On the Prime Spectrum of Torsion Modules

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ABSTRACT. The paper uses a new approach to investigate prime submodules and minimal prime submodules of certain modules such as Artinian and torsion modules. In particular, we introduce a concrete formula for the radical of submodules of Artinian modules.

1. INTRODUCTION

Throughout the article, R is a commutative ring with a nonzero identity and all modules are unitary. We recall some definitions.

Definition 1.1. Let M be an R-module and N be a submodule of M.

(1) $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(0_M :_R M)$. If there is no ambiguity, we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$).

(2) N is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$), then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $\mathfrak{p} := (N : M)$ is a prime ideal of R. In this case, N is said to be $\mathfrak{p}$-*prime* (see [15, 25]).

(3) The set of all prime submodules of M is called the *prime spectrum* of M and is denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$.

(4) The set of all prime submodules of M containing N is denoted by $V^*(N)$ (see [26]). Following [18], we define $V(N)$ as

$$\{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}.$$

By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M. Set $\mathsf{Z}(M) = \{V(N) | N \leq M\}$. Then the elements of the set $\mathsf{Z}(M)$ satisfy the axioms for closed sets in a topological space $\mathrm{Spec}(M)$. The resulting topology due to $\mathsf{Z}(M)$ is called the *Zariski topology relative to M* and denoted by τ (see [18]).

In recent decades, the theory of prime submodules has been widely considered as a generalization of the theory of prime ideals in commutative rings. There are many articles that seek to generalize the various properties of the prime ideals of a ring to the prime submodules of a module (see [5, 7, 9, 11, 12, 13, 15]).

In Section 2, we recall briefly definitions and basic properties of certain topological spaces. In Section 3, we will characterize prime submodules and minimal prime submodules of certain modules such as Artinian modules over arbitrary rings and torsion modules over Dedekind domains (Theorem 3.2). This is a generalization of [6, Corollary 2.4, Proposition 2.5 and Corollary 2.6]. The prime radical of special submodules of these classes of modules is studied (Theorem 3.5). In particular, we introduce a concrete formula for the radical of submodules of Artinian modules. We are going to extend *Anderson's theorem* to minimal prime submodules in some classes of modules (Theorem 3.11). Also, we will show that any Artinian module contains only finitely many minimal prime submodules (Proposition 3.15).

2. Preliminaries

In the present section, we recall briefly definitions and basic properties of certain topological spaces that we shall use.

Remark 2.1. Let M be an R-module and N be a submodule of M.

(1) Note that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and that $\mathrm{Spec}(M)$ may be empty for some nonzero R-module M. For example, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module has no prime submodule for any prime number p (see [17]). Such a module is said to be *primeless*.

(2) M is called *primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the *natural map* $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, is surjective. Finitely generated modules and free modules are primeful (see [18, 20]).

(3) The *radical* of N, denoted by $\mathrm{rad}_M(N)$ or briefly $\mathrm{rad}(N)$, is defined to be the intersection of all prime submodules of M containing N. In the case where there are no such prime submodules, $\mathrm{rad}(N)$ is defined as M. If $\mathrm{rad}(N) = N$, we say that N is a *radical submodule* (see [16, 24]).

(4) We recall that the *Zariski radical* of N, denoted by $z\mathrm{rad}(N)$, is the intersection of all members of $V(N)$, that is $z\mathrm{rad}(N) = \cap_{P \in V(N)} P$ (see [21, Definitions 1.3]).

(5) M is called *weak multiplication* if every prime submodule P of M is of the form IM for some ideal I of R (see [2] and [4]).

(6) For an ideal I of R we recall that the *I-torsion submodule of M* is

$$\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \in \mathbb{N}\}$$

and M is said to be *I-torsion* if $M = \Gamma_I(M)$ (see [8]).

(7) The following statements are equivalent: (1) $(\mathrm{Spec}(M), \tau)$ is a T_0-space; (2) $|\mathrm{Spec}_{\mathfrak{p}}(M)| \leq 1$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$ (see [18, Theorem 6.1]).

