Iranian Journal of Mathematical Sciences and Informatics
Vol. 16, No. 1 (2021), pp 169-180
DOI: 10.29252/ijmsi.16.1.169

Tame Loci of Generalized Local Cohomology Modules

Fatemeh Dehghani-Zadeh[a], Maryam Jahangiri[b,c*]

[a]Department of Mathematics, Islamic Azad University,
Yazd branch, Yazd, Iran.
[b]Faculty of Mathematical Sciences and Computer, Kharazmi University,
Tehran, Iran.
[c]School of Mathematics, Institute for Research in Fundamental Sciences
(IPM), P.O. Box: 19395-5746, Tehran, Iran.

E-mail: dehghanizadeh@iauyazd.ac.ir, fdzadeh@gmail.com
E-mail: jahangiri@khu.ac.ir, jahangiri.maryam@gmail.com

Abstract. Let M and N be two finitely generated graded modules over a standard graded Noetherian ring $R = \bigoplus_{n\geq 0} R_n$. In this paper we show that if R_0 is semi-local of dimension ≤ 2 then, the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ is asymptotically stable for $n \to -\infty$ in some special cases. Also, we study the torsion-freeness of graded generalized local cohomology modules $H^i_{R_+}(M,N)$. Finally, the tame loci $T^i(M,N)$ of (M,N) will be considered and some sufficient conditions are proposed for the openness of these sets in the Zariski topology.

Keywords: Graded modules, Generalized local cohomology modules, Associated prime ideals, Tame loci.

2010 Mathematics subject classification: 13D45, 13A02.

1. Introduction

Throughout, R is a commutative Noetherian ring with identity and all modules are unitary. Let $\mathfrak{a}$ be an ideal of R and $R-Mod$ denotes the category of

*Corresponding Author

Received 07 May 2017; Accepted 24 October 2019

all R-modules and R-homomorphisms. We also denote by $\mathbb{N}_0$ and $\mathbb{N}$ the sets of non-negative and positive integers, respectively.

For $i \in \mathbb{N}_0$, the i-th generalized local cohomology functor with respect to $\mathfrak{a}$ is a generalization of the i-th local cohomology functor with respect to $\mathfrak{a}$, i.e. $H^i_{\mathfrak{a}}(-) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, -)$ ([5], [6]). It is defined, by Herzog ([10]), as follows:

$$H^i_{\mathfrak{a}}(-,-) : R - Mod \times R - Mod \to R - Mod$$

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

For all R-modules M and N, $H^i_{\mathfrak{a}}(M,N)$ is called the i-th generalized local cohomology module of M and N with respect to $\mathfrak{a}$. These functors coincide when $M = R$ and have been studied by many authors (see for instance [11], [19], [20] and [21]).

Now, let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a standard graded Noetherian ring and let M and N be two finitely generated graded R-modules. Also, assume that $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ denotes the irrelevant ideal of R. It is well known that for each $i \in \mathbb{N}_0$, $H^i_{R_+}(M,N)$ carries a natural grading. Furtheremore, according to [12], the n-th graded component $H^i_{R_+}(M,N)_n$ of $H^i_{R_+}(M,N)$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and it vanishes for all sufficiently large values of n. Therefore, the R_0-modules $H^i_{R_+}(M,N)_n$ are asymptotically trivial when $n \to +\infty$.

One basic question in this respect is to ask for the asymptotic behavior of the graded components $H^i_{R_+}(M,N)_n$ for $n \to -\infty$ and it attracts lots of interests (see [3], [1], [7] and [18]).

One concept of this asymptotic behavior is the stability of the set of associated prime ideals $\{\operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)\}_{n\in\mathbb{Z}}$ when $n \to -\infty$ (see [2], [3] and [12]). In the second section of this paper, among other things, we consider this problem and study the asymptotic behavior of $\operatorname{Ass}_{R_0}\big(H^i_{R_+}(M,N)_n\big)$ as $n \to -\infty$. More precisely, we show that if R_0 is semi-local and $\dim R_0 \leq 2$ then the set $\operatorname{Ass}_{R_0}\Big(H^i_{R_+}(M,N)_n\Big)$ is asymptotically stable, this means that there exists $n \in \mathbb{Z}$ such that $\operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n) = \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_{n_0})$ for all $n \leq n_0$, in each of the following cases:

(1) $\operatorname{depth}(R_0) > 0$ and $\Gamma_{\mathfrak{m}_0}(M) = 0 = \Gamma_{\mathfrak{m}_0}(N)$, for all maximal ideal $\mathfrak{m}_0$ of R_0.

(2) $\dim_{R_0}\big(H^{i-1}_{R_+}(M,N)_n\big) \leq 1$ for all $n \ll 0$ (Theorem 2.14).

Section 3 deals with the torsion-freeness of $H^i_{R_+}(M,N)$ over R_0. In [3, Theorem 2.5] the authors show that if R_0 is a domain then there is some $t \in R_0 - \{0\}$ such that the $(R_0)_t$-module $H^i_{R_+}(N)_t$ is torsion-free for all $i \in \mathbb{N}_0$. In this section, we made an extension of this theorem, under certain additional hypothesis. In particular, we show that if R_0 is a domain and $\dim H^i_{R_+}(N) \leq 2$ for all $i \in \mathbb{N}$ then, for a given finitely generated graded R-module M, there is some $t \in R_0 - \{0\}$ such that the $(R_0)_t$-module $H^i_{R_+}(M,N)_t$ is torsion-free (or vanishes) for each $i \in \mathbb{N}_0$ (Theorem 3.3).

