

The Dynamical Analysis of a Delayed prey-Predator Model with a Refuge-Stage Structure Prey Population

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ABSTRACT. A mathematical model describing the dynamics of a delayed stage structure prey-predator system with prey refuge is considered. The existence, uniqueness and boundedness of the solution are discussed. All the feasible equilibrium points are determined. The stability analysis of them are investigated. By employing the time delay as the bifurcation parameter, we observed the existence of Hopf bifurcation at the positive equilibrium. The stability and direction of the Hopf bifurcation are determined by utilizing the normal form method and the center manifold reduction. Numerical simulations are given to support the analytic results.

Keywords: Delayed Prey-Predator System, Stage-Structure, Refuge, Stability, Hopf Bifurcation.

2010 Mathematics Subject Classification: 92D25.

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1. INTRODUCTION

During the last few decades, the prey - predator models have been received great interest in population dynamics, [1-6]. The dynamics of the prey - predator models describe the relationships between species and the outer environment and the connections between different species. These models become famous from the traditional work given by Lotka [7] and Volterra [8]. Although the traditional Lotka-Volterra model serves as a basis for many models used today to analyze population dynamics, it is unfit cannot be neglected. Because it has an unavoidable limitations to describe many realistic phenomena in biology, moreover it is assumed that each individual prey admits the same ability to be attacked by the predators. This assumption is obviously unrealistic for many animals because there are many of them have two stages, immature and mature. So, in order to describe the real biological interactions between the individuals of prey-predator systems, some mathematical researchers proposed the stage-structured prey-predator models, see for example [9-16] and the references therein. In general, the time delays in mathematical models of population dynamics are due to maturation time, gestation time, capturing time or some other reasons and since in most applications of delay differential equations in biology, the need for incorporating time delays is often due to the existence of some stage structures. So, some works of stage-structure prey - predator models with time delay have been provided in the literatures [17-26]. Bandyopadhyaya and Banerjee in [20], Yuanyuan and Changming in [21], and Wang et al in [22] proposed three mathematical models of stage-structure prey-predator involving time delay for gestation, which based on the fact that the reproduction of predator will not be instantaneous after eating the prey but mediated by some time delay needed for gestation of predator, in their models, they supposed that the predator feeds on the immature prey only or mature prey only, and ignored the predation of the other prey. In nature, the predator feeds on both of the prey, mature and immature. From this viewpoint and since the addition of refugees can be controlled on the prey extinction where it will be out of sight of predators, Naji and Majeed in [16] was proposed the following mathematical model:

$$\begin{aligned}\dot{x} &= ry - \delta_1 x^2 - d_1 x - \beta x - \gamma_1(1 - m)xz, \\ \dot{y} &= \beta x - \delta_2 y^2 - d_2 y - \gamma_2(1 - m)yz, \\ \dot{z} &= e_1 \gamma_1(1 - m)xz + e_2 \gamma_2(1 - m)yz - \delta_3 z^2 - d_3 z,\end{aligned}$$

where $x(T)$ represents the population size of the immature prey at time T ; $y(T)$ represents the population size of the mature prey at time T , while $z(T)$ denotes to the population size of the predator species at time T . Clearly the above model does not considers the effect of delay on the gestation of predator further than that it consider the predation from both the prey and the existence

of refuge as a defenses factor against the predation. In this paper the Naji and Majeed model is modified so that it involves the delay for the gestation of the predator, consequently the above model can be written as follows:

$$\begin{aligned}\dot{x} &= ry - \delta_1 x^2 - d_1 x - \beta x - \gamma_1(1-m)xz, \\ \dot{y} &= \beta x - \delta_2 y^2 - d_2 y - \gamma_2(1-m)yz, \\ \dot{z} &= e_1 \gamma_1(1-m)x(T-\tau)z(T-\tau) + e_2 \gamma_2(1-m)y(T-\tau)z(T-\tau) \\ &\quad - \delta_3 z^2 - d_3 z.\end{aligned}$$

Now, in order to simplify the analysis of the proposed model, the above model takes the following dimensionless form:

$$\begin{aligned}\dot{y}_1 &= a_1 y_2 - a_2 y_1^2 - a_3 y_1 - a_4 y_1 y_3, \\ \dot{y}_2 &= b_1 y_1 - b_2 y_2^2 - y_2 - b_3 y_2 y_3, \\ \dot{y}_3 &= y_1(t-\tau)y_3(t-\tau) + y_2(t-\tau)y_3(t-\tau) - y_3^2 - b_4 y_3,\end{aligned}\tag{1.1}$$

with the dimensionless variables and parameters given by

$$\begin{aligned}y_1 &= \frac{e_1 \gamma_1(1-m)}{d_2} x, y_2 = \frac{e_2 \gamma_2(1-m)}{d_2} y, y_3 = \frac{\delta_3}{d_2} z, t = d_2 T \\ a_1 &= \frac{e_1 \gamma_1 r}{e_2 \gamma_2 d_2}, a_2 = \frac{\delta_1}{e_1 \gamma_1(1-m)}, a_3 = \frac{d_1 + \beta}{d_2}, a_4 = \frac{\gamma_1(1-m)}{\delta_3} \\ b_1 &= \frac{e_2 \gamma_2 \beta}{e_1 \gamma_1 d_2}, b_2 = \frac{\delta_2}{e_2 \gamma_2(1-m)}, b_3 = \frac{\gamma_2(1-m)}{\delta_3}, b_4 = \frac{d_3}{d_2}.\end{aligned}$$

2. POSITIVENESS AND BOUNDEDNESS

In this section, we study the positivity and boundedness of the solutions of system (1.1).

Theorem 2.1. *All solutions of system (1.1) are positive for $t \geq 0$.*

Proof. From the first equation of system (1.1), we have for $t \geq 0$

$$\dot{y}_1 \geq -y_1(a_2 y_1 + a_4 y_3 + a_3)$$

Straight forward computation gives that

$$y_1 \geq y_1(0) \exp\left\{-\int_0^t (a_2 y_1(s) + a_4 y_3(s) + a_3) ds\right\} > 0.$$

Since $y_1(0) > 0$, we get $y_1(t) > 0$ for all $t \geq 0$. Similarly we can see that $y_2(t) > 0, y_3(t) > 0$ for all $t \geq 0$. Hence the proof of theorem is complete. \square

In order to prove the boundedness of system (1.1), we need to recall the following Lemma from [11].

Lemma 2.2. *Consider the following equations*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t),$$

where $a, b, c, \tau > 0, x(t) > 0$ for $t \in [-\tau, 0]$.

(i): if $a > b$, then $\lim_{t \rightarrow +\infty} x(t) = (a-b)/c$.

(ii): if $a < b$, then $\lim_{t \rightarrow +\infty} x(t) = 0$.

Theorem 2.3. *All solutions of system (1.1) with positive initial values are bounded.*

Proof. Given any solution $(y_1(t), y_2(t), y_3(t))$ of system (1.1) with the initial condition, $y_1(0) > 0, y_2(0) > 0, y_3(0) > 0$. Then from the first two equations of system (1.1) we obtain

$$\frac{d}{dt}(y_1 + y_2) + \sigma_1(y_1 + y_2) \leq a_1 y_2 \left(1 - \frac{y_2}{b_2}\right) + b_1 y_1 \left(1 - \frac{y_1}{a_1}\right) \leq \sigma_2,$$

where $\sigma_1 = \min\{1, a_3\}$ and $\sigma_2 = \frac{a_1^2}{4b_2} + \frac{b_1^2}{4a_2}$. Moreover by using Gronwall lemma [18], we get that

$$0 < y_1(t) + y_2(t) \leq (y_1(0) + y_2(0))e^{-\sigma_1 t} + \frac{\sigma_2}{\sigma_1}(1 - e^{-\sigma_1 t}).$$

Therefore, for $t \rightarrow \infty$ we have $0 < y_1(t) + y_2(t) < \frac{\sigma_2}{\sigma_1} := M^*$. Thus, there exists a constant $T_1 > 0$ and $M_1^* > M^*$ such that for any $t > T_1$ we have $y_1(t) \leq M_1^*$ and $y_2(t) \leq M_1^*$. In addition, from third equation of system (1.1) with $t > T_1 + \tau$ it is easy to verify that

$$\frac{dy_3}{dt} \leq M_1^* y_3(t - \tau) - y_3^2 - b_4 y_3.$$

Then by using lemma(2.2), it follows that as $t \rightarrow \infty$ we have

$$y_3 = 0 \text{ or } y_3 \leq M_1^* - b_4 := M_2^*.$$

Hence, all solutions of system (1.1), which initiate in R_+^3 are bounded and therefore we have finished the proof. \square

3. LOCAL STABILITY ANALYSIS AND HOPF BIFURCATION.

In this section, we will study the local stability and Hopf bifurcation of system (1.1). It is known that the location and number of equilibrium points do not change with time delay. Accordingly, from [16] system (1.1) have two boundary equilibrium points, say $E_0 = (0, 0, 0)$ and $E_1 = (\tilde{y}_1, \tilde{y}_2, 0)$, with one interior equilibrium point given by $E_2 = (y_1^*, y_2^*, y_3^*)$, where

$$\tilde{y}_1 = \frac{1}{b_1}(b_2 \tilde{y}_2^2 + \tilde{y}_2) \quad (3.1)$$

while \tilde{y}_2 is a positive root of

$$A_1 y_2^3 + A_2 y_2^2 + A_3 y_2 + A_4 = 0, \quad (3.2)$$

here, $A_1 = \frac{a_2 b_2^2}{b_1^2}, A_2 = \frac{2a_2 b_2}{b_1^2}, A_3 = \frac{a_2 + a_3 b_1 b_2}{b_1^2}$ and $A_4 = \frac{a_3 - a_1 b_1}{b_1}$.