3. MAIN RESULTS

Lemma 3.1. *Let M be an R-module and let P be a $\mathfrak{p}$-prime submodule of M for some prime ideal $\mathfrak{p}$ of R. Then, for each ideal J of R with $J \nsubseteq \mathfrak{p}$, we have $\Gamma_J(M) \subseteq P$. Hence, $\sum_{J \nsubseteq \mathfrak{p}} \Gamma_J(M) \subseteq P$.*

Proof. The proof is easy. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The next theorem, as one of the main results of the paper, is a generalization of [6, Corollary 2.4, Proposition 2.5 and Corollary 2.6]. Moreover, it characterizes the prime submodules of the class of Artinian modules over arbitrary rings and the class of torsion modules over Dedekind domains.

Recall that an R-module M is called *catenary* if for any prime submodules P and Q of M with $P \subsetneq Q$, all the saturated chains of the prime submodules of M starting from P and ending at Q have the same length (see [29]).

Theorem 3.2. *Let $\{\mathfrak{m}_\lambda\}_{\lambda \in \Lambda}$ be a collection of distinct maximal ideals of R and M_λ be an $\mathfrak{m}_\lambda$-torsion R-module for each $\lambda \in \Lambda$. Also, let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$. Then, the following statements hold:*

(1) *A proper submodule N of M is a prime submodule if and only if $(N : M) = \mathfrak{m}_h$ for some $h \in \Lambda$.*

(2) *M is catenary.*

(3) *If $(\mathrm{Spec}(M), \tau)$ is a T_0-space, then*

$$\mathrm{Spec}(M) = \mathrm{Max}(M) = \{\mathfrak{m}_\lambda M \mid \lambda \in \Lambda, \mathfrak{m}_\lambda M \neq M\}.$$

Proof.

(1) ($\Leftarrow$) If N is a proper submodule of M such that $(N : M) = \mathfrak{m}_h \in \mathrm{Max}(R)$, then N is a prime submodule of M by [15, Proposition 2].

($\Rightarrow$) Let N be a $\mathfrak{p}$-prime submodule of M. Then $N \cap M_h \neq M_h$ for some $h \in \Lambda$. By [26, Lemma 1.6], $N \cap M_h \in \mathrm{Spec}_\mathfrak{p}(M_h)$. Lemma 3.1 implies that

$$(N : M) = \mathfrak{p} = \mathfrak{m}_h \in \mathrm{Max}(R).$$

(2) Consider a chain of the prime submodules $P \subsetneq Q$ of M. Then by (1), $\mathfrak{p} := (P :_R M)$ is a maximal ideal of R. Let N be a prime submodule of M such that $P \subseteq N \subseteq Q$. Then $(N :_R M) = \mathfrak{p}$ and N/P is a (0)-prime submodule of $R/\mathfrak{p}$-vector space M/P. Therefore, any chain of the prime submodules

$$P \subset N_1 \subset N_2 \subset \cdots \subset Q$$

of M is a saturated chain if and only if

$$P/P \subset N_1/P \subset N_2 \subset \cdots \subset Q/P$$

is a saturated chain of $R/\mathfrak{p}$-subspaces of M/P. Consequently, the length of any saturated chain of the prime submodules of M starting from P and ending at Q is equal to $rank_{R/\mathfrak{p}}(Q/P)$.

(3) Clearly $\mathrm{Max}(M) \subseteq \mathrm{Spec}(M)$. Let $P \in \mathrm{Spec}(M)$. Then, by (1), there is a maximal ideal $\mathfrak{m}_\lambda$ of R such that $\mathfrak{m}_\lambda = (P : M)$. Suppose that L is a proper submodule of M such that $P \subseteq L$. Then

$$\mathfrak{m}_\lambda = (P : M) = (L : M).$$

By [15, p.63, Proposition 4], $\mathfrak{m}_\lambda M$ and L are $\mathfrak{m}_\lambda$-prime submodules of M. Since $(\mathrm{Spec}(M), \tau)$ is a T_0-space, $P = L = \mathfrak{m}_\lambda M$ by Remark 2.1(7). Consequently, $P = \mathfrak{m}_\lambda M \in \mathrm{Max}(M)$.