The concept of " tameness " is the most fundamental concept related to the asymptotic behavior of cohomology. A graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ is said to be tame, or asymptotically gap free, if either $T_n \neq 0$ for all $n \ll 0$ or else $T_n = 0$ for all $n \ll 0$.

In this paper we are also interested in the study of the tame property of graded generalized local cohomology modules $H^i_{R_+}(M,N)$. In particular, in Section 4 we consider the "tame loci" $T^i(M,N)$ with respect to a pair of modules (M,N):

$$T^i(M,N) := \{\mathfrak{p}_0 \in \operatorname{Spec}(R_0) | H^i_{R_+}(M,N)_{\mathfrak{p}_0} \text{ is tame}\}$$

and study whether these sets are open in the Zariski topology. In the case where $M = R$, this subject has been studied in [4]. In this section we use the results in previous sections to show that the sets $T^i(M,N)$ are open in the Zariski topology in some special cases (Theorem 4.4).

Throughout the paper, unless other case stated, $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a standard graded Noetherian ring, $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ is the irrelevant ideal of R and M and N denote two finitely generated graded R-modules.

2. ASSOCIATED PRIME IDEALS

In this section, we assume that the base ring R_0 is semi-local and study the stability of the set $\{\operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)\}_{n\in\mathbb{Z}}$ when $n \to -\infty$. To this end, we need to consider tameness and Artinianness of graded R-modules $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)\big)$ for all maximal ideal $\mathfrak{m}_0$ of R_0.

Definition and Remark 2.1. *Let $T = \bigoplus_{n\in\mathbb{Z}} T_n$ be a graded R-module. Then the following statements hold.*

1. *If T is finitely generated, then in view of* [13]*, one can see that $T_n = 0$ for all $n \ll 0$, T_n is a finitely generated R_0-module for all $n \in \mathbb{Z}$, and there exists $X \subseteq \operatorname{Spec}(R_0)$ such that $\operatorname{Ass}_{R_0}(T_n) = X$ for all $n \gg 0$.*
2. *Following* [1, Definition 4.1]*, T is called tame, or asymptotically gap free, if there exists an integer n_0 such that either $T_n = 0$ for all $n < n_0$ or, $T_n \neq 0$ for all $n < n_0$. One can see that any Noetherian or Artinian graded R-module is tame.*
3. *We say that $\{\operatorname{Ass}_{R_0}(T_n)\}_{n\in\mathbb{Z}}$ is asymptotically stable (when $n \to -\infty$) if there exists an integer n_0 and $X \subseteq \operatorname{Spec}(R_0)$ such that $\operatorname{Ass}_{R_0}(T_n) = X$ for all $n < n_0$.*
4. *For each $i \in \mathbb{N}_0$, it is straightforward to see that*

$$\operatorname{Ass}_R\left(H^i_{R_+}(M,N)\right) = \{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \bigcup_{n\in\mathbb{Z}} \operatorname{Ass}_{R_0}\big(H^i_{R_+}(M,N)_n\big)\}.$$

The next definition and lemma will be useful in the proof of Proposition 2.4.

Definition 2.2. *Let $\mathfrak{a}$ be an ideal of R and T be an R-module. Then T is said to be $\mathfrak{a}$-cofinite if* $\operatorname{Supp}(T) \subseteq V(\mathfrak{a})$ *and* $\operatorname{Ext}^i_R(R/\mathfrak{a}, T)$ *is a finite R-module for all $i \in \mathbb{N}_0$, where* $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) | \mathfrak{p} \supseteq \mathfrak{a}\}$.

Lemma 2.3. *(*[9, Theorem 2.5]*). Let $\mathfrak{a}$ be an ideal of R with* $\dim R/\mathfrak{a} \leq 1$ *and M and N be two finitely generated R-modules. Then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i \in \mathbb{N}_0$.*

In [14, Corollary 2.2.5] it is shown that if $\dim R_0 \leq 2$ then the set $\operatorname{Ass}_R \left(H^i_{R_+}(N)\right)$ is finite. The following proposition is an extension of it to generalized local cohomology modules.

