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The Banach Type Contraction for mappings on Algebraic Cone Metric Spaces Associated with an Algebraic Distance and Endowed with a Graph

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Abstract. In this work, we define the notion of algebraic distance in algebraic cone metric spaces defined by Niknam et al. [A. Niknam, S. Shamsi Gamchi and M. Janfada, Some results on TVS-cone normed spaces and algebraic cone metric spaces, Iranian J. Math. Sci. Infor. 9 (1) (2014) 71–80] and introduce some its elementary properties. Then we prove the existence and uniqueness of fixed point for a Banach contractive type mapping in algebraic cone metric spaces associated with an algebraic distance and endowed with a graph.

Keywords: Algebraic cone metric space, Algebraic distance, Banach contraction, Orbitally G-continuous mapping.

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1. Introduction and Preliminaries

In 2008, Jachymski [8] proved some fixed point results in metric spaces endowed with a graph and generalized simultaneously the Banach contraction principle from metric and partially ordered metric spaces. In 2011, Nicolae et al. [9] presented some fixed point results for a new type of contractions using orbits

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and also for G-asymptotic contractions in metric spaces endowed with a graph. In 2012, Bojor [1] defined the notion of G-Reich type mappings and obtained a fixed point theorem for such mappings in a metric space endowed with a graph. Finally, Cholamjiak [4] presented fixed point theorems for a Banach contractive type mapping on complete tvs-cone metric spaces associated with w-distance and endowed with a graph.

On the other hand, very recently, Niknam et al. [11] defined the concept of algebraic cone metric spaces, studied some of its elementary properties and presented some fixed point results in this space. This definition is different of Huang and Zhang's definition [7]. Moreover, very recently, Rahimi et al. [14] defined an algebraic cone b-metric space.

In this paper, we first define an algebraic distance in algebraic cone metric spaces and study some of its elementary properties. Also we prove some fixed point theorems for a Banach contractive type mapping in algebraic cone metric spaces associated with an algebraic distance and endowed with a graph. Consistent with Niknam et al. [11], the following definitions and results will be needed in the sequel.

Let Y be a real vector space and $\mathcal{P}$ be a convex subset of Y. A point $x \in \mathcal{P}$ is said to be an algebraic interior point of $\mathcal{P}$ if there exists $\epsilon > 0$ such that $x + ty \in \mathcal{P}$ for all $t \in [0, \epsilon]$ and for each $y \in Y$. This definition is equivalent to a point x is called an algebraic interior point of the convex set $\mathcal{P} \subseteq Y$ if $x \in \mathcal{P}$ and there exists $\epsilon > 0$ such that $[x, x + \epsilon y] \subset \mathcal{P}$ for each $y \in Y$, where $[x, x+\epsilon y] = \big\{\lambda x + (1-\lambda)(x+\epsilon y) : \forall \lambda \in [0,1]\big\}$. The set of all algebraic interior points of $\mathcal{P}$ is called algebraic interior and is denoted by algebraic-*int* $\mathcal{P}$ (or *aint* $\mathcal{P}$). Also, $\mathcal{P}$ is called algebraically open if $\mathcal{P} = aint\ \mathcal{P}$. Let Y be vector space with the zero vector θ. A proper nonempty and convex subset $\mathcal{P}$ of Y is called an algebraic cone if $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$, $\lambda\mathcal{P} \subseteq \mathcal{P}$ for $\lambda \geq 0$ and $\mathcal{P} \cap (-\mathcal{P}) = \{\theta\}$. Given an algebraic cone $\mathcal{P} \subseteq Y$, a partial ordering $\preceq_a$ with respect to $\mathcal{P}$ is defined by $x \preceq_a y$ if and only if $y - x \in \mathcal{P}$. We shall write $x \prec_a y$ to mean $x \preceq_a y$ and $x \neq y$. Also, we write $x \ll_a y$ if and only if $y - x \in aint\ \mathcal{P}$, where $aint\ \mathcal{P}$ is the algebraic interior of $\mathcal{P}$. Also, $\mathcal{P}$ is said to be Archimedean if for each $x, y \in \mathcal{P}$ there exists $n \in \mathbb{N}$ such that $x \preceq_a ny$. For example, $\mathcal{P} = \big\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\big\}$ is an algebraic cone with the Archimedean property in the real vector space $\mathbb{R}^2$. In the sequel, we assume that $(Y, \mathcal{P})$ has the Archimedean property. We next recall the concept of algebraic cone metric space.

