph. The set M_1 for graph G_1 of Fig. 1 disproves any such intuition. However, there exists graphs for which a uniform set is either the center or detour-center of the graph, as illustrated by the set

M_3 of graph G_2 of Fig. 1. It must also be noted that a uniform set need not be contained in the center or detour center of the graph, as can be seen for the set M_4 of G_2 of Fig. 1. In general, a uniform set of a graph need not be contained in a block of a connected graph which can be seen for the set M_2 of G_1 in Fig. 1. The following problem is yet to be investigated.

Problem 4.2. Characterize the graphs for which every uniform set induces a connected subgraph.

Further note that, M_1 is both minimum and minimal uniform set, whereas M_2 is minimal but not a minimum uniform set of G_1. The graph G_1 suggests the existence of graphs for which minimal uniform sets need not be a minimum. It also indicates the possibility of the existence of graphs for which no two minimal uniform sets are of the same cardinality. The graph G_2 suggests the existence of the graphs for which every minimal uniform set is also a minimum uniform set. It also suggests the existence of graphs for which the complement of a uniform set is also a uniform set and the existence of graphs for which the uniform set partition the vertex set of the graph. Such an investigation is yet to be done. We summarize the foregoing observations as follows.

Problem 4.3. Characterize the graphs for which all the uniform sets of the same cardinality.

Problem 4.4. Characterize the graphs for which the uniform sets partitions the vertex set of the graph.

Further, it is easy to see that the sets M_1 and M_2 are minimal dominating sets and the set $D = \{2, 3\}$ is the minimum dominating set of G_1. It can also be verified that $D = \{2, 3\}$ is not a uniform set of G_1. Hence, $\varsigma(G_1) \neq \gamma(G_1)$. Also, the sets M_3 and M_4 are minimum dominating sets of the graph G_2. Thus, $\varsigma(G_2) = 3 = \gamma(G_2)$. It follows from the above discussion that minimum dominating set of a graph need to be a uniform set or vice-versa. Hence, it becomes imperative to recognize the graphs for which a minimum uniform set is also a minimum dominating set. Hence we propose the following problem.

Problem 4.5. Characterize the graphs for which $\varsigma(G) = \gamma(G)$.

Furthermore, the set M_2 in G_1 is a minimum uniform set as well as maximum independent set. The same is true for the set M_4 but not for M_3 in G_2. However, the set M in P_5 is not a maximum independent set. It follows from the above discussion that a maximum independent set of a graph need not be a uniform set or vice-versa. To recognize such graphs for which a minimum uniform set is also a maximum independent set is open.

Problem 4.6. Characterize the graphs for which $\varsigma(G) = \beta(G)$.

Last but not the least, K_2 is the largest clique in G_1 and $\varsigma(G_1) = 3$, hence $\omega(G_1) \neq \varsigma(G_1)$. K_3 is the largest clique in G_2 and $\langle M_3 \rangle \cong K_3$, hence $\varsigma(G_2) = \omega(G_2) = 3$. As $\langle M_4 \rangle \ncong K_3$, it follows that it is not necessary for a minimum uniform set to induce the maximum clique in a graph. K_2 is the largest clique in P_5 and $\langle M \rangle \ncong K_2$, hence $\varsigma(P_5) \neq \omega(P_5)$. It would be interesting to investigate the following problems.

Problem 4.7. Characterize the graphs for which $\varsigma(G) = \omega(G)$.

Problem 4.8. Characterize the graphs for which the minimum uniform set induces the maximal clique.

Finally, it is easy to see that $2 = \chi(G_1) \neq \varsigma(G_1) = 3$ and $\chi(G_2) = 3 = \varsigma(G_2)$. This observation raises the following problem.

Problem 4.9. Characterize the graphs for which $\chi(G) = \varsigma(G)$.

The problem of determining domination number $\gamma(G)$, clique number $\omega(G)$, independence number $\beta(G)$ and chromatic number $\chi(G)$ are known to be NP-complete [7]. The problems raised in the foregoing paragraphs opens the flood-gates to explore the unexplored notion of uniform number of a graph $\varsigma(G)$ and its interaction with the well known graph theoretic parameters such as $\gamma(G), \omega(G), \beta(G)$ and $\chi(G)$.

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