Proposition 2.4. *Let* $\dim R_0 = 2$. *Then* $\operatorname{Ass}_R \left(H^i_{R_+}(M,N)\right)$ *is a finite set for all $i \in \mathbb{N}_0$.*

Proof. Let

$$x \in \bigcap_{\mathfrak{m}_0 \in \max(R_0)} \mathfrak{m}_0 - \bigcup_{\mathfrak{p}_0 \in \min(R_0)} \mathfrak{p}_0 \text{ and } A := \{\mathfrak{p}_0 + R_+ \mid \mathfrak{p}_0 \in \operatorname{Spec}(R_0) \text{ and } x \in \mathfrak{p}_0\}.$$

By our hypotheses, A is a finite set, $\operatorname{ht}(xR_0) = 1$ and $\dim((R_0)_x) \leq 1$. Using [12, §2 (4)] and Lemma 2.3, to deduce that $H^i_{R_+}(M,N)_x \cong H^i_{(R_x)_+}(M_x, N_x)$ is $(R_x)_+$-cofinite. So, $\operatorname{Ass}_{R_x}\left(H^i_{R_+}(M,N)_x\right)$ is a finite set. Now, the result follows by using the facts that

$$\operatorname{Ass}_R(H^i_{R_+}(M,N)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p}R_x \in \operatorname{Ass}_{R_x}(H^i_{R_+}(M,N)_x)\} \cup A.$$

□

Lemma 2.5. *Let $S_1, S_2, \ldots, S_k$ be multiplicative closed subsets of R_0 with* $\operatorname{Spec}(R_0) = \bigcup_{j=1}^k \{\mathfrak{p}_0 \in \operatorname{Spec}(R_0) \mid \mathfrak{p}_0 \bigcap S_j = \varnothing\}$. *Then the following hold:*

(1) *If for all $j = 1, \cdots, k$,* $\operatorname{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M, S_j^{-1}N)_n\right)$ *is asymptotically stable, then so is* $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$.

(2) *If for all $j = 1, \cdots, k$,* $H^i_{S_j^{-1}R_+}(S_j^{-1}M, S_j^{-1}N)$ *is tame, then so is* $H^i_{R_+}(M,N)$.

Proof. One can use the same argument as used in the [14, Lemma 2.2.1] to prove the claim. □

Lemma 2.6. *(* [15, §18 Lemma 2]*) Let A be a ring and M and N be two A-modules. Let $x \in A$ be both A-regular and N-regular, and assume that* $xM = 0$. *Then* $\operatorname{Hom}_A(M,N) = 0$ *and* $\operatorname{Ext}^{n+1}_A(M,N) \cong \operatorname{Ext}^n_{A/xA}(M, N/xN)$ *for all $n \in \mathbb{N}_0$.*

Using the above Lemma we have the following, which will be used in the proof of the next theorem.

Lemma 2.7. *Let $x \in R_0$ be both R-regular and N-regular. Then*

$$H^{i+1}_{R_+}(M/xM, N) \cong H^i_{(R/xR)_+}(M/xM, N/xN),$$

for all $i \in \mathbb{N}_0$.

Lemma 2.8. *(*[16, Corollary 1.5] *). Let M be $\mathfrak{a}$-cofinite. Then for every maximal ideal $\mathfrak{m}$ of R, $\Gamma_{\mathfrak{m}}(M)$ is Artinian and $\mathfrak{a}$-cofinite.*

In the next two theorems we study the tame property of some submodules of $H^i_{R_+}(M, N)$.

Theorem 2.9. *Assume that* $\dim(R_0) \leq 2$, $\operatorname{depth}(R_0) > 0$ *and* $\Gamma_{\mathfrak{m}_0 R}(M) = 0 = \Gamma_{\mathfrak{m}_0 R}(N)$ *for all* $\mathfrak{m}_0 \in \max(R_0)$. *Then the graded R-module* $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N)\big)$ *is tame, for all $i \in \mathbb{N}_0$ and all maximal ideal $\mathfrak{m}_0$ of R_0.*

Proof. Using Lemma 2.5(2), we may assume that $(R_0, \mathfrak{m}_0)$ is local. In view of Lemma 2.3 and Lemma 2.8, the assertion holds for $\dim(R_0) \leq 1$. So, let $\dim(R_0) = 2$. By Proposition 2.4 and Remark 2.1(1) and (4), the set

$$A := \left(\bigcup_{n \in \mathbb{Z}} \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M, N)_n\right) \bigcup \operatorname{Ass}_{R_0}(M) \bigcup \operatorname{Ass}_{R_0}(N) \bigcup \operatorname{Ass}_{R_0}(R) \right) - \{\mathfrak{m}_0\}$$

is finite. Therefore, there is some $x \in \mathfrak{m}_0 \setminus A$. Hence, $\dim(R_0/xR_0) = 1$, x is R, M and N-regular and moreover $H^i_{R_+}(M, N)_n / \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M, N)_n)$-regular, for all $n \in \mathbb{Z}$.
It follows that $\Gamma_{\mathfrak{m}_0}\big(H^i_{R_+}(M, N)_n\big) = \Gamma_{xR_0}\big(H^i_{R_+}(M, N)_n\big)$ for all $n \in \mathbb{Z}$ and hence

$$\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N)\big) = \Gamma_{xR_0}\big(H^i_{R_+}(M, N)\big).$$

Now, consider the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ to get the following exact sequence

$$0 \longrightarrow H^i_{R_+}(M, N)/xH^i_{R_+}(M, N) \longrightarrow H^{i+1}_{R_+}(M/xM, N) \longrightarrow H^{i+1}_{R_+}(M, N).$$

Application of the functor $\Gamma_{\mathfrak{m}_0 R}(-)$ to this sequence induces the following exact sequence

$$0 \to \Gamma_{\mathfrak{m}_0 R}\Big(H^i_{R_+}(M, N)/xH^i_{R_+}(M, N)\Big) \to \Gamma_{\mathfrak{m}_0 R}\big(H^{i+1}_{R_+}(M/xM, N)\big) \to \Gamma_{\mathfrak{m}_0 R}\big(H^{i+1}_{R_+}(M, N)\big).$$