$\square$

An associated prime of a module M over a ring R is a prime ideal of R that arises as an annihilator of a nonzero element of M. The set of associated primes is usually denoted by $\mathrm{Ass}(M)$. There are well-known types of modules that satisfy the assumptions of Theorem 3.2. For example, if R is a Noetherian ring and M is an R-module such that $\mathrm{Ass}(M) \subseteq \mathrm{Max}(R)$, then it is easy to see that

$$M = \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(M)} \Gamma_\mathfrak{p}(M).$$

For example, this situation happens if R is a Dedekind domain and M is a torsion R-module. Another examples are Artinian modules. Note that, if M is an Artinian R-module, then there exist finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of R such that

$$M = \Gamma_{\mathfrak{m}_1}(M) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(M)$$

(see [32, p.166]). We record these examples as a corollary.

Corollary 3.3. *Let M be an R-module and one of the following holds:*

(a) *R is a Noetherian ring and* $\mathrm{Ass}(M) \subseteq \mathrm{Max}(R)$.

(b) *R is a Dedekind domain and M is torsion.*

(c) *M is Artinian.*

Then the following statements hold:

(1) *A proper submodule N of M is a prime submodule of M if and only if* $(N : M) \in \mathrm{Max}(R)$.

(2) *M is catenary.*

(3) *If* $(\mathrm{Spec}(M), \tau)$ *is a* T_0*-space, then*

$$\mathrm{Spec}(M) = \mathrm{Max}(M) = \{\mathfrak{m}M \,|\, \mathfrak{m} \in \mathrm{Ass}(M), \mathfrak{m}M \neq M\}.$$

Proof. Use Theorem 3.2. □

We remark that prime submodules of Artinian modules are not necessarily maximal. For example, any finite dimensional vector space is an Artinian module, in which every proper submodule is (0)-prime but not always maximal ([15, p .62, Result 1]).

EXAMPLE 3.4. Let N be an R-module and $M = \bigoplus_{\mathfrak{m} \in \mathrm{Max}(R)} \mathrm{Hom}(R/\mathfrak{m}, N)$. For each $\mathfrak{m} \in \mathrm{Max}(R)$, $\mathrm{Hom}(R/\mathfrak{m}, N)$ is certainly an $\mathfrak{m}$-torsion R-module because it is an $R/\mathfrak{m}$-module. By Theorem 3.2, a proper submodule P of M is prime if and only if $(P : M) \in \mathrm{Max}(R)$.

Regarding the radical of a submodule, it is defined in the algebra textbook of Zariski and Samuel [35, p. 252]. In 1986, McCasland and Moore adjusted the definition to the M-radical given in [23, p. 37]. Afterward, an extensive study of radical theory for modules was begun, which has continued with the more recent work [1, 16, 20, 22, 24, 28, 31]. Many algebraists (for example see [10], [20] and [30]) tried to find a relationship between $\mathrm{rad}(N)$ and $\sqrt{(N : M)}$, where N is a submodule of an R-module M. For example, in [30], it is shown if F is a free R-module and I any ideal of R, then $\mathrm{rad}(IF) = \sqrt{I}F$. In [20], some conditions have been obtained under which $\mathrm{rad}(N) = \sqrt{(N : M)}M$, for each submodule N of M. Finding an explicit formula for the radical of submodule is an interesting subject of many papers (see [1, 10, 19, 28]). In the sequel we introduced some expressions for the radical of special submodules of certain modules.

Theorem 3.5. *Let* $\{\mathfrak{m}_\lambda\}_{\lambda \in \Lambda}$ *be a collection of distinct maximal ideals of R and* M_λ *be an* $\mathfrak{m}_\lambda$*-torsion R-module for each* $\lambda \in \Lambda$. *Also, let* $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$. *For each ideal I of R, we have*

$$\mathrm{rad}(IM) = \bigcap_{\mathfrak{m}_\lambda \supseteq I} \mathfrak{m}_\lambda M.$$