Definition 1.1. [11] Let X be a nonempty set and $(Y, \mathcal{P})$ be an algebraic cone space with $aint\ \mathcal{P} \neq \emptyset$. Suppose that a vector valued function $d_a : X \times X \to Y$ satisfies the following conditions:

(ACM1) $\theta \preceq_a d_a(x, y)$ for all $x, y \in X$ and $d_a(x, y) = \theta$ if and only if $x = y$;
(ACM2) $d_a(x, y) = d_a(y, x)$ for all $x, y \in X$;

(ACM3) $d_a(x,z) \preceq_a d_a(x,y) + d_a(y,z)$ for all $x,y,z \in X$.

Then d_a is called an algebraic cone metric and (X,d_a) is called an algebraic cone metric space.

Definition 1.2. [11] Let (X,d_a) be an algebraic cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

- **(i)** $\{x_n\}$ is said converges to x if for every $c \in Y$ with $c \in aint\ \mathcal{P}$ there exists a $N_0 \in \mathbb{N}$ such that $d_a(x_n,x) \ll_a c$ for all $n > N_0$. We denote this by $d_a - \lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{d_a} x$ as $n \to \infty$;
- **(ii)** $\{x_n\}$ is called a Cauchy sequence if for every $c \in Y$ with $c \in aint\ \mathcal{P}$ there exists a $N_0 \in \mathbb{N}$ such that $d_a(x_n,x_m) \ll_a c$ for all $m,n > N_0$;
- **(iii)** (X,d_a) is called a complete algebraic cone metric space if every Cauchy sequence in X is convergent.

Let (X,d_a) be an algebraic cone metric space. Then the following properties are often useful and simple to prove.

Lemma 1.3. *Let X be a nonempty set, $(Y,\mathcal{P})$ be an algebraic cone space with aint $\mathcal{P} \neq \emptyset$ and (X,d_a) be an algebraic cone metric space. Then, for all $u,v,w,c \in Y$, the following assertions are true:*

- (p_1) *If $u \preceq_a v$ and $v \ll_a w$, then $u \ll_a w$.*
- (p_2) *Let $\{b_n\}$ be sequence in Y such that algebraic convergent to θ (or $b_n \xrightarrow{a} \theta$), $\theta \preceq_a b_n$ and $c \in aint\ \mathcal{P}$. Then there exists positive integer N_0 such that $b_n \ll_a c$ for each $n > N_0$.*
- (p_3) *Let $\theta \ll_a c$. If $\theta \preceq_a d_a(x_n,x) \preceq_a b_n$ and $b_n \xrightarrow{a} \theta$, then eventually $d_a(x_n,x) \ll_a c$, where x_n, x are sequence and given point in X.*

Lemma 1.4. *Let (X,d_a) be an algebraic cone metric space. Then the family $\{N_a(x,c) : x \in X, \theta \ll_a c\}$, where $N_a(x,c) = \{y \in X : d_a(x,y) \ll_a c\}$, is a subbasis for topology on X (see [11]). We denote this algebraic cone topology by τ_a, and note that τ_a is a Hausdorff topology.*

Now, we define algebraic distance and introduce some its properties.

Definition 1.5. Let (X,d_a) be an algebraic cone metric space. A function $q_a : X \times X \to Y$ is called a c-algebraic distance (or briefly, an algebraic distance) on X if the following are satisfied:

- $\mathbf{q_1}$) $\theta \preceq_a q_a(x,y)$ for all $x,y \in X$;
- $\mathbf{q_2}$) $q_a(x,z) \preceq_a q_a(x,y) + q_a(y,z)$ for all $x,y,z \in X$;
- $\mathbf{q_3}$) for $x \in X$, if $q_a(x,y_n) \preceq_a u$ for some $u = u_x$ and all $n \geq 1$, then $q_a(x,y) \preceq_a u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
- $\mathbf{q_4}$) for all $c \in Y$ with $\theta \ll_a c$, there exists $e \in Y$ with $\theta \ll_a e$ such that $q_a(z,x) \ll_a e$ and $q_a(z,y) \ll_a e$ imply $d_a(x,y) \ll_a c$.