In view of Lemma 2.7, $H^{i+1}_{R_+}(M/xM, N) \cong H^i_{(R/xR)_+}(M/xM, N/xN)$. As $\dim((R/xR)_0) = 1$, by Lemmas 2.3 and 2.8, $\Gamma_{\mathfrak{m}_0 R}(H^{i+1}_{R_+}(M/xM, N))$ is Artinian.
Therefore, $\Gamma_{\mathfrak{m}_0 R}\Big(H^i_{R_+}(M, N)/xH^i_{R_+}(M, N)\Big)$ is Artinian. Hence

$$\Theta = \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M, N)) + xH^i_{R_+}(M, N)/xH^i_{R_+}(M, N)$$

is Artinian and consequently, Θ is tame. It follows that either $\Theta_n = 0$ for all $n \ll 0$ or $\Theta_n \neq 0$ for all $n \ll 0$. In the first case $\Big(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M, N)) +$

$xH^i_{R_+}(M,N)\Big)_n \subseteq xH^i_{R_+}(M,N)_n$ for all $n \ll 0$. Then $\Gamma_{xR_0}\Big(H^i_{R_+}(M,N)_n\Big) = \Big(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)\Big) \subseteq xH^i_{R_+}(M,N)_n$. It follows that $\Gamma_{xR_0}\Big(H^i_{R_+}(M,N)_n\Big) = x\Gamma_{xR_0}\Big(H^i_{R_+}(M,N)_n\Big)$ for all $n \ll 0$. Now, in view of Nakayama's Lemma, we get $\Gamma_{xR_0}\Big(H^i_{R_+}(M,N)_n\Big) = \Gamma_{\mathfrak{m}_0}\Big(H^i_{R_+}(M,N)_n\Big) = 0$, for all $n \ll 0$. In the second case, $\Big(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)\Big) \nsubseteq xH^i_{R_+}(M,N)_n$, for all $n \ll 0$. This implies that $\Gamma_{\mathfrak{m}_0}\Big(H^i_{R_+}(M,N)_n\Big) \neq 0$ for all $n \ll 0$, as desired.

□

By [7], there is a standard graded domain R with local base ring $(R_0, \mathfrak{m}_0)$ of dimension 3 and a finitely generated graded R-module T such that

$$\Gamma_{\mathfrak{m}_0}(H^2_{R_+}(T)_n) \begin{cases} \neq 0 & \text{if } n \text{ is even,} \\ = 0 & \text{if } n \text{ is odd.} \end{cases}$$

This example shows that the condition $\dim(R_0) \leq 2$ in the above theorem is necessary. Also, note that, by [11, Proposition 4.6], $\Gamma_{\mathfrak{m}_0 R}\Big(H^1_{R_+}(M,N)\Big)$ is always Artinian.

In the next theorem, as in 2.9, we show that $\Gamma_{\mathfrak{m}_0}\Big(H^i_{R_+}(M,N)\Big)$ is tame in some other conditions.

Theorem 2.10. *Let* $\dim(R_0) \leq 2$ *and* $i \in \mathbb{N}_0$ *such that* $\dim(H^{i-1}_{R_+}(M,N)_n) \leq 1$ *for all* $n \ll 0$. *Then* $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)\big)$ *is an Artinian* R*-module for all maximal ideal* $\mathfrak{m}_0$ *of* R_0.

Proof. Let $\mathfrak{m}_0 \in \max(R_0)$. Using [17, Theorem 11.38] , we consider the Grothendieck graded spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{m}_0 R}(H^q_{R_+}(M,N)) \stackrel{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}_0 + R_+}(M,N).$$

By the assumption on $H^{i-1}_{R_+}(M,N)_n$, we have $(E_2^{p,q})_n = 0$ for all $n \ll 0$ if $p \geq 2$ and $q = i-1$. Also, since $\dim(R_0) \leq 2$, $(E_2^{p,q})_n = 0$ for all $p \geq 3$ and all $q, n \in \mathbb{Z}$. It follows that

$$(E_2^{0,i})_n = (E_\infty^{0,i})_n \ \text{ for all } n \ll 0.$$

By the concept of the convergence of spectral sequences, $E_\infty^{0,i}$ is a subquotient of the graded Artinian R-module $H^i_{\mathfrak{m}_0 + R_+}(M,N)$. Therefore, by [13, Theorem 1],

$$\Big(0 :_{\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)} R_1\Big) = \big(0 :_{(E_\infty^{0,i})_n} R_1\big) = 0$$

for all $n \ll 0$. Also, using [5, Theorem 7.1.3] , $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)_n)$ is Artinian for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$. Therefore, again in view of [13, Theorem 1], $\Gamma_{\mathfrak{m}_0 R}\big((H^i_{R_+}(M,N)\big)$ is Artinian. □