Proof. If $V^*(IM) = \emptyset$, then by [15, p.63, Proposition 4], $\mathfrak{m}_\lambda M = M$ for any maximal ideal $\mathfrak{m}_\lambda \supseteq I$. Hence,

$$\operatorname{rad}(IM) = M = \bigcap_{\mathfrak{m}_\lambda \supseteq I} \mathfrak{m}_\lambda M.$$

Thus, we suppose that $V^*(IM) \neq \emptyset$. Let $P \in V^*(IM)$. Then, $\mathfrak{m}_\lambda = (P : M)$ for some $\lambda \in \Lambda$ by Theorem 3.2(1). This implies that

$$IM \subseteq \mathfrak{m}_\lambda M \subseteq P \neq M.$$

Again by [15, p.63, Proposition 4], $\mathfrak{m}_\lambda M$ is a prime submodule of M. So, $\mathfrak{m}_\lambda M$ is a minimal element of $\operatorname{Spec}_{\mathfrak{m}_\lambda}(M)$. Therefore, $\operatorname{rad}(IM) = \bigcap_{\mathfrak{m}_\lambda \supseteq I}(\mathfrak{m}_\lambda M)$. $\square$

Corollary 3.6. *Let R be a Noetherian ring and M be an R-module such that* $\operatorname{Ass}(M) \subseteq \operatorname{Max}(R)$ *(e.g. R is a Dedekind domain and M is torsion). Then, for each ideal I of R, we have*

$$\operatorname{rad}(IM) = \bigcap_{\mathfrak{m} \in \operatorname{Ass}(M) \cap V(I)} \mathfrak{m}M.$$

In particular,

$$\operatorname{rad}(0) = \bigcap_{\mathfrak{m} \in \operatorname{Ass}(M)} \mathfrak{m}M.$$

Proof. Use Theorem 3.5 $\square$

As a consequence of Theorem 3.5, we introduce a concrete formula for the radical of submodules of Artinian modules.

Corollary 3.7. *Let M be an Artinian R-module and let N be a submodule of M. Then there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \in \operatorname{Ass}(M)$ such that*

$$\operatorname{rad}(N) = \bigcap_{i=1}^{r}(\mathfrak{m}_i M + N).$$

In particular, $M/\operatorname{rad}(IM)$ is a Noetherian R-module for each ideal I of R.

Proof. By assumption, M/N is Artinian and so there exist finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of R such that $M/N = \Gamma_{\mathfrak{m}_1}(M/N) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_r}(M/N)$. Hence, Theorem 3.5 implies that

$$\operatorname{rad}(N)/N = \operatorname{rad}(0_{M/N}) = \bigcap_{i=1}^{r}(\mathfrak{m}_i(M/N)) = \bigcap_{i=1}^{r}\frac{\mathfrak{m}_i M + N}{N}.$$

Therefore, $\operatorname{rad}(N) = \bigcap_{i=1}^{r}(\mathfrak{m}_i M + N)$. For the second assertion, it is enough for us to set $N := IM$. Then there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r \in \operatorname{Ass}(M) \cap V(I)$ such that

$$\operatorname{rad}(IM) = \bigcap_{i=1}^{r} \mathfrak{m}_i M.$$

This implies that $M/\mathrm{rad}(IM)$ is annihilated by $\mathfrak{m}_{\lambda_1} \cdots \mathfrak{m}_{\lambda_t}$. Therefore, $M/\mathrm{rad}(IM)$ is a Noetherian R-module. $\qquad\square$

The next proposition is a generalization of [28, Theorem 1.8].

Proposition 3.8. *Let I be an ideal of R with $\sqrt{I} \in \mathrm{Spec}(R)$ and let M be a flat R-module. Then $\mathrm{rad}(IM) = \sqrt{I}M$. Moreover, if M is a nonzero primeful flat R-module, then for any primary ideal $\mathfrak{q}$ of R, $\mathrm{rad}(\mathfrak{q}M)$ is a prime submodule. In particular, for each primary submodule Q of M, $\mathrm{rad}((Q:M)M)$ is a prime submodule.*

