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On Subclasses of Analytic and m-Fold Symmetric bi-Univalent Functions

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Abstract. The purpose of the present paper is to introduce and investigate two new subclasses $K_{\Sigma_m}(\lambda,\gamma;\alpha)$ and $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions defined in the open unit disk U. We obtain upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to these subclasses. Many of the well-known and new results are shown to follow as special cases of our results.

Keywords: Analytic functions, Univalent functions, m-Fold symmetric bi-univalent functions, Coefficient estimates.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f that are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. And normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

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Let S be the subclass of $\mathcal{A}$ consisting of the form (1.1) which are also univalent in U. The Koebe one-quarter theorem (see [4]) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of bi-univalent functions in U given by (1.1). For a brief history and interesting examples in the class Σ see [18], (see also [6, 7, 8, 10, 14, 15, 21, 22]).

For each function $f \in S$, the function $h(z) = (f(z^m))^{\frac{1}{m}}$, $(z \in U, m \in N)$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [9, 12]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \ (z \in U, m \in N). \tag{1.3}$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the class S are one-fold symmetric.

In [19] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in N$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1}$$
$$- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots , \tag{1.4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \ \textit{and} \ \left[-\log\left(1-z^m\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [1, 2, 5, 16, 17, 19, 20]).

The aim of the present paper is to introduce the new subclasses $K_{\Sigma_m}(\lambda,\gamma;\alpha)$ and $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ of Σ_m and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to prove our main results, we require the following lemma (see [4]).

Lemma 1.1. *If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each $k \in N$, where $\mathcal{P}$ is the family of all functions h analytic in U for which*

$$Re\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

2. Coefficient Estimates for the Functions Class $K_{\Sigma_m}(\lambda,\gamma;\alpha)$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the class $K_{\Sigma_m}(\lambda,\gamma;\alpha)$ if it satisfies the following conditions:

$$\left|arg\left[\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \alpha\pi, \quad (z \in U) \qquad (2.1)$$

and

$$\left|arg\left[\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \alpha\pi, \quad (w \in U), \qquad (2.2)$$

$$\left(0 < \alpha \le 1,\ 0 < \lambda \le 1,\ \gamma \ge 0,\ m \in N\right),$$

where the function $g = f^{-1}$ is given by (1.4).

Theorem 2.2. *Let $f \in K_{\Sigma_m}(\lambda,\gamma;\alpha)$ $(0 < \alpha \le 1,\ 0 < \lambda \le 1,\ \gamma \ge 0,\ m \in N)$ be given by (1.3). Then*

$$|a_{m+1}| \le \frac{4\lambda\alpha}{(m+\gamma)\sqrt{(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right) + (1-\alpha)(\lambda+1)\right) + 2\alpha(1-\lambda)}} \qquad (2.3)$$

and

$$|a_{2m+1}| \le \frac{8\lambda^2\alpha^2(m+1)}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\lambda\alpha}{(2m+\gamma)(\lambda+1)}. \qquad (2.4)$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = [p(z)]^{\alpha} \qquad (2.5)$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)=[q(w)]^{\alpha}, \tag{2.6}$$

where $g=f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z)=1+p_m z^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \tag{2.7}$$

and

$$q(w)=1+q_m w^m+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$\frac{(m+\gamma)(\lambda+1)}{2\lambda}a_{m+1}=\alpha p_m, \tag{2.9}$$

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left(2a_{2m+1}+(\gamma-1)a_{m+1}^2\right)+\frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2}a_{m+1}^2 =\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-\frac{(m+\gamma)(\lambda+1)}{2\lambda}a_{m+1}=\alpha q_m \tag{2.11}$$

and

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left((2m+\gamma+1)a_{m+1}^2-2a_{2m+1}\right)+\frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2}a_{m+1}^2 =\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