Definition and Remark 2.11. *Let R_0 be a domain. Then $x \in M$ is called a torsion element of M if $\operatorname{Ann}_{R_0}(x) \neq 0$. The set of all torsion elements of M forms a submodule of M and is denoted by $T(M)$. If $T(M) = 0$ then M is said to be torsion-free. Clearly, flat modules and in particular free and projective modules are torsion-free.*

Let $k \in \mathbb{N}_0$ and $A \subseteq \operatorname{Spec}(R_0)$. Then we set

$$A^{\leq k} := \{\mathfrak{p}_0 \in A \mid \operatorname{ht}(\mathfrak{p}_0) \leq k\}.$$

Theorem 2.12. *(*[2, Theorem 4.1] *). Let R_0 be essentially of finite type over a field and $i \in \mathbb{N}_0$. Then, the set $\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)^{\leq 2}$ is asymptotically stable for $n \to -\infty$.*

Lemma 2.13. *(*[12, Theorem 4.4] *). Let $i \in \mathbb{N}_0$ and assume that $\dim(R_0) \leq 1$. Then the set $\operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)$ is asymptotically stable for $n \to -\infty$.*

The next theorem, which deals with the asymptotic stability of the set $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ when $n \to -\infty$, can be viewed as an extension, under certain additional hypotheses, of Theorem 2.12.

Theorem 2.14. *Let $\dim(R_0) \leq 2$, $i \in \mathbb{N}_0$ and assume that*

(1) $\operatorname{depth}(R_0) > 0$ *and M and N are torsion-free over R_0.*
or
(2) $\dim_{R_0}\left(H^{i-1}_{R_+}(M,N)_n\right) \leq 1$ *for all $n \ll 0$.*

Then the set $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ is asymptotically stable, when $n \to -\infty$.

Proof. In the case where $\dim(R_0) \leq 1$ the result follows from Lemma 2.13. So let $\dim(R_0) = 2$.

For all $n \in \mathbb{Z}$ set

$$\omega^i_n := \max(R_0) \bigcap \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$$

and

$$A^i_n := \{\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n) \mid \dim(R_0/\mathfrak{p}_0) \geq 1\}.$$

Then $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right) = A^i_n \cup \omega^i_n$, for each $n \in \mathbb{Z}$ and using Proposition 2.4 and Remark 2.1(4), $\cup_{n\in\mathbb{Z}} A^i_n$ is a finite set. Now, let $\mathfrak{p}_0 \in \cup_{n\in\mathbb{Z}} A^i_n$, then $(R_0)_{\mathfrak{p}_0}$ is a local ring of dimension ≤ 1. It follows by Lemma 2.13 that $\{\operatorname{Ass}_{(R_0)_{\mathfrak{p}_0}}\left(H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})_n\right)\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$. In view of the natural isomorphisms of $(R_0)_{\mathfrak{p}_0}$-modules $\left(H^i_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} \cong H^i_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})_n$ ([12, §2 (4)]), we thus get that either $\mathfrak{p}_0 \notin \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ for all $n \ll 0$ or $\mathfrak{p}_0 \in \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ for all $n \ll 0$. In the first case $\mathfrak{p}_0 \notin A^i_n$ for all $n \ll 0$ and in the second case $\mathfrak{p}_0 \in A^i_n$ for all $n \ll 0$. As $\cup_{n\in\mathbb{Z}} A^i_n$ is finite, $\{A^i_n\}_{n\in\mathbb{Z}}$ is stable for $n \to -\infty$.

On the other hand, $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N))$ is tame for all $\mathfrak{m}_0 \in \max(R_0)$, by Theorems 2.9 and 2.10. It follows that either $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ for all $n \ll 0$ or $\mathfrak{m}_0 \notin \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ for all $n \ll 0$. This, in conjunction with Lemma 2.5 prove our claim.

□

3. Torsion-Freeness

We keep the hypotheses introduced in the introduction and assume, in addition, that the base ring R_0 is semi-local and a domain. In this section we show that, in a special case, there is an element $t \in R_0 - \{0\}$ such that the localized generalized local cohomology module $H^i_{R_+}(M,N)_t$ is torsion-free or vanishes over $(R_0)_t$ for all $i \in \mathbb{N}_0$. The torsion-freeness of $H^i_{R_+}(R,N)$ was studied in [3].

Using [12, §2 (4)] and 2.1(4), it is straightforward to see that:

Lemma 3.1. *Let $t \in R_0 - \{0\}$ and $i \in \mathbb{N}_0$. Then $R_t \cong (R_0)_t \bigotimes_{R_0} R$ is a homogeneous Noetherian ring with irrelevant ideal $(R_t)_+ = R_+R_t = (R_+)_t$. Also, the following statements are equivalent:*

(1) *$H^i_{R_+}(M,N)_t$ is a torsion-free $(R_0)_t$-module;*
(2) *$H^i_{(R_t)_+}(M_t,N_t)$ is a torsion-free $(R_0)_t$-module;*
(3) *If $\mathfrak{p} \in \mathrm{Ass}_R(H^i_{R_+}(M,N))$, then $t \in \mathfrak{p}$ or $\mathfrak{p} \cap R_0 = 0$.*