Example 1.6. Let $(Y, \mathcal{P})$ be an algebraic cone space with $aint\ \mathcal{P} \neq \emptyset$ and (X, d_a) be an algebraic cone metric space such that the metric $d_a(\cdot,\cdot)$ is a continuous function in second variable. Then, $q_a(x, y) = d_a(x, y)$ is an algebraic distance. In fact, $(\mathbf{q_1})$ and $(\mathbf{q_2})$ are immediate. But, property $(\mathbf{q_3})$ is nontrivial and it follows from $q_a(x, y_n) = d_a(x, y_n) \preceq u$, passing to the limit when $n \to \infty$ and using continuity of d_a. Let $c \in Y$ with $c \in aint\ \mathcal{P}$ be given and put $e = \frac{c}{2}$. Suppose that $q_a(z, x) \ll_a e$ and $q_a(z, y) \ll_a e$. Then $d_a(x, y) = q_a(x, y) \preceq q_a(x, z) + q_a(z, y) \ll_a e + e = c$. Using (p_1), this shows that $d_a(x, y) \ll_a c$ and thus q_a satisfies $(\mathbf{q_4})$. Hence, q_a is an algebraic distance.

In Example 1.6, we introduced a known algebraic distance in an algebraic cone metric space. There are some other examples about distance in [3, 5, 13] that reader can consider them in algebraic version. Also, similar to Example 3 of Djordevicć [6], one can consider algebraic distances which are not c-distances in cone metric spaces of [3, 13].

We will recall a sequence $\{u_n\}$ in algebraic cone $\mathcal{P}$ is c-sequence if for every $c \in aint\ \mathcal{P}$ there exists $N_0 \in \mathbb{N}$ such that $u_n \ll_a c$ for $n > N_0$. It is easy to prove that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in Y and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is c-sequence. Note that in the case where cone $\mathcal{P}$ is normal, a sequence in Y is c-sequence if and only if it is θ-sequence (see property (p_2)). However, a c-sequence need not be a θ-sequence in algebraic cone metric spaces.

Lemma 1.7. *Let (X, d_a) be an algebraic cone metric space and q_a be an algebraic distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{u_n\}$ and $\{v_n\}$ be c-sequences in algebraic cone $\mathcal{P}$ converging to θ, and let $x, y, z \in X$. Then the following properties holds:*

- qp_1) *If $q_a(x_n, y) \preceq_a u_n$ and $q_a(x_n, z) \preceq_a v_n$ for $n \in \mathbb{N}$, then $y = z$. Specifically, if $q_a(x, y) = \theta$ and $q_a(x, z) = \theta$, then $y = z$.*
- qp_2) *If $q_a(x_n, y_n) \preceq_a u_n$ and $q_a(x_n, z) \preceq_a v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.*
- qp_3) *If $q_a(x_n, x_m) \preceq_a u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
- qp_4) *If $q_a(y, x_n) \preceq_a u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. qp_1) In order to prove that $y = z$, it is enough to show that $d_a(y, z) \ll_a c$ for each $c \in aint\ \mathcal{P}$. For the given c, choose $e \in aint\ \mathcal{P}$ such that property $(\mathbf{q_4})$ is satisfied. Since $\{u_n\}$ and $\{v_n\}$ are c-sequence, so there exists $N_0 \in \mathbb{N}$ such that $u_n \ll_a e$ and $v_n \ll_a e$ for each $n \geq N_0$. By properties (p_1), since $u_n \ll_a e$ and $q_a(x_n, y) \preceq u_n$, we have $q_a(x_n, y) \ll_a e$. Similarly, we get $q_a(x_n, z) \ll_a e$. Now, using $(\mathbf{q_4})$, we have $d_a(y, z) \ll_a c$.

qp_3) Let $c \in Y$ with $c \in aint\ \mathcal{P}$. As in the proof of (qp_1), choose $e \in Y$ with $e \in aint\ \mathcal{P}$. Then there exists positive integer $N_0 \in \mathbb{N}$ such that $q_a(x_n, x_{n+1}) \ll_a e$ and $q_a(x_n, x_m) \ll_a e$ for any $m > n \geq N_0$

and hence $d_a(x_{n+1}, x_m) \ll_a c$ (by $(\mathbf{q_4})$). This implies that $\{x_n\}$ is a Cauchy sequence in X.