And

$$y_1^* = \frac{[(b_2 + b_3)y_2^* + (1 - b_3 b_4)]y_2^*}{b_1 - b_3 y_2^*}, y_3^* = y_1^* + y_2^* - b_4, \quad (3.3)$$

while y_2^* is a positive root of

$$B_1 y_2^3 + B_2 y_2^2 + B_3 y_2 + B_4 = 0, \quad (3.4)$$

where,

$$\begin{aligned} B_1 &= a_2(b_2 + b_3)^2 + a_4b_2(b_2 + b_3) > 0, \\ B_2 &= [2a_2(b_2 + b_3) + 2a_4b_2 + a_4b_3](1 - b_3b_4) - [a_3b_2b_3 + b_3^2(a_3 + a_1)], \\ B_3 &= 2a_1b_1b_3 + (a_2 + a_4)(1 - b_3b_4)^2 + b_1(a_3 - a_4b_4)(b_2 + b_3) \\ &\quad + [b_1a_4 - b_3(a_3 - a_4b_4)](1 - b_3b_4), \\ B_4 &= b_1[a_3 - a_1b_1 - a_3b_3b_4 - a_4b_4(1 - b_3b_4)]. \end{aligned}$$

Clearly, the equilibrium point E_0 always exists, while E_1 exists uniquely in the interior of y_1y_2 -plane provided that

$$a_3 < a_1b_1, \quad (3.5)$$

However the interior equilibrium point E_2 exists uniquely under the following set of conditions

$$B_4 < 0 \text{ with } (B_2 > 0 \text{ or } B_3 < 0), \quad (3.6)$$

$$y_1^* + y_2^* > b_4, \quad (3.7)$$

$$\frac{b_3b_4 - 1}{b_2 + b_3} < y_2^* < \frac{b_1}{b_3} \text{ or } \frac{b_1}{b_3} < y_2^* < \frac{b_3b_4 - 1}{b_2 + b_3}. \quad (3.8)$$

It is well known that, the variational matrix of system (1.1) at any equilibrium point $\check{E} = (\check{y}_1, \check{y}_2, \check{y}_3)$, takes the form

$$J(\check{E}) = \begin{pmatrix} -2a_2\check{y}_1 - a_3 - a_4\check{y}_3 & a_1 & -a_4\check{y}_1 \\ b_1 & -2b_2\check{y}_2 - 1 - b_3\check{y}_3 & -b_3\check{y}_2 \\ \check{y}_3e^{-\lambda\tau} & \check{y}_3e^{-\lambda\tau} & (\check{y}_1 + \check{y}_2)e^{-\lambda\tau} - 2\check{y}_3 - b_4 \end{pmatrix} \quad (3.9)$$

while its associated characteristic equation takes the form

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \quad (3.10)$$

here $P(\lambda)$ and $Q(\lambda)$ are polynomials of λ . Accordingly the local stability properties of system (1.1) at all feasible equilibrium points are determined by the roots of the above equation for all $\tau \geq 0$.

For the equilibrium point E_0 , Eq.(3.9) reduces to

$$J(E_0) = \begin{pmatrix} -a_3 & a_1 & 0 \\ b_1 & -1 & 0 \\ 0 & 0 & -b_4 \end{pmatrix}. \quad (3.11)$$

Then the associated characteristic equation of the variational matrix (3.11) is given by

$$(\lambda + b_4)[\lambda^2 + (a_3 + 1)\lambda + a_3 - a_1b_1] = 0. \quad (3.12)$$

Clearly, all roots of (3.12) have negative real parts if and only if the following condition holds:

$$a_3 > a_1b_1. \quad (3.13)$$

Therefore, E_0 is locally asymptotically stable for any $\tau \geq 0$ provided that condition (3.13) holds.

For the equilibrium point E_1 , the variational matrix Eq.(3.9) reduces to

$$J(E_1) = \begin{pmatrix} -2a_2\tilde{y}_1 - a_3 & a_1 & -a_4\tilde{y}_1 \\ b_1 & -2b_2\tilde{y}_2 - 1 & -b_3\tilde{y}_2 \\ 0 & 0 & (\tilde{y}_1 + \tilde{y}_2)e^{-\lambda\tau} - b_4 \end{pmatrix} = (a_{ij}), \quad (3.14)$$

while its characteristic equation is given by

$$[\lambda + b_4 - (\tilde{y}_1 + \tilde{y}_2)e^{-\lambda\tau}][\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}] = 0. \quad (3.15)$$

Obviously, all roots of the equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$

have negative real parts for any $\tau \geq 0$ if the following condition holds

$$(2a_2\tilde{y}_1 + a_3)(2b_2\tilde{y}_2 + 1) > a_1b_1, \quad (3.16)$$

while all other roots of Eq.(3.15) are given by the roots of

$$\lambda + b_4 - (\tilde{y}_1 + \tilde{y}_2)e^{-\lambda\tau} = 0. \quad (3.17)$$

Obviously, for $\tau = 0$, equation (3.17) has only one root given by $\lambda = (\tilde{y}_1 + \tilde{y}_2) - b_4$, which is negative under the condition

$$\tilde{y}_1 + \tilde{y}_2 < b_4 \quad (3.18)$$

Consequently, for $\tau = 0$, E_1 is locally asymptotically stable under the conditions (3.16) and (3.18). This stability may be lost, as τ increases, if Eq.(3.17) has a pair of purely imaginary roots, that cross the imaginary axis.

Now suppose that $\lambda = iw(\tau)$ is a root of Eq. (3.17), where $w(\tau)$ is real positive, then by substituting iw into Eq. (3.17) and separating real and imaginary parts, we obtain

$$\begin{aligned} (\tilde{y}_1 + \tilde{y}_2) \cos w\tau &= b_4, \\ (\tilde{y}_1 + \tilde{y}_2) \sin w\tau &= -w. \end{aligned} \quad (3.19)$$

Squaring each equation and then adding them, we get that

$$w = \mp \sqrt{(\tilde{y}_1 + \tilde{y}_2)^2 - b_4^2}$$

Note that, under the condition (3.18), $w(\tau)$ with $\tau > 0$ cannot be real, which contradicts with the assumption. Therefore, the characteristic Eq. (3.17) can't have purely imaginary root, and E_1 is locally asymptotically stable for all $\tau \geq 0$ if the conditions (3.16) and (3.18) hold. We can summarize the above discussion by the following theorem on the local stability of the boundary equilibrium points.

Theorem 3.1. (i): *The equilibrium E_0 is locally asymptotically stable for all $\tau \geq 0$ provided that condition (3.13) holds.*

(ii): *If the equilibrium point E_1 exists then it is locally asymptotically stable for all $\tau \geq 0$ provided that conditions (3.16) and (3.18) hold.*

Now for the interior equilibrium point E_2 , the variational matrix given by Eq.(3.9) reduces to

$$J(E_2) = (c_{ij})_{3 \times 3} = \begin{pmatrix} -(2a_2y_1^* + a_4y_3^* + a_3) & a_1 & -a_4y_1^* \\ b_1 & -(2b_2y_2^* + b_3y_3^* + 1) & -b_3y_2^* \\ R_1e^{-\lambda\tau} & R_1e^{-\lambda\tau} & R_2e^{-\lambda\tau} + R_3 \end{pmatrix} \quad (3.20)$$

where, $R_1 = y_3^*$, $R_2 = (y_1^* + y_2^*) > 0$, $R_3 = -(y_1^* + y_2^* + y_3^*) < 0$.

However, the associated characteristic equation of (3.20) is given by

$$\lambda^3 + M_1\lambda^2 + M_2\lambda + M_3 + (N_1\lambda^2 + N_2\lambda + N_3)e^{-\lambda\tau} = 0, \quad (3.21)$$

with

$$M_1 = -(c_{11} + c_{22} + R_3) > 0,$$

$$M_2 = c_{11}c_{22} - c_{12}c_{21} + R_3(c_{11} + c_{22}), \quad M_3 = R_3(c_{12}c_{21} - c_{11}c_{22}),$$

$$N_1 = -R_2 < 0, \quad N_2 = R_2(c_{11} + c_{22}) - R_1(c_{13} + c_{23}),$$

$$N_3 = R_2(c_{12}c_{21} - c_{11}c_{22}) + R_1[(c_{11} - c_{12})c_{23} + (c_{22} - c_{21})c_{13}].$$

Now, when $\tau = 0$, Eq.(3.21) becomes

$$\lambda^3 + (M_1 + N_1)\lambda^2 + (M_2 + N_2)\lambda + M_3 + N_3 = 0 \quad (3.22)$$

From [16] Eq.(3.22) has three roots with negative real parts provided the following conditions are satisfied

$$(2a_2y_1^* + a_4y_3^* + a_3)(2b_2y_2^* + b_3y_3^* + 1) > a_1b_1, \quad (3.23)$$

$$y_3^* > a_1 \text{ and } y_3^* > b_1, \quad (3.24)$$

Thus, for $\tau = 0$ the equilibrium point E_2 is locally asymptotically stable provided that the conditions (3.23) and (3.24) are satisfied. On the other hand for $\tau > 0$ straightforward computation shows that Eq.(3.21) has at least a pair of purely imaginary roots represented by $\lambda = \pm i\omega$ ($\omega > 0$) if in addition to conditions (3.23)-(3.24) the following condition holds

$$N_3 > M_3 \quad (3.25)$$

By substituting $\lambda = i\omega$ in to Eq(3.21) we obtain that

$$-i\omega^3 - M_1\omega^2 + iM_2\omega + M_3 + (-N_1\omega^2 + iN_2\omega + N_3)(\cos \omega\tau - i \sin \omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$(N_3 - N_1\omega^2) \cos \omega\tau + N_2\omega \sin \omega\tau = M_1\omega^2 - M_3,$$

$$N_2\omega \cos \omega\tau - (N_3 - N_1\omega^2) \sin \omega\tau = \omega^3 - M_2\omega, \quad (3.26)$$