Proof. Let $\mathfrak{p} := \sqrt{I} \in \mathrm{Spec}(R)$. If $V^*(IM) = \emptyset$, then by assumption and [15, p.66, Theorem 3], $\mathfrak{p}M = \sqrt{I}M = M$. So, in this case we have

$$\mathrm{rad}(IM) = M = \sqrt{I}M.$$

Now, suppose that $P \in V^*(IM)$. Then

$$IM \subseteq \mathfrak{p}M \subseteq (P:M)M \subseteq P \neq M.$$

Since M is flat, $\mathfrak{p}M$ is a prime submodule of M by [15, p.66, Theorem 3]. Therefore,

$$\mathrm{rad}(IM) = \mathfrak{p}M = \sqrt{I}M.$$

Now, suppose that M is primeful. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of R. Then $\mathrm{rad}(\mathfrak{q}M) = \mathfrak{p}M$. Note that by [20, Corollary 3.3], $\mathfrak{p}M \neq M$ whence $\mathfrak{p}M$ is a prime submodule of M. $\qquad\square$

If $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R, where $\mathfrak{m} \in \mathrm{Max}(R)$, then we can omit the flatness of the R-module M in Proposition 3.8.

Proposition 3.9. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R, where $\mathfrak{m} \in \mathrm{Max}(R)$. Then for any nonzero R-module M we have $\mathrm{rad}(\mathfrak{q}M) = \mathfrak{m}M$. Moreover, if M is a nonzero primeful R-module, then $\mathrm{rad}(\mathfrak{q}M)$ is a prime submodule.*

Proof. If $V^*(\mathfrak{q}M) = \emptyset$, then according to [15, Proposition 2], $\mathfrak{m}M = M$. So, in this case we have $\mathrm{rad}(\mathfrak{q}M) = \mathfrak{m}M$. Now, suppose that $P \in V^*(\mathfrak{q}M)$. Then

$$\mathfrak{q}M \subseteq \mathfrak{m}M \subseteq (P:M)M \subseteq P \neq M.$$

Again by [15, Proposition 2], $\mathfrak{m}M$ is a prime submodule of M. Therefore, $\mathrm{rad}(\mathfrak{q}M) = \mathfrak{m}M$. $\qquad\square$

The notion of minimal prime submodule was investigated by many authors [14, 19, 24, 34]. McCasland and Smith [27] showed that any Noetherian module M contains only finitely many minimal prime submodules. D. D. Anderson [3] generalized the well-known counterpart of this result for commutative rings, i.e., he abandoned the Noetherianness and showed that if every prime ideal minimal over an ideal I is finitely generated, then R contains only finitely many prime ideals minimal over I. We are going to extend Anderson's Theorem to minimal

prime submodules in some classes of modules (see Theorem 3.11). Also, we show that any Artinian module M contains only finitely many minimal prime submodules.

Lemma 3.10. *Let M be a primeful R-module and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be a subset of minimal elements of $V(\operatorname{Ann}(M))$. If $\mathfrak{p}_1 \cdots \mathfrak{p}_t M \subseteq \operatorname{rad}(0)$, then $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are the only minimal elements of $V(\operatorname{Ann}(M))$.*

Proof. Suppose that $\mathfrak{p}$ is a minimal element of $V(\operatorname{Ann}(M))$. Since M is primeful, there exists a prime submodule P of M such that $(P : M) = \mathfrak{p}$. By assumption we have

$$\mathfrak{p}_1 \cdots \mathfrak{p}_t M \subseteq \operatorname{rad}(0) \subseteq P.$$

Indeed, $\mathfrak{p}_1 \cdots \mathfrak{p}_t \subseteq (P : M) = \mathfrak{p}$. By minimality of $\mathfrak{p}$, $\mathfrak{p} = \mathfrak{p}_j$ for some $j \in \{1, \ldots, t\}$. □

Theorem 3.11. *Let M be a primeful R-module such that every minimal prime submodule of M is of the form $\mathfrak{p}M$, for some minimal element $\mathfrak{p}$ of $V(\operatorname{Ann}(M))$. If every minimal prime submodule of M is finitely generated, then M has only finitely many minimal prime submodules.*

Proof. We define

$$Y := \{\mathfrak{p}_1 \cdots \mathfrak{p}_t M \,|\, t \in \mathbb{N}, \mathfrak{p}_i \text{ is a minimal element of } V(\operatorname{Ann}(M))\}.$$