As in the proofs of (qp_1) and (qp_3), one can prove (qp_2) and (qp_4). This completes the proof. $\square$

We next review some basic notions of graph theory in relation to an algebraic cone metric space. Consider a directed graph G with $V(G) = X$ such that the set $E(G)$ consisting of the edges of G contains all loops, that is, $\Delta(X) \subseteq E(G)$ where $\Delta(X) = \{(x, x) \in X \times X : \ x \in X\}$ and let G have no parallel edges. Then G can be denoted by the ordered pair $(V(G), E(G))$, and also it is said that the algebraic cone metric space (X, d_a) is endowed with the graph G. Aa algebraic cone metric space (X, d_a) may also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$, $E(G^{-1}) = \big\{(x, y) : (y, x) \in E(G)\big\}$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$. If x and y are two vertices in a graph G, then a path in G from x to y is a finite sequence $(x_i)_{i=0}^{N}$ consisting of N + 1 vertices of G such that $x_0 = x$, $x_N = y$, and (x_{i-1}, x_i) is an edge of G for $i = 1, \cdots, N$ and $N \in \mathbb{N}$. A graph G is said to be connected if there exists a path in G between every two vertices of G. For more details on the theory of graphs, see [2, 8].

Definition 1.8. [10] Let $(X, \preceq)$ be a poset. A mapping $T : X \to X$ is called nondecreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

Following Petruşel and Rus [12, Definitions 3.1 and 3.6], we define the concept of Picard operator in an algebraic cone metric space.

Definition 1.9. Let (X, d_a) be an algebraic cone metric space and $T : X \to X$ be a mapping. Then T is called a Picard operator if T has an unique fixed point $x^* \in X$ and $T^n x \xrightarrow{d_a} x^*$ for all $x \in X$.

Following Jachymski [8, Definition 2.4], we define the concept of orbitally G-continuous for self-map T on algebraic cone metric spaces.

Definition 1.10. Let (X, d_a) be an algebraic cone metric space endowed with a graph G. A mapping $T : X \to X$ is called orbitally G-continuous on X if for all $x, y \in X$ and all sequences $\{p_n\}$ of positive integers with $(T^{p_n} x, T^{p_{n+1}} x) \in E(G)$ for all $n \geq 1$, the convergence $T^{p_n} x \xrightarrow{d_a} y$ implies $T(T^{p_n} x) \xrightarrow{d_a} Ty$.

Trivially, a continuous mapping on a algebraic cone metric space is orbitally G-continuous for all graphs G, but the converse is not generally true.

2. Main Results

In this section, let (X, d_a) be an algebraic cone metric space associated with an algebraic distance q_a and endowed with a directed graph G with $V(G) = X$

and $\Delta(X) \subseteq E(G)$. Throughout this section, we use X_T to denote the set of all points $x \in X$ such that $(x, Tx) \in E(G)$. In other words,

$$X_T = \{x \in X : (x, Tx) \in E(G)\}.$$

Motivated by [8, Definition 2.1], we introduce GA-Banach contraction in algebraic cone metric spaces associated with an algebraic distance q_a and endowed with a graph as follows:

Definition 2.1. Let (X, d_a) be an algebraic cone metric space associated with an algebraic distance q_a and endowed with a graph G. We say that a mapping $T : X \to X$ is a G-algebraic Banach contraction (or GA-contraction) if

- GA1) T preserves the edges of G; that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;
- GA2) there exists a $\alpha \in [0, 1)$ such that $q_a(Tx, Ty) \preceq_a \alpha q_a(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$.

Example 2.2. Let (X, d_a) be an algebraic cone metric space associated with the algebraic distance q_a and endowed with a graph G. Since $E(G)$ contains all loops, it follows that any constant mapping $T : X \to X$ preserves the edges of G, and since d_a on the diagonal of X is equal θ_X, it follows that T satisfies (GA2) for any constant $\alpha \in [0, 1)$. Hence, each constant mapping with domain X is a GA-contraction.