Squaring these two equations and then adding them, we get

$$\omega^6 + h_1\omega^4 + h_2\omega^2 + h_3 = 0 \quad (3.27)$$

where

$$\begin{aligned} h_1 &= M_1^2 - N_1^2 - 2M_2 = R_1^2 + 2R_1R_2 + c_{11}^2 + c_{22}^2 + 2c_{12}c_{21} > 0, \\ h_2 &= M_2^2 - N_2^2 - 2M_1M_3 + 2N_1N_3, \\ h_3 &= M_3^2 - N_3^2 = (M_3 + N_3)(M_3 - N_3). \end{aligned}$$

Obviously due to conditions (3.23) - (3.25), we have $h_3 < 0$. So, according to Descartes rule of sign there is a unique positive root say ω_0 satisfying Eq.(3.27). Therefore Eq.(3.21) has a pair of purely imaginary roots represented by $\pm i\omega_0$. Moreover, by substituting ω_0 in Eq.(3.26) and solving the resulting system for τ , we can have

$$\tau_0 = \frac{1}{\omega_0} \cos^{-1} \frac{(N_2 - N_1M_1)\omega_0^4 + (N_3M_1 + N_1M_3 - N_2M_2)\omega_0^2 - N_3M_3}{N_1^2\omega_0^4 + (N_2^2 - 2N_1N_3)\omega_0^2 + N_3^2} \quad (3.28)$$

Keeping the above condition in view, we can obtain the following lemma:

Lemma 3.2. : *Assume that the conditions (3.23)-(3.25) hold, then when $\tau \in [0, \tau_0)$ all roots of Eq.(3.21) have negative real parts, and when $\tau = \tau_0$ Eq.(3.21) has a pair of purely imaginary roots $\pm i\omega_0$ while all other roots have negative real parts.*

Next, in the following lemma, we will show the transversal condition of Hopf bifurcation of system (1.1) near the interior equilibrium point E_2 where using τ as bifurcation parameter.

Lemma 3.3. : *Suppose that $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ is a root of Eq.(3.21) satisfying $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. Then the following transversal condition holds:*

$$\text{sign}\left[\frac{d(\text{Re}\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} > 0, \quad (3.29)$$

if

$$M_2^2 - 2M_1M_3 > N_2^2 - 2N_1N_3. \quad (3.30)$$

Proof. : by using $\lambda(\tau)$ in Eq.(3.21) and differentiating the resulting equation with respect to τ , we get that

$$\begin{aligned} &\{3\lambda^2 + 2M_1\lambda + M_2 + (2N_1\lambda + N_2)e^{-\lambda\tau} - \tau(N_1\lambda^2 + N_2\lambda + N_3)e^{-\lambda\tau}\} \frac{d\lambda}{d\tau} \\ &= \lambda(N_1\lambda^2 + N_2\lambda + N_3)e^{-\lambda\tau}. \end{aligned} \quad (3.31)$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2M_1\lambda + M_2)e^{\lambda\tau}}{(N_1\lambda^2 + N_2\lambda + N_3)\lambda} + \frac{2N_1\lambda + N_2}{(N_1\lambda^2 + N_2\lambda + N_3)\lambda} - \frac{\tau}{\lambda}. \quad (3.32)$$

Since for $\tau = \tau_0$ and $\lambda = i\omega_0$, we have

$$\frac{\tau}{\lambda} = -i \frac{\tau_0}{\omega_0}, \quad (3.33)$$

$$2N_1\lambda + N_2 = N_2 + 2iN_1\omega_0, \quad (3.34)$$

$$(N_1\lambda^2 + N_2\lambda + N_3)\lambda = -N_2\omega_0^2 + i\omega_0(N_3 - N_1\omega_0^2), \quad (3.35)$$

and

$$\begin{aligned} (3\lambda^2 + 2M_1\lambda + M_2)e^{\lambda\tau} &= (M_2 - 3\omega_0^2 + i2M_1\omega_0)(\cos \omega_0\tau_0 + i \sin \omega_0\tau_0) \\ &= [(M_2 - 3\omega_0^2) \cos \omega_0\tau_0 - 2M_1\omega_0 \sin \omega_0\tau_0] \\ &\quad + i[2M_1\omega_0 \cos \omega_0\tau_0 + (M_2 - 3\omega_0^2) \sin \omega_0\tau_0]. \end{aligned} \quad (3.36)$$

Then

$$\begin{aligned} Re\left[\frac{d(\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0}^{-1} &= Re\left[\frac{(2N_1\lambda + N_2) + (3\lambda^2 + 2M_1\lambda + M_2)e^{\lambda\tau}}{(N_1\lambda^2 + N_2\lambda + N_3)\lambda} - \frac{\tau}{\lambda}\right]_{\lambda=i\omega_0} \\ &= \frac{1}{M_0}[3\omega_0^6 + 2(M_1^2 - N_1^2 - 2M_2)\omega_0^4 + (M_2^2 - 2M_1M_3 + 2N_1N_3 - N_2^2)\omega_0^2] \\ &= \frac{\omega_0^2}{M_0}[3\omega_0^4 + 2h_1\omega_0^2 + h_2]. \end{aligned}$$

where $M_0 = N_2^2\omega_0^4 + \omega_0^2(N_3 - N_1\omega_0^2)^2 > 0$, $h_1 = M_1^2 - N_1^2 - 2M_2$ and $h_2 = M_2^2 - 2M_1M_3 + 2N_1N_3 - N_2^2$. So, we have

$$sign\left[\frac{d(Re\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0} = sign Re\left[\frac{d(\lambda(\tau))}{d\tau}\right]_{\tau=\tau_0}^{-1} = sign[h(\varpi)],$$

where, $h(\varpi) = 3\varpi^2 + 2h_1\varpi + h_2$ and $\varpi = \omega_0^2 > 0$. Since $h'(\varpi) = 6\varpi + 2h_1 > 0$. Hence, we obtain that $h(\varpi)$ monotonously increases in $[0, +\infty)$. Furthermore, under condition (3.30), we gain $h(0) > 0$ and hence $h(\omega) > 0$ for $\omega > 0$. Consequently, we have the transversal condition (3.29) signifies. This completes the proof. \square

The transversal condition(3.29) signify that Eq.(3.21) has at least one root with positive real part for $\tau \in (\tau_0, \infty)$. Moreover, a Hopf bifurcation occurs when τ passes through the critical value τ_0 .

We summarize the above conclusion on the local stability of interior equilibrium point E_2 and Hopf bifurcation of system (1.1) by the following theorem.

Theorem 3.4. *Assume that the conditions (3.23)-(3.25) and (3.30) hold, then:*

- E_2 is locally asymptotically stable for $\tau < \tau_0$.
- E_2 is unstable for $\tau > \tau_0$.
- System (1.1) undergoes Hopf bifurcations at E_2 for $\tau = \tau_0$.

where τ_0 is defined in equation (3.28).

4. THE DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In the following the direction of the Hopf bifurcations and the stability of the periodic solutions, which arising through the occurrence of Hopf bifurcation around the interior equilibrium point of system (1.1) as the delay parameter passes through the value τ_0 , are investigated with the help of normal form theory and center manifold theorem introduced by Hassard in [27].

Accordingly by normalizing the delay τ by scaling $t \rightarrow \frac{t}{\tau}$ and taking $Y_i(t) = y_i(\tau t) - y_i^*$, $i = 1, 2, 3$ then system(1.1) is transformed to

$$\begin{aligned}\dot{Y}_1 &= \tau[a_1(Y_2 + y_2^*) - a_2(Y_1 + y_1^*)^2 - a_3(Y_1 + y_1^*) - a_4(Y_1 + y_1^*)(Y_3 + y_3^*)], \\ \dot{Y}_2 &= \tau[b_1(Y_1 + y_1^*) - b_2(Y_2 + y_2^*)^2 - (Y_2 + y_2^*) - b_3(Y_2 + y_2^*)(Y_3 + y_3^*)], \\ \dot{Y}_3 &= \tau[(Y_1(t-1) + y_1^*)(Y_3(t-1) + y_3^*) + (Y_2(t-1) + y_2^*)(Y_3(t-1) + y_3^*) \\ &\quad - (Y_3 + y_3^*)^2 - b_4(Y_3 + y_3^*)].\end{aligned}$$

So by taking $\tau = \tau_0 + \mu$, and linearize the system around $(0, 0, 0)$, we get

$$\begin{aligned}\dot{Y}_1 &= (\tau_0 + \mu)[c_{11}Y_1 + c_{12}Y_2 + c_{13}Y_3], \\ \dot{Y}_2 &= (\tau_0 + \mu)[c_{21}Y_1 + c_{22}Y_2 + c_{23}Y_3], \\ \dot{Y}_3 &= (\tau_0 + \mu)[R_1Y_1(t-1) + R_1Y_2(t-1) + R_2Y_3(t-1) + R_3Y_3(t)].\end{aligned}$$

where c_{ij} , R_1, R_2 and R_3 are given in Eq.(3.20) with τ_0 defined in Eq.(3.28) and $\mu \in R$. Then system(??)is transformed into a functional differential equation in $C = C([-1, 0], R^3)$ as

$$\dot{Y} = L_\mu(Y_t) + F(\mu, Y_t), \quad (4.1)$$

here $Y(t) = (Y_1(t), Y_2(t), Y_3(t))^T \in R^3$, and $L_\mu : C \rightarrow R^3$, $F : R \times C \rightarrow R^3$ are given by

$$L_\mu(\phi) = (\tau_k + \mu)[H_1\phi(0) + H_2\phi(-1)],$$

And

$$F(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} -a_2\phi_1^2(0) - a_4\phi_1(0)\phi_3(0) \\ -b_2\phi_2^2(0) - b_3\phi_2(0)\phi_3(0) \\ \phi_1(-1)\phi_3(-1) + \phi_2(-1)\phi_3(-1) - \phi_3^2(0) \end{pmatrix},$$

where H_1 and H_2 are defined as

$$H_1 = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & R_3 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_1 & R_1 & R_2 \end{pmatrix},$$

while $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C^1([-1, 0], R^3)$.