Lemma 3.2. *(*[3, Theorem 2.5]*) Let R_0 be a domain. Then, there is an element $s \in R_0 - \{0\}$ such that $H^i_{R_+}(M)_s$ is a torsion-free $(R_0)_s$-module for all $i \in \mathbb{N}_0$.*

Theorem 3.3. *Assume that $\dim(H^i_{R_+}(N)) \leq 2$ for all $i \in \mathbb{N}_0$. Then, given any finitely generated graded R-module M, there exists $t \in R_0 - \{0\}$ such that $H^i_{R_+}(M,N)_t$ is a torsion-free $(R_0)_t$-module for all $i \in \mathbb{N}_0$.*

Proof. In view of [17, Theorem 11.38], consider the convergence of the graded spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_R\left(M, H^j_{R_+}(N)\right) \overset{i}{\Longrightarrow} H^{i+j}_{R_+}(M,N).$$

By definition, $E_\infty^{i,j}$ is a subqotient of $E_2^{i,j}$ for all i,j. Therefore,

$$\mathrm{Supp}(E_\infty^{i,j}) \subseteq \mathrm{Supp}\left(\mathrm{Ext}^i_R\left(M, H^j_{R_+}(N)\right)\right) \text{ for all } i,j \in \mathbb{N}_0.$$

On the other hand, by the concept of the convergence of spectral sequences, for all $n \in \mathbb{N}_0$, there exists a finite filtration

$$0 \subseteq \phi^n \subseteq \cdots \subseteq \phi^1 \subseteq \phi^0 = H^n_{R_+}(M,N)$$

of submodules of $H^n_{R_+}(M,N)$ such that $E_\infty^{i,n-i} \cong \phi^i/\phi^{i+1}$ for all $i = 0,1,\cdots,n$. Now, the exact sequence

$$0 \longrightarrow \phi^1 \longrightarrow H^n_{R_+}(M,N) \longrightarrow E_\infty^{0,n} \longrightarrow 0$$

yields

$$\begin{aligned}\mathrm{Supp}\ \left(H^n_{R_+}(M,N)\right) &\subseteq \mathrm{Supp}\ (\phi^1)\bigcup \mathrm{Supp}\ (E^{0,n}_\infty)\\ &\subseteq \mathrm{Supp}\ (\phi^1)\bigcup \mathrm{Supp}\ \left(\mathrm{Ext}^0_R\left(M, H^n_{R_+}(N)\right)\right)\\ &\subseteq \mathrm{Supp}\ (\phi^1)\bigcup \mathrm{Supp}\ \left(H^n_{R_+}(N)\right).\end{aligned}$$

By induction, one can see that

$$\mathrm{Supp}\ (\phi^i) \subseteq \mathrm{Supp}\ (\phi^{i+1})\bigcup \mathrm{Supp}\ (E^{i,n-i}_\infty) \subseteq \mathrm{Supp}\ (\phi^{i+1})\bigcup \mathrm{Supp}\ (H^{n-i}_{R_+}(N))$$

for all $i = 0, 1, \cdots, n$. Therefore,

$$\mathrm{Supp}\ \left(H^n_{R_+}(M,N)\right) \subseteq \mathrm{Supp}\ (H^0_{R_+}(N))\bigcup \mathrm{Supp}\ (H^1_{R_+}(N))\bigcup\cdots\bigcup \mathrm{Supp}\ (H^n_{R_+}(N)).$$

Now, let $\mathfrak{p} \in \mathrm{Ass}_R\left(H^n_{R_+}(M,N)\right) - \{0\}$. Then, there exists $0 \leq i \leq n$ such that $\mathfrak{p} \in \mathrm{Supp}\ (H^i_{R_+}(N))$. So, there is some $\mathfrak{q} \in \mathrm{Ass}_R\left(H^i_{R_+}(N)\right)$ with $\mathfrak{q} \subseteq \mathfrak{p}$. On the other hand, by Lemma 3.2, there exists $t \in R_0 - \{0\}$ such that $H^i_{R_+}(N)_t$ is a torsion-free $(R_0)_t$-module for all $i \in \mathbb{N}_0$.

If $\mathfrak{q} \cap R_0 \neq 0$ then, by Lemma 3.1, $t \in \mathfrak{q} \cap R_0 \subseteq \mathfrak{p} \cap R_0$. Otherwise, if $\mathfrak{q} \cap R_0 = 0$, then

$$\dim(R_0) = \dim(R_0/\mathfrak{q} \cap R_0) = \dim(R/\mathfrak{q}) \leq \dim\left(H^i_{R_+}(N)\right) \leq 2.$$

In view of Proposition 2.4 and Remark 2.1(4), $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M,N)\right)$ is a finite set. Hence there is $s \in R_0 - \{0\}$ such that $s \in \bigcap_{\mathfrak{p}' \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M,N))\setminus\{0\}} \mathfrak{p}'$. Now, we are done by Lemma 3.1. □

4. Tame Loci

Let R, M and N be as in the introduction. In the following we define the i-th tame loci of M and N and look for the cases for which these sets are open in the Zariski topology. When $M = R$ these sets were studied in [4].