Example 2.3. Let (X, d_a) be an algebraic cone metric space associated with the algebraic distance q_a and $T : X \to X$ satisfies

$$q_a(Tx, Ty) \preceq_a \alpha q_a(x, y) \tag{2.1}$$

for all $x, y \in X$, where $\alpha \in [0, 1)$. Consider the complete graph G_0 whose vertex set coincides with X; that is, $V(G_0) = X$ and $E(G_0) = X \times X$. Assume that (X, d_a) is endowed with the graph G_0. Then it is clear that T preserves the edges of G_0 and T satisfies (GA2) if and only if T satisfies in condition (2.1). Therefore, T is G_0A-contraction with constant α.

Example 2.4. Let $(X, \preccurlyeq)$ be a partially ordered set and d_a be an algebraic cone metric on X. Define a graph G_1 by $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \preccurlyeq y\}$. Since $x \preccurlyeq x$ for all $x \in X$, it follows that $E(G_1)$ contains all loops and so it is allowed to consider the algebraic cone metric space (X, d_a) associated with the algebraic distance q_a and endowed with the graph G_1. Now, a mapping $T : X \to X$ preserves the edges of G_1 if and only if T is order-preserving, and T satisfies (GA2) for the graph G_1 if and only if there exists $\alpha \in [0, 1)$ such that

$$q_a(Tx, Ty) \preceq_a \alpha q_a(x, y) \tag{2.2}$$

for all elements $x, y \in X$ such that $x \preccurlyeq y$.

Example 2.5. Let $(X, \preccurlyeq)$ be a partially ordered set, d_a be an algebraic cone metric on X and q_a be an algebraic distance on X. Define a graph G_2 by $V(G_2) = X$ and $E(G_2) = \{(x,y) \in X \times X : x \preccurlyeq y \ \vee \ y \preccurlyeq x\}$; that is, an ordered pair $(x,y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \preccurlyeq)$. Since each element of $(X, \preccurlyeq)$ is comparable to itself, it follows that $E(G_2)$ contains all loops and one can consider the metric space (X, d_a) with the graph G_2. In addition, it is obvious that $\widetilde{G_2} = \widetilde{G_1} = G_2$. Now, a mapping $T : X \to X$ preserves the edges of G_2 if and only if T maps comparable elements of $(X, \preccurlyeq)$ onto comparable elements, and T satisfies (GA2) for the graph G_2 if and only if there exists $\alpha \in [0,1)$ such that

$$q_a(Tx, Ty) \preceq_a \alpha q_a(x,y) \tag{2.3}$$

for all comparable elements $x, y \in X$. In particular, if T is a G_1A-contraction, then T is a G_2A-contraction.

Example 2.6. Suppose that (X, d_a) is an algebraic cone metric space associated with the algebraic distance q_a and $e \in aint\ \mathcal{P}$ is a fixed. Recall that two elements $x, y \in X$ are said to be e-close if $d_a(x,y) \preceq_a e$. Define the e-graph G_3 by $V(G_3) = X$ and $E(G_3) = \big\{(x,y) \in X \times X : d_a(x,y) \preceq_a e\big\}$. Since d_a on the diagonal of X is equal to θ_X, it follows that $E(G_3)$ contains all loops. Assume that (X, d_a) is endowed with the graph G_3. Then a mapping $T : X \to X$ preserves the edges of G_3 if and only if T maps the e-close elements of X onto e-close elements, and T satisfies (GA2) for the graph G_3 if and only if

$$q_a(Tx, Ty) \preceq_a \alpha q_a(x,y) \tag{2.4}$$

for all e-close elements $x, y \in X$, where $\alpha \in [0,1)$.

To prove the existence of a fixed point for a GA-contraction in a complete algebraic cone metric space associated with an algebraic distance and endowed with a graph, we need the following lemma.