Making use of (2.9) and (2.11), we obtain

$$p_m=-q_m \tag{2.13}$$

and

$$\frac{(m+\gamma)^2(\lambda+1)^2}{2\lambda^2}a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we find that

$$\begin{aligned}&\left(\frac{(2m+\gamma)(m+\gamma)(\lambda+1)}{2\lambda}+\frac{(m+\gamma)^2(1-\lambda)}{2\lambda^2}\right)a_{m+1}^2\\ &=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right)\\ &=\alpha(p_{2m}+q_{2m})+\frac{(\alpha-1)(m+\gamma)^2(\lambda+1)^2}{4\lambda^2\alpha}a_{m+1}^2.\end{aligned}$$

Therefore, we have

$$a_{m+1}^2=\frac{4\lambda^2\alpha^2(p_{2m}+q_{2m})}{(m+\gamma)^2\left[(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right)+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)\right]}. \tag{2.15}$$

Now, taking the absolute value of (2.15) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{4\lambda\alpha}{(m+\gamma)\sqrt{(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right)+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.3).
In order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$\frac{(2m+\gamma)(\lambda+1)}{\lambda}a_{2m+1} - \frac{(2m+\gamma)(m+1)(\lambda+1)}{2\lambda}a_{m+1}^2$$
$$= \alpha\left(p_{2m}-q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{2.16}$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1} = \frac{\lambda^2\alpha^2(m+1)\left(p_m^2+q_m^2\right)}{(m+\gamma)^2(\lambda+1)^2} + \frac{\lambda\alpha\left(p_{2m}-q_{2m}\right)}{(2m+\gamma)(\lambda+1)}. \tag{2.17}$$

Taking the absolute value of (2.17) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{8\lambda^2\alpha^2(m+1)}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\lambda\alpha}{(2m+\gamma)(\lambda+1)},$$

which completes the proof of Theorem 2.2. □

3. Coefficient Estimates for the Functions Class $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$

Definition 3.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the class $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ if it satisfies the following conditions:

$$Re\left\{\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta, \quad (z \in U) \tag{3.1}$$

and

$$Re\left\{\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta, \quad (w \in U), \tag{3.2}$$

$$\left(0 \leq \beta < 1,\ 0 < \lambda \leq 1,\ \gamma \geq 0,\ m \in N\right),$$

where the function $g = f^{-1}$ is given by (1.4).

Theorem 3.2. *Let $f \in K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ $(0 \leq \beta < 1,\ 0 < \lambda \leq 1,\ \gamma \geq 0,\ m \in N)$ be given by (1.3). Then*

$$|a_{m+1}| \leq \frac{2\lambda}{m+\gamma}\sqrt{\frac{2(1-\beta)}{\left(\frac{m}{m+\gamma}+1\right)\lambda^2+\frac{m}{m+\gamma}\lambda+1}} \tag{3.3}$$

and

$$|a_{2m+1}| \leq \frac{8\lambda^2(m+1)(1-\beta)^2}{(m+\gamma)^2(\lambda+1)^2} + \frac{4(1-\beta)}{(2m+\gamma)(\lambda+1)}. \tag{3.4}$$

Proof. It follows from conditions (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$\frac{(m+\gamma)(\lambda+1)}{2\lambda}a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left(2a_{2m+1} + (\gamma-1)a_{m+1}^2\right) + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2}a_{m+1}^2 = (1-\beta)p_{2m}, \tag{3.8}$$

$$-\frac{(m+\gamma)(\lambda+1)}{2\lambda}a_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left((2m+\gamma+1)a_{m+1}^2 - 2a_{2m+1}\right) + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2}a_{m+1}^2 = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m = -q_m \tag{3.11}$$

and

$$\frac{(m+\gamma)^2(\lambda+1)^2}{2\lambda^2}a_{m+1}^2 = (1-\beta)^2(p_m^2 + q_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$\left(\frac{(2m+\gamma)(m+\gamma)(\lambda+1)}{2\lambda} + \frac{(m+\gamma)^2(1-\lambda)}{2\lambda^2}\right)a_{m+1}^2 = (1-\beta)(p_{2m} + q_{2m}).$$