By the Riesz representation theorem, there exists a matrix $\eta(\theta, \mu)$ whose components are bounded variation functions such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\mu), \quad \phi \in C^1([-1, 0], R^3).$$

In fact we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)H_1\delta(\theta) - (\tau_0 + \mu)H_2\delta(\theta + 1),$$

where δ in the Dirac delta function. Now, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu)\phi(s) = L_\mu(\phi), & \theta = 0. \end{cases}$$

and

$$B(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0); \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Thus system (4.1) is equivalent to

$$\dot{Y} = A(\mu)Y_t + B(\mu)Y_t, \quad (4.2)$$

where $Y_t(\theta) = Y(t + \theta)$, $\theta \in [-1, 0]$. Further for $\varphi(s) = (\varphi_1(s), \varphi_2(s), \varphi_3(s))^T \in C^1([-1, 0], \mathbb{R}^3)$, we define

$$A^*(\mu)\varphi = \begin{cases} -\frac{d\varphi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0)\varphi(-t), & s = 0. \end{cases}$$

Then we define bilinear inner product by

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}^T(0)\phi(0) - \int_{-1}^0 \int_0^\theta \bar{\varphi}^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \quad (4.3)$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A = A(0)$ and $A^* = A^*(0)$ are adjoint operators. Moreover, for $\mu = 0$ it is clear that $\pm i\omega_0\tau_0$ are the eigenvalues of A . Thus, $\pm i\omega_0\tau_0$ are the eigenvalues of A^* . Furthermore the corresponding eigenvectors are established in the following theorem.

Theorem 4.1. *Let $q(\theta)$ be the eigenvector of A associated with the eigenvalue $i\omega_0\tau_0$ and $q^*(\theta)$ be the eigenvector of A^* associated with the eigenvalue $-i\omega_0\tau_0$. Then*

$$\begin{aligned} q(\theta) &= (1, \alpha_1, \alpha_2)^T e^{i\theta\omega_0\tau_0}, \\ q^*(s) &= D(1, \alpha_1^*, \alpha_2^*) e^{is\omega_0\tau_0}, \end{aligned}$$

where,

$$\begin{aligned} \alpha_1 &= \frac{(c_{21}c_{13} - c_{11}c_{23}) + i\omega_0c_{23}}{(c_{12}c_{23} - c_{22}c_{13}) + i\omega_0c_{13}}, \\ \alpha_2 &= \frac{R_1(1 + \alpha_1)e^{-i\omega_0\tau_0}}{-(R_3 + R_2e^{i\omega_0\tau_0} - i\omega_0)}, \\ \alpha_1^* &= \frac{(c_{11} - c_{12}) + i\omega_0}{(c_{22} - c_{21}) + i\omega_0}, \\ \alpha_2^* &= \frac{c_{13} + c_{23}\alpha_1^*}{-(R_3 + R_2e^{i\omega_0\tau_0} + i\omega_0)}. \end{aligned}$$

So that the following quantities hold:

$$\langle q^*, q \rangle = 1,$$

$$\langle q^*, \bar{q} \rangle = 0.$$

Proof. Suppose that $q(\theta) = (1, \alpha_1, \alpha_2)^T e^{i\theta\omega_0\tau_0}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_0$, then

$$A(0)q(\theta) = i\omega_0\tau_0q(\theta).$$

So, we obtain

$$A(0)q(0)e^{i\theta\omega_0\tau_0} = i\omega_0\tau_0q(0)e^{i\theta\omega_0\tau_0}$$

From the definition of $A(0)$ we have

$$\tau_0 \begin{pmatrix} i\omega_0 - c_{11} & -c_{12} & -c_{13} \\ -c_{21} & i\omega_0 - c_{22} & -c_{23} \\ -R_1 e^{-i\omega_0\tau_0} & -R_1 e^{-i\omega_0\tau_0} & i\omega_0 - R_3 - R_2 e^{-i\omega_0\tau_0} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which yields $q(0) = (1, \alpha_1, \alpha_2)^T$, where

$$\alpha_1 = \frac{(c_{21}c_{13} - c_{11}c_{23}) + i\omega_0c_{23}}{(c_{12}c_{23} - c_{22}c_{13}) + i\omega_0c_{13}},$$

$$\alpha_2 = \frac{R_1(1 + \alpha_1)e^{-i\omega_0\tau_0}}{-(R_3 + R_2e^{-i\omega_0\tau_0} - i\omega_0)}.$$

On the other hand, suppose that $q^*(s) = D(1, \alpha_1^*, \alpha_2^*)^T e^{is\omega_0\tau_0}$ is the eigenvector of A^* corresponding to $-i\omega_0\tau_0$, then

$$A^*q^*(s) = -i\omega_0\tau_0q^*(s).$$

From the definition of A^* we have

$$\tau_0 \begin{pmatrix} i\omega_0 + c_{11} & c_{21} & R_1 e^{i\omega_0\tau_0} \\ c_{12} & i\omega_0 + c_{22} & R_1 e^{i\omega_0\tau_0} \\ c_{13} & c_{23} & i\omega_0 + R_3 + R_2 e^{i\omega_0\tau_0} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_1^* \\ \alpha_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which yields

$$\alpha_1^* = \frac{(c_{11} - c_{12}) + i\omega_0}{(c_{22} - c_{21}) + i\omega_0}, \quad \alpha_2^* = \frac{c_{13} + c_{23}\alpha_1^*}{-(R_3 + R_2 e^{i\omega_0\tau_0} + i\omega_0)}.$$

Now to compute the parameter D , and show that $\langle q^*, q \rangle = 1$, $\langle q^*, \bar{q} \rangle = 0$, Eq.(4.3) can be used and then we get

$$\begin{aligned}
\langle q^*, q \rangle &= \bar{q}^{*T}(0)q(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi - \theta) d\eta(\theta) q(\xi) d\xi \\
&= \bar{D}(1, \bar{\alpha}_1^*, \bar{\alpha}_2^*)(1, \alpha_1, \alpha_2)^T - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha}_1^*, \bar{\alpha}_2^*) e^{-i(\xi-\theta)\omega_0\tau_0} \times \\
&\quad d\eta(\theta)(1, \alpha_1, \alpha_2)^T e^{i\xi\omega_0\tau_0} d\xi \\
&= \bar{D}\{1 + \bar{\alpha}_1^*\alpha_1 + \bar{\alpha}_2^*\alpha_2 - \int_{-1}^0 (1, \bar{\alpha}_1^*, \bar{\alpha}_2^*)\theta e^{i\theta\omega_0\tau_0} d\eta(\theta)(1, \alpha_1, \alpha_2)^T\} \\
&= \bar{D}\{1 + \bar{\alpha}_1^*\alpha_1 + \bar{\alpha}_2^*\alpha_2 + (1, \bar{\alpha}_1^*, \bar{\alpha}_2^*)(\tau_k H_2 e^{-i\omega_0\tau_0})(1, \alpha_1, \alpha_2)^T\} \\
&= \bar{D}\{1 + \bar{\alpha}_1^*\alpha_1 + \bar{\alpha}_2^*\alpha_2 + \tau_0 \bar{\alpha}_2^*(R_1 + \alpha_1 R_1 + \alpha_2 R_2) e^{-i\omega_0\tau_0}\}
\end{aligned}$$

Thus, if we take

$$\bar{D} = \{1 + \bar{\alpha}_1^*\alpha_1 + \bar{\alpha}_2^*\alpha_2 + \tau_0 \bar{\alpha}_2^*(R_1 + \alpha_1 R_1 + \alpha_2 R_2) e^{-i\omega_0\tau_0}\}^{-1},$$

or

$$D = \{1 + \alpha_1^* \bar{\alpha}_1 + \alpha_2^* \bar{\alpha}_2 + \tau_0 \alpha_2^*(R_1 + \bar{\alpha}_1 R_1 + \bar{\alpha}_2 R_2) e^{i\omega_0\tau_0}\}^{-1},$$

Then we obtain $\langle q^*, q \rangle = 1$.

Moreover, by using the adjoint property $\langle \varphi, A\phi \rangle = \langle A^* \varphi, \phi \rangle$ it is easy to verify that

$$\begin{aligned}
i\omega_0\tau_0 \langle q^*, \bar{q} \rangle &= \langle -i\omega_0\tau_0 q^*, \bar{q} \rangle \\
&= \langle A^* q^*, \bar{q} \rangle \\
&= \langle q^*, A\bar{q} \rangle \\
&= \langle q^*, i\omega_0\tau_0 \bar{q} \rangle \\
&= -i\omega_0\tau_0 \langle q^*, \bar{q} \rangle.
\end{aligned}$$

Therefore, we obtain $\langle q^*, \bar{q} \rangle = 0$, and the proof is complete.