Lemma 2.7. *Let (X, d_a) be an algebraic cone metric space associated with the algebraic distance q_a and endowed with a connected graph G. Let $T : X \to X$ be a GA-contraction. Then, for any $x \in X$, $\{T^n x\}$ is a Cauchy sequence in X.*

Proof. Let $x \in X$. We divide the proof into two steps.
Step 1. If $(x, Tx) \in E(G)$, since T preserves the edges of G, then it follows that $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$ (by induction). Because of T is a GA-contraction on X, we have

$$q_a(T^n x, T^{n+1} x) \preceq_a \alpha q_a(T^{n-1} x, T^n x) \preceq_a \cdots \preceq_a \alpha^n q_a(x, Tx)$$

for all $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, we get

$$q_a(T^n x, T^{n+1} x) \preceq_a \alpha^n q_a(x, Tx).$$

Now, let $m > n$. From Definition 1.5.($\mathbf{q_2}$), we have

$$\begin{aligned} q_a(T^nx, T^mx) &\preceq_a q_a(T^nx, T^{n+1}x) + q_a(T^{n+1}x, T^{n+2}x) + \cdots + q_a(T^{m-1}x, T^mx) \\ &\preceq_a \alpha^n q_a(x, Tx) + \alpha^{n+1} q_a(x, Tx) + \cdots + \alpha^{m-1} q_a(x, Tx) \\ &\preceq_a \Big(\frac{\alpha^n}{1-\alpha}\Big) q_a(x, Tx). \end{aligned}$$

Since $\big\{\big(\frac{\alpha^n}{1-\alpha}\big)q_a(x,Tx)\big\}$ is a c-sequence in algebraic cone $\mathcal{P}$, hence from Lemma 1.7.(qp_3), $\{T^nx\}$ is a Cauchy sequence in X.

Step 2. If $(x, Tx) \notin E(G)$, since G is connected, then there exists path $(x_i)_{i=0}^N$ from x to Tx such that $x = x_0$, $x_N = Tx$ and $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, \cdots, N$. Because of T is a GA-contraction, we get by induction that $(T^nx_{i-1}, T^nx_i) \in E(G)$ for $i = 1, 2, \cdots, N$ and for all $n \geq 1$. Furthermore, using contractive condition (GA2), we get

$$\begin{aligned} q_a(T^nx, T^{n+1}x) &= q_a(T^nx_0, T^nx_N) \\ &\preceq_a q_a(T^nx_0, T^nx_1) + q_a(T^nx_1, T^nx_2) + \cdots + q_a(T^nx_{N-1}, T^nx_N) \\ &\preceq_a \alpha^n q_a(x_0, x_1) + \alpha^n q_a(x_1, x_2) + \cdots + \alpha^n q_a(x_{N-1}, x_N) \\ &= \alpha^n \cdot r, \end{aligned}$$

where $r = q_a(x_0, x_1) + q_a(x_1, x_2) + \cdots + q_a(x_{N-1}, x_N)$. Thus, for $m > n$, we can show that $q_a(T^nx, T^mx) \preceq_a (\frac{\alpha^n}{1-\alpha}) \cdot r$. Since $\big\{\big(\frac{\alpha^n}{1-\alpha}\big) \cdot r\big\}$ is a c-sequence, then $\{T^nx\}$ is a Cauchy sequence in X by Lemma 1.7.(qp_3). □

Proposition 2.8. *Let (X, d_a) be a complete algebraic cone metric space associated with the algebraic distance q_a and endowed with a connected graph G and let $T : X \to X$ be a GA-contraction. Then there exists a unique point $x^* \in X$ such that $\{T^nx\}$ converges to x^* for all $x \in X$.*

Proof. Fix a point $x \in X$. By Lemma 2.7, $\{T^nx\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $T^nx \xrightarrow{d_a} x^*$. Now, if $y \in X$, then in similar way, there exists $x^{**} \in X$ such that $T^ny \xrightarrow{d_a} x^{**}$. We show that $x^* = x^{**}$.

Now, let $(x, Tx) \in E(G)$. By Definition 1.5.($\mathbf{q_3}$) and since $\{T^nx\}$ converges to x^* and $q_a(T^nx, T^mx) \preceq_a (\frac{\alpha^n}{1-\alpha}) q_a(x, Tx)$ for all $m \geq 1$, we have

$$q_a(T^nx, x^*) \preceq_a \big(\frac{\alpha^n}{1-\alpha}\big) q_a(x, Tx) \tag{2.5}$$

for all $n \in \mathbb{N}$.