Therefore, we have

$$a_{m+1}^2 = \frac{2\lambda^2(1-\beta)(p_{2m}+q_{2m})}{(m+\gamma)^2\left[\left(\frac{m}{m+\gamma}+1\right)\lambda^2 + \frac{m}{m+\gamma}\lambda + 1\right]}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\lambda}{m+\gamma}\sqrt{\frac{2(1-\beta)}{\left(\frac{m}{m+\gamma}+1\right)\lambda^2 + \frac{m}{m+\gamma}\lambda + 1}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).
In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$\frac{(2m+\gamma)(\lambda+1)}{\lambda}a_{2m+1}-\frac{(2m+\gamma)(m+1)(\lambda+1)}{2\lambda}a_{m+1}^2=(1-\beta)\left(p_{2m}-q_{2m}\right),$$

or equivalently

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{\lambda(1-\beta)\left(p_{2m}-q_{2m}\right)}{(2m+\gamma)(\lambda+1)}.$$

Upon substituting the value of a_{m+1}^2 from (3.12), it follows that

$$a_{2m+1}=\frac{\lambda^2(m+1)\left(1-\beta\right)^2\left(p_m^2+q_m^2\right)}{\left(m+\gamma\right)^2\left(\lambda+1\right)^2}+\frac{\lambda\left(1-\beta\right)\left(p_{2m}-q_{2m}\right)}{(2m+\gamma)(\lambda+1)}.$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{8\lambda^2(m+1)\left(1-\beta\right)^2}{\left(m+\gamma\right)^2\left(\lambda+1\right)^2}+\frac{4\left(1-\beta\right)}{(2m+\gamma)(\lambda+1)}.$$

which completes the proof of Theorem 3.2. ☐

4. Corollaries and Consequences

This section is devoted to the presentation of some special cases of the main results.

For one-fold symmetric bi-univalent functions, Theorems 2.2 and 3.2 reduce to the following corollaries:

Corollary 4.1. *Let* $f\in K_\Sigma(\lambda,\gamma;\alpha)$ $(0<\alpha\leq 1,\ 0<\lambda\leq 1,\ \gamma\geq 0)$ *be given by (1.1). Then*

$$|a_2|\leq\frac{4\lambda\alpha}{(1+\gamma)\sqrt{(\lambda+1)\left(\frac{2\lambda\alpha(2+\gamma)}{1+\gamma}+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)}}$$

and

$$|a_3|\leq\frac{16\lambda^2\alpha^2}{\left(1+\gamma\right)^2\left(\lambda+1\right)^2}+\frac{4\lambda\alpha}{(2+\gamma)(\lambda+1)}.$$

Corollary 4.2. *Let* $f\in K_\Sigma^*(\lambda,\gamma;\beta)$ $(0\leq\beta<1,\ 0<\lambda\leq 1,\ \gamma\geq 0)$ *be given by (1.1). Then*

$$|a_2|\leq\frac{2\lambda}{1+\gamma}\sqrt{\frac{2(1-\beta)}{\frac{2+\gamma}{1+\gamma}\lambda^2+\frac{1}{1+\gamma}\lambda+1}}$$

and

$$|a_3|\leq\frac{16\lambda^2\left(1-\beta\right)^2}{\left(1+\gamma\right)^2\left(\lambda+1\right)^2}+\frac{4\left(1-\beta\right)}{(2+\gamma)(\lambda+1)}.$$

The classes $K_\Sigma(\lambda,\gamma;\alpha)$ and $K_\Sigma^*(\lambda,\gamma;\beta)$ are defined in the following way:

Definition 4.3. A function $f \in \Sigma$ given by (1.1) is said to be in the class $K_\Sigma(\lambda,\gamma;\alpha)$ if it satisfies the following conditions:

$$\left| arg\left[\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}+\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \frac{\alpha\pi}{2}, \quad (z \in U)$$

and

$$\left| arg\left[\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \frac{\alpha\pi}{2}, \quad (w \in U),$$

$$(0<\alpha\le 1,\ 0<\lambda\le 1,\ \gamma\ge 0),$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 4.4. A function $f \in \Sigma$ given by (1.1) is said to be in the class $K^*_\Sigma(\lambda,\gamma;\beta)$ if it satisfies the following conditions:

$$Re\left\{\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}+\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta, \quad (z \in U)$$

and

$$Re\left\{\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta, \quad (w \in U),$$

$$(0\le\beta< 1,\ 0<\lambda\le 1,\ \gamma\ge 0),$$

where the function $g = f^{-1}$ is given by (1.2).