Keeping the above in view, to study the stability of the periodic solutions those bifurcates as $\tau = \tau_0$, we start with computing the coordinate that describes the center manifold C_1 at $\mu = 0$. Let $v_t = (v_{1t}, v_{2t}, v_{3t})$ be the solution of Eq.(4.1) at $\mu = 0$ and $Z(t) = \langle q^*, v_t \rangle$. Define

$$W(t, \theta) = v_t(\theta) - Z(t)q(\theta) - \bar{Z}(t)\bar{q}(\theta) = v_t(\theta) - 2\text{Re}\{Z(t)q(\theta)\}. \quad (4.4)$$

On the center manifold C_1 we have $W(t, \theta) = W(Z(t), \bar{Z}(t), \theta)$ where

$$W(Z, \bar{Z}, \theta) = W_{20}(\theta) \frac{Z^2}{2} + W_{11}(\theta) Z\bar{Z} + W_{02}(\theta) \frac{\bar{Z}^2}{2} + \dots \quad (4.5)$$

Z and \bar{Z} are local coordinates of center manifold C_1 in the direction of q^* and \bar{q}^* . Clearly W is real when v_t is real. Hence only real solutions are considered. According to Eq.(4.4) we have

$$\langle q^*, W \rangle = \langle q^*, v_t - Zq - \bar{Z}\bar{q} \rangle = \langle q^*, v_t \rangle - Z(t)\langle q^*, q \rangle - \bar{Z}(t)\langle q^*, \bar{q} \rangle = 0.$$

Now for a solution $v_t \in C_1$ of Eq.(22), with $\mu = 0$ and Eq.(25) we have

$$\begin{aligned}
\dot{Z}(t) &= \langle q^*, \dot{v}_t \rangle = \langle q^*, A(0)v_t + B(0)v_t \rangle \\
&= \langle q^*, A(0)v_t \rangle + \langle q^*, B(0)v_t \rangle \\
&= \langle A^* q^*, v_t \rangle + \langle q^*, F(0, v_t) \rangle \\
&= \langle A^* q^*, v_t \rangle + \langle q^*, F(0, W(Z, \bar{Z}, 0) + 2Re\{Z(t)q(0)\}) \rangle \\
&= i\omega_0\tau_0 \langle q^*, v_t \rangle + \bar{q}^{*T} F(0, W(Z, \bar{Z}, 0) + 2Re\{Z(t)q(0)\}) \\
&= i\omega_0\tau_0 Z(t) + \bar{q}^{*T} f_0(Z, \bar{Z}).
\end{aligned} \tag{4.6}$$

Rewrite the above equation as

$$\dot{Z}(t) = i\omega_0\tau_0 Z(t) + g(Z, \bar{Z}), \tag{4.7}$$

where

$$\begin{aligned}
g(Z, \bar{Z}) &= \bar{q}^{*T}(0)f_0(Z, \bar{Z}) = \bar{q}^{*T}(0)F(0, W(Z, \bar{Z}, 0) + 2Re\{Z(t)q(0)\}) \\
&= g_{20} \frac{Z^2}{2} + g_{11} Z\bar{Z} + g_{02} \frac{\bar{Z}^2}{2} + \dots
\end{aligned} \tag{4.8}$$

From (4.4), (4.6) and definition of B, the following is obtained

$$\begin{aligned}
\dot{W} &= \dot{v}_t - \dot{Z}q - \dot{\bar{Z}}\bar{q} \\
&= Av_t + Bv_t - i\omega_0\tau_0 Zq - \bar{q}^{*T} f_0(Z, \bar{Z})q + i\omega_0\tau_0 \bar{Z}\bar{q} - \bar{q}^{*T} \overline{f_0(Z, \bar{Z})\bar{q}} \\
&= Av_t + Bv_t - AZq - A\bar{Z}\bar{q} - 2Re\{\bar{q}^{*T} f_0(Z, \bar{Z})q\} \\
&= AW + Bv_t - 2Re\{\bar{q}^{*T} f_0(Z, \bar{Z})q\} \\
&= \begin{cases} AW - 2Re\{\bar{q}^{*T} f_0(Z, \bar{Z})q\}, & \theta \in [-1, 0), \\ AW + f_0(Z, \bar{Z}) - 2Re\{\bar{q}^{*T} f_0(Z, \bar{Z})q\}, & \theta = 0. \end{cases}
\end{aligned} \tag{4.9}$$

The above equation can be rewritten as

$$\dot{W} = AW + H(Z, \bar{Z}, \theta), \tag{4.10}$$

where

$$H(Z, \bar{Z}, \theta) = H_{20}(\theta) \frac{Z^2}{2} + H_{11}(\theta) Z\bar{Z} + H_{02}(\theta) \frac{\bar{Z}^2}{2} + \dots \tag{4.11}$$

On the other hand , on C_1 , we know that

$$\dot{W} = W_Z \dot{Z} + W_{\bar{Z}} \dot{\bar{Z}}.$$

Now by using Eq.(4.5) and Eq.(4.7) in the above equation , we get

$$W = i\omega_0\tau_0 W_{20}(\theta) Z^2 - i\omega_0\tau_0 W_{02}(\theta) \bar{Z}^2 + \dots$$

This equation , together with Eq.(4.5) and Eq.(4.10) , give that

$$H(Z, \bar{Z}, \theta) = (2i\omega_0\tau_0 - A)W_{20}(\theta) \frac{Z^2}{2} - AW_{11}(\theta) Z\bar{Z} - (2i\omega_0\tau_0 + A)W_{02}(\theta) \frac{\bar{Z}^2}{2} + \dots$$

By comparing the coefficients in the last equation with those in Eq.(4.11) , we obtain

$$(A - 2i\omega_0\tau_0)W_{20}(\theta) = -H_{20}(\theta) \quad (4.12)$$

$$AW_{11}(\theta) = -H_{11}(\theta) \quad (4.13)$$

$$(A + 2i\omega_0\tau_0)W_{02}(\theta) = -H_{02}(\theta) \quad (4.14)$$

Moreover,from (4.8)-(4.10), we have for $\theta \in [-1, 0)$

$$\begin{aligned} H(Z, \bar{Z}, \theta) &= -\bar{q}^{*T}(0)f_0(Z, \bar{Z})q(\theta) - q^{*T}(0)\overline{f_0(Z, \bar{Z})}\bar{q}(\theta) \\ &= -g(Z, \bar{Z})q(\theta) - \overline{g(Z, \bar{Z})}\bar{q}(\theta) \\ &= -\left\{g_{20}\frac{Z^2}{2} + g_{11}Z\bar{Z} + g_{02}\frac{\bar{Z}^2}{2} + \dots\right\}q(\theta) \\ &\quad - \left\{\bar{g}_{02}\frac{Z^2}{2} + \bar{g}_{11}Z\bar{Z} + \bar{g}_{20}\frac{\bar{Z}^2}{2} + \dots\right\}\bar{q}(\theta). \end{aligned}$$

Again comparing the coefficients with those in Eq.(4.11), gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \quad (4.15)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), \quad (4.16)$$

$$H_{02}(\theta) = -g_{02}q(\theta) - \bar{g}_{20}\bar{q}(\theta). \quad (4.17)$$

Therefore using definition of A together with (4.12) and (4.15) gives that

$$\begin{aligned} \dot{W}_{20}(\theta) &= AW_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) - H_{20}(\theta) \\ &= 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta) \\ &= 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(0)e^{i\omega_0\tau_0\theta} + \bar{g}_{02}\bar{q}(0)e^{-i\omega_0\tau_0\theta}. \end{aligned}$$

Solving the above equation for $W_{20}(\theta)$ gives

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta}. \quad (4.18)$$

Similarly, Eq.(4.13) and (4.16)give that

$$\dot{W}_{11}(\theta) = AW_{11}(\theta) = g_{11}q(0)e^{i\omega_0\tau_0\theta} + \bar{g}_{11}q(0)e^{-i\omega_0\tau_0\theta},$$

and then we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}q(0)e^{-i\omega_0\tau_0\theta} + E_2, \quad (4.19)$$

where E_1, E_2 are both three dimensional arbitrary constant vectors and can be found by setting $\theta = 0$ in $H(Z, \bar{Z}, \theta)$.

Now in view of Eq.(4.7), we have

$$\begin{pmatrix} v_1(t+\theta) \\ v_2(t+\theta) \\ v_3(t+\theta) \end{pmatrix} = Z \begin{pmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} e^{i\omega_0\tau_0\theta} + \bar{Z} \begin{pmatrix} 1 \\ \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{pmatrix} e^{-i\omega_0\tau_0\theta} + \begin{pmatrix} W^{(1)}(Z, \bar{Z}, \theta) \\ W^{(2)}(Z, \bar{Z}, \theta) \\ W^{(3)}(Z, \bar{Z}, \theta) \end{pmatrix}.$$

Thus , we can get that

$$\begin{aligned}\nu_{1t} &= Ze^{i\omega_0\tau_0\theta} + \bar{Z}e^{-i\omega_0\tau_0\theta} + W_{20}^{(1)}\frac{Z^2}{2} + W_{11}^{(1)}Z\bar{Z} + W_{02}^{(1)}\frac{Z^2}{2} + \dots, \\ \nu_{2t} &= Z\alpha_1e^{i\omega_0\tau_0\theta} + \bar{Z}\bar{\alpha}_1e^{-i\omega_0\tau_0\theta} + W_{20}^{(2)}\frac{Z^2}{2} + W_{11}^{(2)}Z\bar{Z} + W_{02}^{(2)}\frac{Z^2}{2} + \dots,\end{aligned}$$

and

$$\nu_{3t} = Z\alpha_2e^{i\omega_0\tau_0\theta} + \bar{Z}\bar{\alpha}_2e^{-i\omega_0\tau_0\theta} + W_{20}^{(3)}\frac{Z^2}{2} + W_{11}^{(3)}Z\bar{Z} + W_{02}^{(3)}\frac{Z^2}{2} + \dots.$$