On the other hand, let $(x, Tx) \notin E(G)$. Since G is connected, then there exists path $(x_i)_{i=0}^N$ from x to Tx such that $x = x_0$, $x_N = Tx$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \cdots, N$. Thus, for all $m > n$,

$$q_a(T^nx, T^mx) \preceq_a \big(\frac{\alpha^n}{1-\alpha}\big) \cdot r$$

where $r = q_a(x, x_1) + q_a(x_1, x_2) + \cdots + q_a(x_{N-1}, Tx)$. By Definition 1.5.($\mathbf{q_3}$) and since $\{T^n x\}$ converges to x^* and $q_a(T^n x, T^m x) \preceq_a (\frac{\alpha^n}{1-\alpha}) \cdot r$ for all $m \geq 1$, we get

$$q_a(T^n x, x^*) \preceq_a \big(\frac{\alpha^n}{1-\alpha}\big) \cdot r \tag{2.6}$$

for all $n \in \mathbb{N}$. Repeating the argument above, if $(y, Ty) \in E(G)$, then

$$q_a(T^n y, x^{**}) \preceq_a \big(\frac{\alpha^n}{1-\alpha}\big) q_a(y, Ty) \tag{2.7}$$

for all $n \in \mathbb{N}$. Furthermore, if $(y, Ty) \notin E(G)$, then there exists path $(y_j)_{j=0}^M$ from y to Ty such that $y = y_0$, $y_M = Ty$ and $(y_{j-1}, y_j) \in E(G)$ for each $j = 1, \cdots, M$. Thus, we have

$$q_a(T^n y, x^{**}) \preceq_a \big(\frac{\alpha^n}{1-\alpha}\big) \cdot s$$

where $s = q_a(y, y_1) + q_a(y_1, y_2) + \cdots + q_a(y_{M-1}, Ty)$.

Next, we divide the proof into two steps.

Step 1. Let $(x, y) \in E(G)$. If $(y, Ty) \in E(G)$, then there exists M_0 such that

$$q_a(T^n x, T^m x) \preceq_a q_a(T^n x, T^n y) + q_a(T^n y, T^m y) \preceq_a \alpha^n q_a(x, y) + \frac{\alpha^n}{1-\alpha} q_a(y, Ty)$$

for all $m > M_0$. By Definition 1.5.($\mathbf{q_3}$) and since $\{T^n x\}$ converges to x^* and $q_a(T^n x, T^m x) \preceq_a \alpha^n q_a(x, y) + \frac{\alpha^n}{1-\alpha} q_a(y, Ty)$ for all $m \geq M_0$, we get

$$q_a(T^n x, x^*) \preceq_a \alpha^n q_a(x, y) + \frac{\alpha^n}{1-\alpha} q_a(y, Ty)$$

for all $n \in \mathbb{N}$. Using Conditions (2.5), (2.6) and Lemma 1.7.(qp_1), we have $x^* = x^{**}$. Similarly, if $(y, Ty) \notin E(G)$, then we have $x^* = x^{**}$.

Step 2. Suppose that $(x, y) \notin E(G)$. Then there exists path $(x_k)_{k=1}^S$ from x to y such that $x_0 = x$, $x_S = y$ and $(x_{k-1}, x_k) \in E(G)$ for $k = 1, 2, \cdots, S$. Now, let $(y, Ty) \in E(G)$. Then, by condition (2.7), we have

$$\begin{aligned} q_a(T^n x, x^{**}) &\preceq_a q_a(T^n x, T^n x_1) + q_a(T^n x_1, T^n x_2) \\ &\quad + \cdots + q_a(T^n x_{S-1}, T^n y) + q_a(T^n y, x^{**}) \\ &\preceq_a \alpha^n q_a(x, x_1) + \cdots + \alpha^n q_a(x_{S-1}, y) + (\frac{\alpha^n}{1-\alpha}) q_a(y, Ty) \\ &= \alpha^n \big(q_a(x, x_1) + q_a(x_1, x_2) + \cdots + q_a(x_{S-1}, y) + \frac{1}{1-\alpha} q_a(y, Ty)\big). \end{aligned}$$

Consequently, since $\alpha^n \big(q_a(x, x_1) + q_a(x_1, x_2) + \cdots + q_a(x_{S-1}, y) + \frac{1}{1-\alpha} q_a(y, Ty)\big)$ is a c-sequence in algebraic cone $\mathcal{P}$, by condition (2.5), (2.6) and Lemma 1.7.(qp_1), we have $x^* = x^{**}$. In this case, let $(y, Ty) \notin E(G)$. Similarly, we obtain $x^* = x^{**}$. □

Now we are ready to prove our main theorem on the existence and uniqueness of fixed points for GA-contractions in complete algebraic cone metric spaces associated with an algebraic distance and endowed with a graph.