Definition 4.1. *For $i \in \mathbb{N}_0$ define the i-th tame locus of M and N as*

$$T^i(M,N) := \{\mathfrak{p}_0 \in \mathrm{Spec}(R_0) \mid H^i_{R_+}(M,N)_{\mathfrak{p}_0} \text{ is tame}\}.$$

Therefore, $T^i(M,N)$ consists of all prime ideals $\mathfrak{p}_0 \in \mathrm{Spec}(R_0)$ such that

$$\exists n_0 \in \mathbb{Z} \text{ with} \begin{cases} \mathfrak{p}_0 \in \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M,N)_n\right) & \text{for all } n \leq n_0,\\ \text{or} & \\ \mathfrak{p}_0 \notin \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M,N)_n\right) & \text{for all } n \leq n_0.\end{cases}$$

Remark 4.2. By definition, if the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M,N)_n\right)$ is asymptotically stable for $n \to -\infty$ then, $T^i(M,N) = \mathrm{Spec}(R_0)$. Hence, in view of [8, Theorem 3.2], Lemma 2.4 and Theorem 2.14, we have $T^i(M,N) = \mathrm{Spec}(R_0)$ in each of the following cases

(1) R_0 is semi-local of dimension≤ 1,
(2) R_0 is semi-local of dimension≤ 2 and
   (a) $\operatorname{depth}(R_0) > 0$ and M and N are torsion free over R_0, or
   (b) $\dim_{R_0}\left(H^{i-1}_{R_+}(M,N)_n\right) \leq 1$ for all $n \ll 0$.
(3) for all $i \leq g(M,N) < \infty$, where

$$g(M,N) := \inf\{i \in \mathbb{N}_0 \mid \operatorname{length}_{R_0}\left(H^i_{R_+}(M,N)_n\right) = \infty \quad \text{for infinitely many } n \in \mathbb{Z}\}.$$

Lemma 4.3. *Let R_0 be a domain and M and N be torsion-free over R_0. Then* $\operatorname{Spec}(R_0)^{\leq 2} \subseteq T^i(M,N)$.

Proof. The result follows using Remark 4.2(2)(a). □

Theorem 4.4. *Let $i \in \mathbb{N}_0$ and the situations be as in the above lemma. Also, assume that $\dim(H^j_{R_+}(N)) \leq 2$ for all $j \in \mathbb{N}_0$ and that $\operatorname{Supp}_{R_0}\left(H^{i+1}_{R_+}(M,N)\right)^{=3}$ is a finite set. Then $T^i(M,N)^{\leq 3}$ is open and dense in the Zariski topology in $\operatorname{Spec}(R_0)^{\leq 3}$ provided that $\dim(H^{i-1}_{R_+}(M,N)) \leq 1$.*

Proof. Using Lemma 4.3, $\operatorname{Spec}(R_0)^{\leq 2} \subseteq T^i(M,N)$. So, let

$$\mathfrak{p}_0 \in \operatorname{Spec}(R_0)^{=3} \setminus \operatorname{Supp}_{R_0}(H^{i+1}_{R_+}(M,N))^{=3}.$$

If $\left(H^i_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} = 0$ for all $n \ll 0$, then $\mathfrak{p}_0 \in T^i(M,N)$. Otherwise, let $\left(H^i_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} \neq 0$ for infinitely many n. Then $\mathfrak{p}_0 \in \min \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)$ for infinitely many n.

In view of Theorem 3.3 and Lemma 3.1, there exists $0 \neq x \in \bigcap_{\mathfrak{q}_0 \in \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N))\setminus\{0\}} \mathfrak{q}_0$. Now, the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ deduces the exact sequence

$$\begin{aligned}\left(H^{i-1}_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} \quad &\to \quad \left(H^i_{R_+}(M/xM,N)_n\right)_{\mathfrak{p}_0} \to \left(H^i_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} \xrightarrow{.x} \left(H^i_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} \\ &\to \quad \left(H^{i+1}_{R_+}(M/xM,N)_n\right)_{\mathfrak{p}_0} \to \left(H^{i+1}_{R_+}(M,N)_n\right)_{\mathfrak{p}_0},\end{aligned}$$

for all $n \in \mathbb{Z}$. Since $x \in \mathfrak{p}_0$, using Nakayama's Lemma and the above exact sequence, we get

$$\left(H^{i+1}_{R_+}(M/xM,N)_n\right)_{\mathfrak{p}_0} \neq 0 \quad \text{for infinitely many } n. \tag{4.1}$$

Therefore, by Lemma 2.7,

$$\left(H^i_{(R/xR)_+}(M/xM,N/xN)_n\right)_{\mathfrak{p}_0/xR_0} \cong \left(H^{i+1}_{R_+}(M/xM,N)_n\right)_{\mathfrak{p}_0} \neq 0$$

for infinitely many n. On the other hand, again using 2.7 and the above exact sequence, we have

$$\begin{aligned}\dim_{(R_0/xR_0)_{\mathfrak{p}_0}}\left((H^{i-1}_{(R/xR)_+}(M/xM,N/xN)_n)_{\mathfrak{p}_0/xR_0}\right) &= \dim_{(R_0)_{\mathfrak{p}_0}}\left(H^i_{R_+}(M/xM,N)_n)_{\mathfrak{p}_0}\right)\\ &\le \max\{\dim_{(R_0)_{\mathfrak{p}_0}}\left(H^{i-1}_{R_+}(M,N)_n)_{\mathfrak{p}_0}\right),\\ &\qquad \dim_{(R_0)_{\mathfrak{p}_0}}\left(H^{i}_{R_+}(M,N)_n)_{\mathfrak{p}_0}\right)\}\\ &\le 1\end{aligned}$$