Hance it is easy to verify that

$$\begin{aligned}\phi_1(0) &= Z + \bar{Z} + W_{20}^{(1)}(0)\frac{Z^2}{2} + W_{11}^{(1)}(0)Z\bar{Z} + W_{02}^{(1)}(0)\frac{Z^2}{2} + \dots, \\ \phi_2(0) &= Z\alpha_1 + \bar{Z}\bar{\alpha}_1 + W_{20}^{(2)}(0)\frac{Z^2}{2} + W_{11}^{(2)}(0)Z\bar{Z} + W_{02}^{(2)}(0)\frac{Z^2}{2} + \dots, \\ \phi_3(0) &= Z\alpha_2 + \bar{Z}\bar{\alpha}_2 + W_{20}^{(3)}(0)\frac{Z^2}{2} + W_{11}^{(3)}(0)Z\bar{Z} + W_{02}^{(3)}(0)\frac{Z^2}{2} + \dots, \\ \phi_1(-1) &= Ze^{-i\omega_0\tau_0} + \bar{Z}e^{i\omega_0\tau_0} + W_{20}^{(1)}(-1)\frac{Z^2}{2} + W_{11}^{(1)}(-1)Z\bar{Z} + W_{02}^{(1)}(-1)\frac{Z^2}{2} + \dots, \\ \phi_2(-1) &= Z\alpha_1e^{-i\omega_0\tau_0} + \bar{Z}\bar{\alpha}_1e^{i\omega_0\tau_0} + W_{20}^{(2)}(-1)\frac{Z^2}{2} + W_{11}^{(2)}(-1)Z\bar{Z} \\ &\quad + W_{02}^{(2)}(-1)\frac{Z^2}{2} + \dots, \\ \phi_3(-1) &= Z\alpha_2e^{-i\omega_0\tau_0} + \bar{Z}\bar{\alpha}_2e^{i\omega_0\tau_0} + W_{20}^{(3)}(-1)\frac{Z^2}{2} + W_{11}^{(3)}(-1)Z\bar{Z} \\ &\quad + W_{02}^{(3)}(-1)\frac{Z^2}{2} + \dots, \\ \phi_1^2(0) &= Z^2 + 2Z\bar{Z} + \bar{Z}^2 + [W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)]Z^2\bar{Z} + \dots, \\ \phi_2^2(0) &= Z^2\alpha_1^2 + 2\alpha_1\bar{\alpha}_1Z\bar{Z} + \bar{Z}^2\bar{\alpha}_1^2 + [\bar{\alpha}_1W_{20}^{(2)}(0) + 2\alpha_1W_{11}^{(2)}(0)]Z^2\bar{Z} + \dots, \\ \phi_3^2(0) &= Z^2\alpha_2^2 + 2\alpha_2\bar{\alpha}_2Z\bar{Z} + \bar{Z}^2\bar{\alpha}_2^2 + [\bar{\alpha}_2W_{20}^{(2)}(0) + 2\alpha_2W_{11}^{(2)}(0)]Z^2\bar{Z} + \dots, \\ \phi_1(0)\phi_3(0) &= Z^2\alpha_2 + (\alpha_2 + \bar{\alpha}_2)Z\bar{Z} + \bar{Z}^2\bar{\alpha}_2 + [\alpha_2W_{11}^{(1)}(0) + \frac{1}{2}\bar{\alpha}_2W_{20}^{(1)}(0) \\ &\quad + \frac{1}{2}W_{20}^3(0) + W_{11}^3(0)]Z^2\bar{Z} + \dots, \\ \phi_2(0)\phi_3(0) &= Z^2\alpha_1\alpha_2 + (\alpha_2\bar{\alpha}_1 + \alpha_1\bar{\alpha}_2)Z\bar{Z} + \bar{Z}^2\bar{\alpha}_1\bar{\alpha}_2 + [\alpha_2W_{11}^{(2)}(0) \\ &\quad + \frac{1}{2}\bar{\alpha}_2W_{20}^{(2)}(0) + \alpha_1W_{11}^3(0) + \frac{1}{2}\bar{\alpha}_1W_{20}^3(0)]Z^2\bar{Z} + \dots, \\ \phi_1(-1)\phi_3(-1) &= Z^2\alpha_2e^{-i2\omega_0\tau_0} + (\alpha_2 + \bar{\alpha}_2)Z\bar{Z} + \bar{Z}^2\bar{\alpha}_2e^{i2\omega_0\tau_0} \\ &\quad + [\alpha_2W_{11}^{(1)}(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}\bar{\alpha}_2W_{20}^{(1)}(-1)e^{i\omega_0\tau_0} \\ &\quad + W_{11}^3(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}W_{20}^3(-1)e^{i\omega_0\tau_0}]Z^2\bar{Z} + \dots,\end{aligned}$$

$$\begin{aligned}
\phi_2(-1)\phi_3(-1) &= Z^2\alpha_1\alpha_2e^{-i2\omega_0\tau_0} + Z\bar{Z}\alpha_2\bar{\alpha}_2 + \bar{Z}^2\bar{\alpha}_1\bar{\alpha}_2e^{i2\omega_0\tau_0} \\
&+ [\alpha_2W_{11}^{(2)}(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}\bar{\alpha}_2W_{20}^{(2)}(-1)e^{i\omega_0\tau_0} \\
&+ \alpha_1W_{11}^3(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}\bar{\alpha}_1W_{20}^3(-1)e^{i\omega_0\tau_0}]Z^2\bar{Z} + \dots,
\end{aligned}$$

Therefore the function f_0 can be expressed as

$$f_0(Z, \bar{Z}) = \tau_0 \begin{pmatrix} P_1Z^2 + P_2Z\bar{Z} + P_3\bar{Z}^2 + P_4Z^2\bar{Z} \\ P_5Z^2 + P_6Z\bar{Z} + P_7\bar{Z}^2 + P_8Z^2\bar{Z} \\ P_9Z^2 + P_{10}Z\bar{Z} + P_{11}\bar{Z}^2 + P_{12}Z^2\bar{Z} \end{pmatrix}$$

where

$$\begin{aligned}
P_1 &= -(a_2 + a_4\alpha_2), P_2 = -[2a_2 + (\alpha_2 + \bar{\alpha}_2)a_4], P_3 = -(a_2 + a_4\bar{\alpha}_2), \\
P_4 &= -[(\alpha_2 + \frac{1}{2}\bar{\alpha}_2a_4)W_{20}^1(0) + (2a_2 + \alpha_2a_4)W_{11}^1(0) + a_4W_{11}^3(0) + \frac{1}{2}a_4W_{20}^3(0)], \\
P_5 &= -(\alpha_1^2b_2 + \alpha_1\alpha_2b_3), P_6 = -2\alpha_1\bar{\alpha}_1b_2 - (\alpha_2\bar{\alpha}_1 + \alpha_1\bar{\alpha}_2)b_3, P_7 = -(\bar{\alpha}_1^2b_2 + \bar{\alpha}_1\bar{\alpha}_2b_3), \\
P_8 &= -[(2\alpha_1b_2 + \alpha_2b_3)W_{11}^2(0) + \alpha_1b_3W_{11}^3(0) + (\bar{\alpha}_1b_2 + \frac{1}{2}\bar{\alpha}_2b_3)W_{20}^2(0) + \frac{1}{2}\bar{\alpha}_1b_3W_{20}^3(0)], \\
P_9 &= (\alpha_2 + \alpha_1\alpha_2)e^{-i2\omega_0\tau_0} - \alpha_2^2, P_{10} = (\alpha_2 + \bar{\alpha}_2 + \alpha_1\bar{\alpha}_2 + \alpha_2\bar{\alpha}_1 - 2\alpha_2\bar{\alpha}_2), P_{11} = -\bar{\alpha}_2^2 \\
&+ (\bar{\alpha}_2 + \bar{\alpha}_1\bar{\alpha}_2)e^{i2\omega_0\tau_0}, P_{12} = [(W_{11}^1(-1) + W_{11}^3(-1))\alpha_2e^{-i\omega_0\tau_0} + (\alpha_1 + 1)W_{11}^3(-1)e^{-i\omega_0\tau_0} \\
&+ \frac{1}{2}(W_{20}^1(-1) + W_{20}^2(-1))\bar{\alpha}_2e^{i\omega_0\tau_0} + \frac{1}{2}(\bar{\alpha}_1 + 1)W_{20}^3(-1)e^{i\omega_0\tau_0} - \bar{\alpha}_2W_{20}^2(0) - 2\alpha_2W_{11}^2(0)].
\end{aligned}$$

Note that since $\bar{q}^{*T}(0) = \bar{D}(1, \bar{\alpha}_1^*, \bar{\alpha}_2^*)$,

Then we get

$$\begin{aligned}
g(Z, \bar{Z}) &= \bar{q}^{*T}(0)f_0(Z, \bar{Z}) \\
&= \tau_0\bar{D}[(P_1 + \bar{\alpha}_1^*P_5 + \bar{\alpha}_2^*P_9)Z^2 + (P_2 + \bar{\alpha}_1^*P_6 + \bar{\alpha}_2^*P_{10})Z\bar{Z} \\
&+ (P_3 + \bar{\alpha}_1^*P_7 + \bar{\alpha}_2^*P_{11})\bar{Z}^2 + (P_4 + \bar{\alpha}_1^*P_8 + \bar{\alpha}_2^*P_{12})Z^2\bar{Z}].
\end{aligned}$$