Suppose that for any element A of Y we have $A \not\subseteq \operatorname{rad}(0)$. Let

$Z = \{I \,|\, I \supseteq \operatorname{Ann}(M) \text{ is an ideal of } R \text{ such that } IM \text{ contains no element of } Y\}.$

Then Z is a non-empty set, since $(\operatorname{rad}(0) : M) \in Z$. Let Δ be a non-empty totally ordered subset of Z. Set $H = \bigcup_{I \in \Delta} I$. Assume on the contrary that $H \notin Z$. Then there are minimal elements $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of $V(\operatorname{Ann}(M))$, for some $s \in \mathbb{N}$, such that $\mathfrak{p}_1 \cdots \mathfrak{p}_s M \subseteq HM$.

We claim that $\mathfrak{p}_i M$ is a minimal prime submodule of M for each $i \in \{1, \ldots, s\}$. Fix $i \in \{1, \ldots, s\}$. Since M is primeful, there exists a prime submodule P_i of M such that $(P_i : M) = \mathfrak{p}_i$. If P_i is a minimal prime submodule of M, then by assumption $P_i := \mathfrak{q}M$ for some minimal element $\mathfrak{q}$ of $V(\operatorname{Ann}(M))$. So, $\mathfrak{p}_i M \subseteq P_i = \mathfrak{q}M$. Hence, $\mathfrak{p}_i \subseteq (\mathfrak{q}M : M) = \mathfrak{q}$ by [20, Corollary 3.3]. The minimality of $\mathfrak{q}$ implies that $P_i = \mathfrak{q}M = \mathfrak{p}_i M$. On the other hand, if P_i is not a minimal prime submodule of M, then by [16, Proposition 1], P_i contains a minimal prime submodule L of M. By assumption, $L := \mathfrak{q}'M$, where $\mathfrak{q}'$ is a minimal element of $V(\operatorname{Ann}(M))$. So,

$$\operatorname{Ann}(M) \subseteq \mathfrak{q}' \subseteq (P_i : M) = \mathfrak{p}_i.$$

Therefore, $\mathfrak{p}_i = \mathfrak{q}'$, by minimality of $\mathfrak{p}_i$. Thus $\mathfrak{p}_i M = L$ is a minimal prime submodule of M.

Hence, by assumption, $\mathfrak{p}_i M$ is finitely generated for each $i \in \{1, \ldots, s\}$. Therefore, $\mathfrak{p}_1 \cdots \mathfrak{p}_s M$ is finitely generated (see, for example [33]). This shows

that there exists an element $K \in \Delta$ such that $\mathfrak{p}_1 \cdots \mathfrak{p}_s M \subseteq KM$, a contradiction. Hence, by Zorn's lemma, we infer that Z has a maximal element $\mathfrak{q}$, say. We claim that $\mathfrak{q}$ is a prime ideal of R. For this, suppose that a and b are two elements of R such that $a \notin \mathfrak{q}$, $b \notin \mathfrak{q}$ and $ab \in \mathfrak{q}$. Thus $\mathfrak{q} + Ra \notin Z$ and $\mathfrak{q} + Rb \notin Z$. Hence, there are minimal elements $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ and $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_l$ of $V(\mathrm{Ann}(M))$, for some $r, l \in \mathbb{N}$, such that $\mathfrak{p}_1 \cdots \mathfrak{p}_r M \subseteq (\mathfrak{q} + Ra)M$ and $\mathfrak{p}'_1 \cdots \mathfrak{p}'_l M \subseteq (\mathfrak{q} + Rb)M$. Therefore,

$$\mathfrak{p}_1 \cdots \mathfrak{p}_r \mathfrak{p}'_1 \cdots \mathfrak{p}'_l M \subseteq \mathfrak{q}M,$$

a contradiction. This yields that $\mathfrak{q}$ is a prime ideal of R. Since $\mathrm{Ann}(M) \subseteq \mathfrak{q}$, there is a minimal element $\mathfrak{q}'$ of $V(\mathrm{Ann}(M))$ such that $\mathfrak{q}'M \subseteq \mathfrak{q}M$. This is a contradiction, because $\mathfrak{q}$ belongs to Z. Therefore, there exists an element A of Y such that $A \subseteq \mathrm{rad}(0)$. By Lemma 3.10, we conclude that M has only finitely many minimal prime submodules. $\square$