Theorem 2.9. *Let (X, d_a) be a complete algebraic cone metric space associated with the algebraic distance q_a and endowed with a connected graph G. Let $T : X \to X$ be a GA-contraction and orbitally G-continuous on X. Then T is Picard operator if and only if $X_T \neq \emptyset$.*

Proof. If $X_T = \emptyset$, then there is nothing to prove. Otherwise, if $x \in X_T$, then $(x, Tx) \in E(G)$ and since T preserves the edges of G, it follows that by induction $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. From Proposition 2.8, there exists a unique point $x^* \in X$ such that $T^n x \xrightarrow{d_a} x^*$.

We next show that x^* is a fixed point for T. Because of T is orbitally G-continuous on X, then $T^{n+1}x \xrightarrow{d_a} Tx^*$ as $n \to \infty$. By Proposition 2.8, the limit of a sequence $\{T^n x\}$ in an algebraic cone metric space is unique, so we have $Tx^* = x^*$; that is, x^* is a fixed point for T.

Let $x^{**} \in X$ be another fixed point of T. Since $x^{**} = T^n x^{**} \xrightarrow{d_a} x^{**}$, again by Proposition 2.8, we have $x^* = x^{**}$. Thus, T is a Picard operator. □

Let $G = G_0$ in Theorem 2.9. Then, as mentioned before, the set X_T related to any arbitrary $T : X \to X$ coincides with the whole set X. Therefore, we get the following version of GA-Banach fixed point theorem in complete algebraic cone metric spaces equipped the algebraic distance q_a on X.

Corollary 2.10. *Let (X, d_a) be a complete algebraic cone metric space associated with the algebraic distance q_a and $T : X \to X$ be a mapping which satisfies in (2.1). Then T is a Picard operator.*

If we set $G = G_1$ in Theorem 2.9, then the following partially ordered version of GA-Banach fixed point theorem in complete algebraic cone metric spaces associated with the algebraic distance q_a and endowed with a partial order is obtained.

Corollary 2.11. *Let $(X, \preccurlyeq)$ be a poset, d_a be an algebraic cone metric on X such that (X, d_a) is a complete algebraic cone metric space associated with the algebraic distance q_a, and $T : X \to X$ be a nondecreasing and orbitally G_1-continuous which satisfies in (2.2). Then T is Picard operator if and only if there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*

If we set $G = G_2$ in Theorem 2.9, then another partially ordered version of GA-Banach fixed point theorem in complete algebraic cone metric spaces associated with an algebraic distance q_a and endowed with a partial order is obtained as follows.

Corollary 2.12. *Let $(X, \preccurlyeq)$ be a poset, d_a be an algebraic cone metric on X such that (X, d_a) is a complete algebraic cone metric space associated with the algebraic distance q_a, and $T : X \to X$ be a mapping which maps comparable elements of X onto comparable elements and satisfies in* (2.3)*. Also let T be a orbitally G_2-continuous on X. Then T is Picard operator if and only if there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable.*

Finally, if we set $G = G_3$ in Theorem 2.9, then we get the following version of GA-Banach fixed point theorem in complete algebraic cone metric spaces associated with an algebraic distance q_a.

Corollary 2.13. *Let (X, d_a) be a complete algebraic cone metric space associated with the algebraic distance q_a, $e \in aint\, \mathcal{P}$ and $T : X \to X$ be a mapping which maps e-close elements of X onto e-close elements and satisfies in* (2.4)*. Also, let T be a orbitally G_3-continuous on X. Then T is Picard operator if and only if there exists $x_0 \in X$ such that $d(x_0, Tx_0) \preceq_a e$.*

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