Comparing the coefficients in the above equation with those of Eq.(4.8), we obtain

$$g_{20} = 2\tau_0\bar{D}(P_1 + \bar{\alpha}_1^*P_5 + \bar{\alpha}_2^*P_9), \quad (4.20)$$

$$g_{11} = \tau_0\bar{D}(P_2 + \bar{\alpha}_1^*P_6 + \bar{\alpha}_2^*P_{10}), \quad (4.21)$$

$$g_{02} = 2\tau_0\bar{D}(P_3 + \bar{\alpha}_1^*P_7 + \bar{\alpha}_2^*P_{11}), \quad (4.22)$$

$$g_{21} = 2\tau_0\bar{D}(P_4 + \bar{\alpha}_1^*P_8 + \bar{\alpha}_2^*P_{12}). \quad (4.23)$$

Now in order to evaluate g_{ij} we need to compute W_{20} and W_{11} . From (4.10) with $\theta = 0$, we have

$$\begin{aligned}
H(Z, \bar{Z}, 0) &= -2Re\{\bar{q}^{*T}f_0(Z, \bar{Z})q\} + f_0(Z, \bar{Z}) \\
&= -\{g_{20}\frac{Z^2}{2} + g_{11}Z\bar{Z} + g_{02}\frac{\bar{Z}^2}{2} + \dots\}q(0) \\
&- \{\bar{g}_{02}\frac{Z^2}{2} + \bar{g}_{11}Z\bar{Z} + \bar{g}_{20}\frac{\bar{Z}^2}{2} + \dots\}\bar{g}(0) + f_0(Z, \bar{Z}).
\end{aligned}$$

Comparing the coefficients here with those in Eq.(4.11) , it shows that

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_0 \begin{pmatrix} P_1 \\ P_5 \\ P_9 \end{pmatrix}, \quad (4.24)$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_0 \begin{pmatrix} P_2 \\ P_6 \\ P_{10} \end{pmatrix}, \quad (4.25)$$

$$H_{02}(0) = -g_{02}q(0) - \bar{g}_{20}\bar{q}(0) + 2\tau_0 \begin{pmatrix} P_3 \\ P_7 \\ P_{11} \end{pmatrix}. \quad (4.26)$$

According to the definition of $A(0)$,together with Eq.(4.18) and Eq.(4.19), we obtain

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0W_{20}(0) - H_{20}(0), \quad (4.27)$$

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0). \quad (4.28)$$

Since $Aq(0) = i\omega_0\tau_0q(0)$ and $q(\theta) = q(0)e^{i\theta\omega_0\tau_0}$, we obtain

$$\int_{-1}^0 d\eta(\theta)q(0)e^{i\theta\omega_0\tau_0} = i\omega_0\tau_0q(0),$$

$$\int_{-1}^0 d\eta(\theta)\bar{q}(0)e^{-i\theta\omega_0\tau_0} = -i\omega_0\tau_0\bar{q}(0).$$

Therefore

$$(i\omega_0\tau_0I - \int_{-1}^0 d\eta(\theta)e^{i\theta\omega_0\tau_0})q(0) = 0, \quad (4.29)$$

and

$$(-i\omega_0\tau_0I - \int_{-1}^0 d\eta(\theta)e^{-i\theta\omega_0\tau_0})\bar{q}(0) = 0. \quad (4.30)$$

Substituting (4.18) and (4.24) into (4.27) and using (4.29), we obtain that

$$(i2\omega_0\tau_0I - \int_{-1}^0 d\eta(\theta)e^{2i\theta\omega_0\tau_0})E_1 = 2\tau_0 \begin{pmatrix} P_1 \\ P_5 \\ P_9 \end{pmatrix},$$

which gives

$$\begin{pmatrix} i2\omega_0 - c_{11} & -c_{12} & -c_{13} \\ -c_{21} & i2\omega_0 - c_{22} & -c_{23} \\ -R_1e^{-2i\theta\omega_0\tau_0} & -R_1e^{-2i\theta\omega_0\tau_0} & i2\omega_0 - R_2e^{-2i\theta\omega_0\tau_0} - R_3 \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} P_1 \\ P_5 \\ P_9 \end{pmatrix},$$

Thus, after solving the above equation and letting

$$\Delta_1 = \begin{vmatrix} i2\omega_0 - c_{11} & -c_{12} & -c_{13} \\ -c_{21} & i2\omega_0 - c_{22} & -c_{23} \\ -R_1 e^{-2i\theta\omega_0\tau_0} & -R_1 e^{-2i\theta\omega_0\tau_0} & i2\omega_0 - R_2 e^{-2i\theta\omega_0\tau_0} - R_3 \end{vmatrix}$$

we obtain that

$$E_1^{(1)} = \frac{2}{\Delta_1} \begin{vmatrix} P_1 & -c_{12} & -c_{13} \\ P_5 & i2\omega_0 - c_{22} & -c_{23} \\ P_9 & -R_1 e^{-2i\theta\omega_0\tau_0} & 2i\omega_0 - R_2 e^{-2i\theta\omega_0\tau_0} - R_3 \end{vmatrix}$$

$$E_1^{(2)} = \frac{2}{\Delta_1} \begin{vmatrix} i2\omega_0\tau_0 - c_{11} & P_1 & -c_{13} \\ -c_{21} & P_5 & -c_{23} \\ -R_1 e^{-2i\theta\omega_0\tau_0} & P_9 & i2\omega_0\tau_0 - R_2 e^{-2i\theta\omega_0\tau_0} - R_3 \end{vmatrix}$$

$$E_1^{(3)} = \frac{2}{\Delta_1} \begin{vmatrix} i2\omega_0\tau_0 - c_{11} & -c_{12} & P_1 \\ -c_{21} & i2\omega_0 - c_{22} & P_5 \\ -R_1 e^{-2i\theta\omega_0\tau_0} & R_1 e^{-2i\theta\omega_0\tau_0} & P_9 \end{vmatrix}$$

By similar discussion, substituting (4.18) and (4.25) into (4.28) and applying (4.30), one can obtain

$$\left(\int_{-1}^0 d\eta(\theta) \right) E_2 = -\tau_0 \begin{pmatrix} P_2 \\ P_6 \\ P_{10} \end{pmatrix}$$

That is

$$\begin{pmatrix} -c_{11} & -c_{12} & -c_{13} \\ -c_{21} & -c_{22} & -c_{23} \\ -R_1 & -R_1 & -R_2 - R_3 \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = 2 \begin{pmatrix} P_2 \\ P_6 \\ P_{10} \end{pmatrix},$$

Hance, we obtain

$$E_2^{(1)} = \frac{1}{\Delta_2} \begin{vmatrix} P_2 & -c_{12} & -c_{13} \\ P_6 & -c_{22} & -c_{23} \\ P_{10} & -R_1 & -R_2 - R_3 \end{vmatrix}$$

$$E_2^{(2)} = \frac{1}{\Delta_2} \begin{vmatrix} -c_{11} & P_2 & -c_{13} \\ -c_{21} & P_6 & -c_{23} \\ -R_1 & P_{10} & -R_2 - R_3 \end{vmatrix}$$

$$E_2^{(3)} = \frac{1}{\Delta_2} \begin{vmatrix} -c_{11} & -c_{12} & P_2 \\ -c_{21} & -c_{22} & P_6 \\ -R_1 & -R_1 & P_{10} \end{vmatrix}$$

where

$$\Delta_2 = \begin{vmatrix} -c_{11} & -c_{12} & -c_{13} \\ -c_{21} & -c_{22} & -c_{23} \\ -R_1 & -R_1 & -R_2 - R_3 \end{vmatrix}$$

Therefore, it is easy to verify that W_{20} and W_{11} can be determined using Eq.(4.18) and Eq.(4.19) respectively and hence we can evaluate all g_{ij} with the help of (4.20) - (4.23). Consequently, we can calculate the following quantities:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0\tau_0}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}} \\ \beta_2 &= 2Re\{C_1(0)\}, \\ T_2 &= -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0}. \end{aligned} \quad (4.31)$$

which determine respectively the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_0 ; While μ_2 determines the direction of the Hopf bifurcation so that for $\mu_2 > 0$ ($\mu_2 < 0$) the Hopf bifurcation is supercritical (subcritical); Further β_2 determines the stability of the bifurcating periodic solutions so that the periodic solutions are stable (unstable) when $\beta_2 < 0$ ($\beta_2 > 0$); Finally T_2 determines the period of the bifurcating solutions so that the periodic increase (decrease) if $T_2 > 0$ ($T_2 < 0$). then we have the following theorem.

Theorem 4.2. *Assume that the conditions (3.25) and (3.30) hold. then, system (1.1) undergoes a stable supercritical Hopf bifurcation as τ crosses τ_0 if $Re\{C_1(0)\} < 0$, while it has unstable subcritical Hopf bifurcation when $Re\{C_0(0)\} > 0$.*

5. NUMERICAL ANALYSIS

In this section, numerical simulation of system (1.1) is applied to confirm our obtained analytical results in the above sections for the following set of biologically feasible hypothetical parameter values

$$S_1 = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 = 3, 0.1, 0.4, 0.5, 0.4, 0.1, 0.5, 0.2\}.$$

System (1.1) is solved numerically with the help of Matlab software. It is observed that for the parameter set given by S_1 with $\tau = 0$ system (1.1) has a globally asymptotically stable interior equilibrium point $E_2 = (0.7483, 0.2135, 0.7618)$ starting from different sets of initial values as show in Fig. (1). Straight forward computation shows that, for the parameters values given by S_1 , Eq.(3.22) has three roots (eigenvalues) with negative real parts.