Remark 3.12. At first sight, it seems that our assumptions on M in Theorem 3.11 are very strange. But, in the next example we show that the class of such modules is not empty. Also, note that there exists an R-module N such that every minimal prime submodule of N is of the form $\mathfrak{p}N$, for some prime ideal $\mathfrak{p}$ of R. For example consider the R-module $N := \bigoplus_{\lambda \in \Lambda} M_\lambda$, where $\{\mathfrak{m}_\lambda\}_{\lambda \in \Lambda}$ is a collection of distinct maximal ideals of R and M_λ is an $\mathfrak{m}_\lambda$-torsion R-module for each $\lambda \in \Lambda$. According to Theorem 3.2, if Q is a minimal $\mathfrak{p}$-prime submodule of N, then $\mathfrak{p} \in \mathrm{Max}(R)$. By [15, p.63, Proposition 4], $\mathfrak{p}N$ is a $\mathfrak{p}$-prime submodules of N. Since $\mathfrak{p}N \subseteq Q$, by minimality of Q, we infer that $Q = \mathfrak{p}N$, as desired.

EXAMPLE 3.13. Let N be an Artinian R-module and consider the R-module $M := N/\mathrm{rad}(0_N)$. By Corollary 3.7, M is Noetherian and there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ such that $\mathrm{rad}(0_N) = \cap_{i=1}^t \mathfrak{m}_i N$. Therefore, M is primeful and its annihilator is equal to

$$\mathrm{Ann}(M) = (\mathrm{rad}(0_N) : N) = \bigcap_{i=1}^t \mathfrak{m}_i.$$

Hence, $V(\mathrm{Ann}(M)) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$. Suppose that $\overline{P} := P/\mathrm{rad}(0_N)$ is a minimal prime submodule of (the finite length R-module) M. As we mentioned in Remark 3.12, $\overline{P} = \mathfrak{p}M$, for some $\mathfrak{p} \in V(\mathrm{Ann}(M))$. Consequently, M is primeful and every minimal prime submodule of M is of the form $\mathfrak{p}M$, for some minimal element $\mathfrak{p} \in V(\mathrm{Ann}(M))$.

Corollary 3.14. *(**Anderson's Theorem**) If all the prime ideals minimal over an ideal I of any ring R are finitely generated, then there are only finitely many prime ideals minimal over I.*

Proof. Set $M := R$ and use Theorem 3.11. $\square$

Proposition 3.15. *Let M be an Artinian R-module. Then M has only finitely many minimal prime submodules.*

Proof. We may assume that $\operatorname{Spec}(M) \neq \emptyset$. Let S denote the collection of submodules of M which are finite intersections of minimal prime submodules. By hypothesis and Remark 3.12, S has a minimal member which has the form $\mathfrak{m}_1 M \cap \cdots \cap \mathfrak{m}_n M$ for some maximal ideals $\mathfrak{m}_i$. We claim that $\mathfrak{m}_1 M, \ldots, \mathfrak{m}_n M$ are the only minimal prime submodules of M. To see this, suppose that Q is a minimal prime submodule of M. Then by Remark 3.12, $Q = \mathfrak{m}M$ for some maximal ideal $\mathfrak{m}$ of R. We have

$$\mathfrak{m}_1 M \cap \cdots \cap \mathfrak{m}_n M = \mathfrak{m}_1 M \cap \cdots \cap \mathfrak{m}_n M \cap \mathfrak{m}M.$$

Thus, $(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)M \subseteq \mathfrak{m}M$. This implies that

$$\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subseteq ((\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)M : M) \subseteq (\mathfrak{m}M : M) = \mathfrak{m}.$$

Therefore, $Q = \mathfrak{m}_j M$ for some $1 \leq j \leq n$. This completes the proof. $\square$

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