Remark 4.5. It should be remarked that the classes $K_{\Sigma_m}(\lambda,\gamma;\alpha)$, $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$, $K_\Sigma(\lambda,\gamma;\alpha)$ and $K^*_\Sigma(\lambda,\gamma;\beta)$ are a generalization of well-known classes consider earlier. These classes are:

(1) For $\gamma = 0$, the classes $K_{\Sigma_m}(\lambda,\gamma;\alpha)$ and $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ reduce to the classes $S_{\Sigma_m}(\alpha,\lambda)$ and $S_{\Sigma_m}(\beta,\lambda)$ which were given recently by Altinkaya and Yalcin [2].

(2) For $\lambda = 1$ and $\gamma = 0$, the classes $K_{\Sigma_m}(\lambda,\gamma;\alpha)$ and $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ reduce to the classes $S^\alpha_{\Sigma_m}$ and $S^\beta_{\Sigma_m}$ which were introduced by Altinkaya and Yalcin [1].

(3) For $\lambda = \gamma = 1$, the classes $K_{\Sigma_m}(\lambda,\gamma;\alpha)$ and $K^*_{\Sigma_m}(\lambda,\gamma;\beta)$ reduce to the classes $\mathcal{H}^\alpha_{\Sigma,m}$ and $\mathcal{H}_{\Sigma,m}(\beta)$ which were introduced by Srivastava et al. [19].

(4) For $\lambda = 1$, the classes $K_\Sigma(\lambda,\gamma;\alpha)$ and $K^*_\Sigma(\lambda,\gamma;\beta)$ reduce to the classes $P_\Sigma(\alpha,\gamma)$ and $P_\Sigma(\beta,\gamma)$ which were considered by Prema and Keerthi [13].

(5) For $\lambda = 1$ and $\gamma = 0$, the classes $K_\Sigma(\lambda,\gamma;\alpha)$ and $K^*_\Sigma(\lambda,\gamma;\beta)$ reduce to the classes $S^*_\Sigma(\alpha)$ and $S^*_\Sigma(\beta)$ which were given by Brannan and Taha [3].

(6) For $\lambda = \gamma = 1$, the classes $K_\Sigma(\lambda, \gamma; \alpha)$ and $K^*_\Sigma(\lambda, \gamma; \beta)$ reduce to the classes $\mathcal{H}^\alpha_\Sigma$ and $\mathcal{H}_\Sigma(\beta)$ which were investigated by Srivastava et al. [18].

Remark 4.6. By specifying the parameters λ and γ, we can derive a number of known results. Some of them are given below:

(1) If we take $\gamma = 0$ in Theorems 2.2 and 3.2, we obtain the results which were proven by Altinkaya and Yalcin [2, Theorems 1 and 2].
(2) If we take $\lambda = \gamma = 1$ in Theorems 2.2 and 3.2, we obtain the results which were proven by Srivastava et al. [19, Theorems 2 and 3].
(3) If we put $\lambda = 1$ in Corollaries 4.1 and 4.2, we have the results which were proven by Prema and Keerthi [13, Theorems 2.2 and 3.2].
(4) If we set $\lambda = 1$ and $\gamma = 0$ in Corollaries 4.1 and 4.2, we have the results obtained by Murugusundaramoorthy et al. [11, Corollaries 6 and 7].
(5) If we take $\lambda = \gamma = 1$ in Corollaries 4.1 and 4.2, we obtain the results obtained by Srivastava et al. [18, Theorems 1 and 2].

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