Moreover, for $\tau > 0$ with set of data S_1 , it is easy to verify that the coefficient of Eq. (3.26) is given by $h_1 = 7.3388 > 0$ and $h_3 = -0.4713 < 0$, and hence Eq. (3.26) has a unique positive root given by $\omega_0 = 0.2064$. Therefore the characteristic equation (3.21) has a unique pair of purely imaginary roots $\omega_0 = 0.2063$ with $\tau_0 = 5.5428$. Consequently, due to theorem (4), the interior equilibrium point E_2 is locally asymptotically stable for $\tau < \tau_0$, as shown in

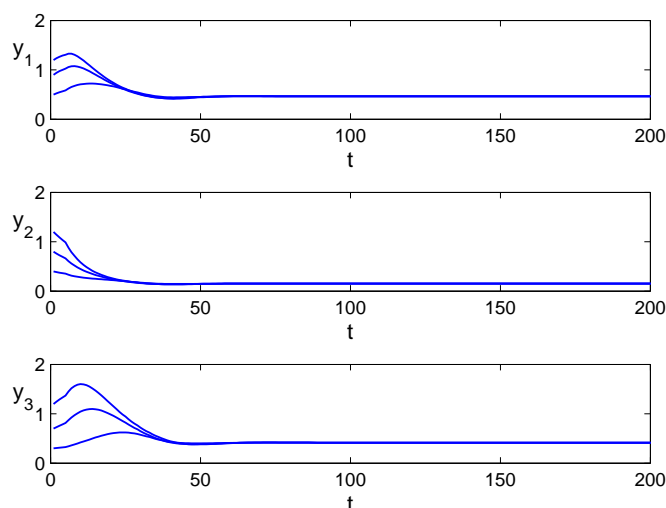


FIGURE 1. Trajectories of system (1) approach asymptotically to the interior equilibrium point $E_2 = (0.7484, 0.2135, 0.7618)$ for data given by S_1 and $\tau = 0$ starting from different initial values.

the typical figure given by Fig (2) for $\tau = 4.5$, while E_2 is unstable point for $\tau > \tau_0$ as shown in the typical figure given by Fig (3) for $\tau = 6.5$ and Fig (4) for $\tau = 20$. Obviously, the obtained numerical trajectories of system (1.1) represented in Fig (2), Fig (3) and Fig (4) confirm our obtained analytical result given by theorem 4. Indeed the trajectory represented by Fig (3) for $\tau = 6.5$ approaches asymptotically to periodic attractor and the period becomes larger with τ increases as shown in Fig (4) for $\tau = 20$ which insures of having a Hopf bifurcation. This confirms our obtained analytical results in lemma 2 and theorem 4 for which we have $h(0) = 10.7546 > 0$ and $h(\omega_0^2) = 13.9106 > 0$ that indicates to satisfying of transversal condition of Hopf bifurcation. On the other hand, substituting the values in S_1 with the value of ω_0 and τ_0 in Eq. (4.31) gives that $C_1 = -0.2318 - 1.4311i$, $\beta_2 = -0.4637 < 0$, $\mu_2 = 117.4314 > 0$ and $T_2 = 1.9203 > 0$. Therefore due to theorem 6 system (1.1) for $\tau > \tau_0$ undergoes a stable supercritical Hopf bifurcation, which is clearly shown in Fig(3) and Fig(4). Finally increasing the value of τ further leads to losing of the stability of periodic dynamics and the trajectory of system (1.1) approaches asymptotically to chaotic attractor as shown in Fig (5) for $\tau = 60.5$.

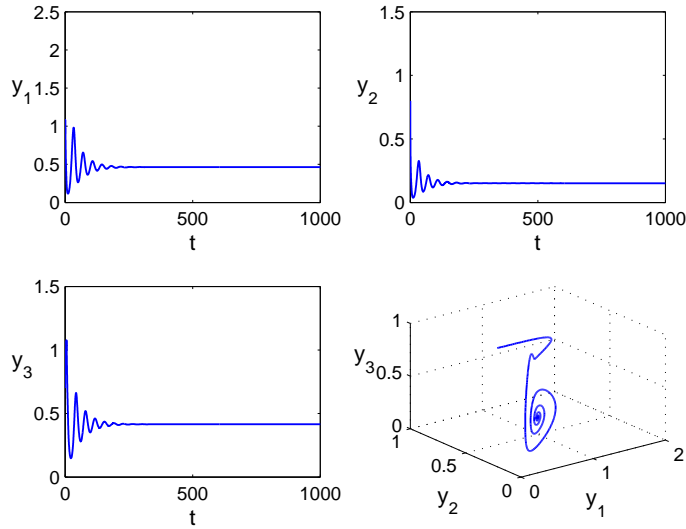


FIGURE 2. Trajectories of system (1.1) approach asymptotically to the interior equilibrium point $E_2 = (0.7484, 0.2135, 0.7618)$ for data given by S_1 and $\tau = 4.5$.

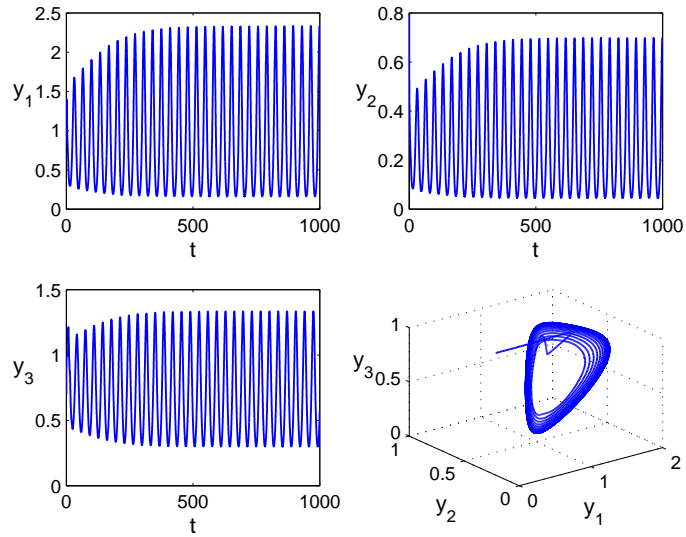


FIGURE 3. Trajectories of system (1.1) approach asymptotically to the periodic dynamic for data given by S_1 and $\tau = 6.5$.

6. DISCUSSION AND CONCLUSIONS

In this paper, we imposed a delay factor in the gestation of predator on the stage-structure prey-predator model given by [16]. Our purpose is to understand the effect of delay on the stability of the model. Stability analysis shows that the existence of discrete time delay does not effect on the stability of the boundary equilibrium points E_0 and E_1 . However it is working as a destabilizing factor of the system around the interior equilibrium point, so that the system still approaches asymptotically to the interior equilibrium point for the value of τ less than the critical value τ_0 . However the system loses its stability at E_2 and the trajectory approaches asymptotically to the periodic dynamics for $\tau > \tau_0$, which indicates to occurrence of Hopf bifurcation at E_2 for $\tau = \tau_0$. Finally, it is observed that increasing the value of τ further leads to losing the stability of the periodic dynamics too and the trajectory of system (1.1) approaches asymptotically to chaotic attractor.

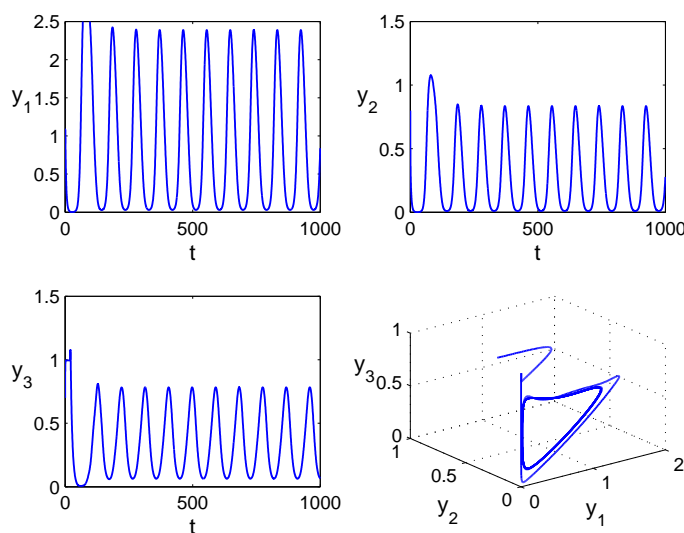


FIGURE 4. Trajectories of system (1.1) approach asymptotically to the periodic dynamic for data given by S_1 and $\tau = 20$.

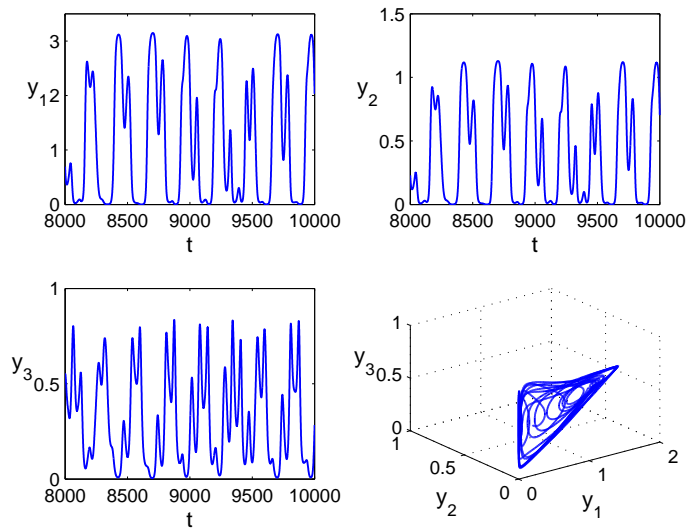


FIGURE 5. Strange attractor of system (1.1) for data given by S_1 with $\tau = 60.5$.

ACKNOWLEDGMENTS

The authors are thankful to referees for their valuable suggestions.

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