for all $n \ll 0$. This, in conjunction with Theorem 2.14, implies that

$$\mathrm{Ass}_{(R_0)_{\mathfrak{p}_0}}\left(H^{i+1}_{R_+}(M/xM,N)_{\mathfrak{p}_0}\right) = \mathrm{Ass}_{(R_0)_{\mathfrak{p}_0}}\left(H^{i}_{(R/xR)_+}(M/xM,N/xN)_{\mathfrak{p}_0/xR_0}\right)$$

is asymptotically stable when $n \to -\infty$. Hence, by 4.1,

$$\left(H^{i+1}_{R_+}(M/xM,N)_n\right)_{\mathfrak{p}_0} \neq 0 \quad \text{for all } n \ll 0.$$

Also, by the assumption on $\mathfrak{p}_0$, $\left(H^{i+1}_{R_+}(M,N)_n\right)_{\mathfrak{p}_0} = 0$ for all $n \in \mathbb{Z}$. It follows, by the above exact sequence, that $\mathfrak{p}_0 \in T^i(M,N)^{\le 3}$. Therefore,

$$\mathrm{Spec}(R_0)^{\le 3} \setminus \mathrm{Supp}_{R_0}(H^{i+1}_{R_+}(M,N)) \subseteq T^i(M,N)^{\le 3}$$

which implies that $T^i(M,N)^{\le 3} = \mathrm{Spec}(R_0)^{\le 3} \setminus X$ for some finite subset X of $\mathrm{Spec}(R_0)$, as desired.

□

Acknowledgments

The authors would like to thank the referee for her/his comments. The second author was in part supported by a grant from IPM (No. 92130111).

References

1. Brodmann, Markus, et al.,*Asymptotic behaviour of cohomology: tameness, supports and associated primes.*, Vol. 390. No. 390. American Mathematical Society, 2005.
2. M. P. Brodmann, A cohomological stability result for projective schemes over surfaces, *J. Reine angew. Math*, **606**, (2007), 179-192.
3. M. P. Brodmann, S. Fumasoli and C. S. Lim, Low-codimensional associated primes of graded components of local cohomology modules, *J. Alg.*, **275**, (2004) 867-882.
4. M. P. Brodmann, M. Jahangiri, Tame loci of certain local cohomology modules, *J. Commut. Alg.*, **4**(1),(2012), 79-100.
5. M. P. Brodmann, R. Y. Sharp, *Local cohomology -An Algebraic introduction with geometric applications*, (Cambridge Studies in Advanced Mathematics 60, Cambridge University Press, 1998.
6. W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in Advanced Mathematics 39, Revised edition, Cambridge University Press, 1998.
7. S. D. Cutkosky, J. Herzog, Failure of tameness for local cohomology, *J. Pure Appl. Alg.*, **211**, (2007) 428-432.
8. F. Dehghani-Zadeh, H. Zakeri, Some Results on Graded Generalized Local Cohomology Modules, *J. Math. Ext.*, **5**(1), (2010), 59-73.

9. K. Divaani-Azar, A. Hajikarimi, Cofiniteness of Generalized Local Cohomology Modules for One-Dimensional Ideals, *Canad. Math. Bull*, (2011), 1-7.
10. J. Herzog, Komplexe, *Auflösungen und Dualität in der Lokalen Algebra, Habilitationsschrift*, Universität Regensburg, 1974.
11. M. Jahangiri, N. Shirmohammadi and sh. Tahamtan, Tameness and Artinianness of graded generalized local cohomology modules, *Alg. Colloq.*, **22**(1), (2015), 131-146.
12. K. Khashyarmanesh, Associated primes of graded components of generalized local cohomology modules, *Comm. Alg.*,**33**, (2005), 3081-3090.
13. D. Kirby, Artinian modules and Hilbert polynomials, *Q. J. Math*, **24**(2), (1973), 17-57.
14. C. S. Lim, Graded local cohomology modules and their associated primes: the Cohen-Macauly case, *J. Pure Appl. Alg.*, **185**, (2003), 225-238.
15. H. Matsumura, *Commutative Ring Theory*, Cambridge, UK:Cambridge University Press, 1986.
16. L. Melkersson, Properties of cofinite modules and applications to local cohomology, *Math. Proc. Camb. Phil. Soc.*, **125** (1999) 417-423.
17. J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, Orlando, 1979.
18. C. Rotthaus, L. M. Sega, Some properties of graded local cohomology modules, *J. Algebra,* **283**, (2005), 232- 247.
19. N. Suzuki, On the generalized local cohomology and its duality, *J. Math. Kyoto. Univ*, **18**, (1978), 71-85.
20. S. Yassemi, Generalized section functors, *J. Pure. Appl. Alg.*, **95**, (1994), 103-119.
21. N. Zamani, On graded generalize local cohomology, *Arch.Math.* **86**, (2006), 321-330.