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Characterization of $\mathrm{PSL}(5,q)$ by its Order and One Conjugacy Class Size

Alireza Khalili Asboei

Department of Mathematics, Farhangian University, Tehran, Iran

E-mail: a.khalili@cfu.ac.ir

Abstract. Let $p = \frac{q^4+q^3+q^2+q+1}{(5,q-1)}$ be a prime number, where q is a prime power. In this paper, we will show $G \cong \mathrm{PSL}(5,q)$ if and only if $|G| = |\mathrm{PSL}(5,q)|$, and G has a conjugacy class size $\frac{|\mathrm{PSL}(5,q)|}{p}$. Further, the validity of a conjecture of J. G. Thompson is generalized to the groups under consideration by a new way.

Keywords: Conjugacy class size, Prime graph, Thompson's conjecture.

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1. Introduction

In mathematics, especially group theory, the elements of any group may be partitioned into conjugacy classes; members of the same conjugacy class share many properties, and study of conjugacy classes of non-abelian groups reveals many important features of their structure. For an abelian group, each conjugacy class is a set containing one element.

Denote by $N(G)$ the set of all conjugacy class sizes of a group G. The starting point for our discussion is from a conjecture of J. G. Thompson. Thompson's conjecture which is Problem 12.38 in the Kourovka notebook [19] is as follows: **Thompson's conjecture**. Let G be a group with trivial center. If M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

In [11, 12], Thompson's conjecture is verified for a few finite simple groups. Recently, Chen and his students contributed to Thompson's conjecture under

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a weak condition. They only used order and one or two special conjugacy class sizes of simple groups and characterized successfully sporadic simple groups, Alt_{10}, $\mathrm{PSL}(4,4)$ and $\mathrm{PSL}(2,p)$ (see [13, 14, 21]).

Similar characterizations have been found in [5], [2] , [8], [6], [7] and [5] for the groups: $\mathrm{PSL}(n,2)$, ${}^2D_n(2)$, ${}^2D_{n+1}(2)$, $C_n(2)$, alternating group of degree p, $p+1$, $p+2$ and symmetric group of degree p, where p is a prime number.

In this paper, we prove that $\mathrm{PSL}(5,q)$ are uniquely determined by one conjugacy class size and its order, where $\frac{q^4+q^3+q^2+q+1}{(5,q-1)}$ is a prime number. In fact, the main theorem of our paper is as follows:

Main Theorem. Let G be a group and q a prime power. Then $G \cong \mathrm{PSL}(5,q)$ if and only if $|G| = |\mathrm{PSL}(5,q)|$ and G has a conjugacy class size $\frac{|\mathrm{PSL}(5,q)|}{p}$, where $p = \frac{q^4+q^3+q^2+q+1}{(5,q-1)}$ is a prime number.

The *prime graph* of a finite group G that is denoted by $\Gamma(G)$ is the graph whose vertices are the prime divisors of G and where prime p is defined to be adjacent to prime q ($\neq p$) if and only if G contains an element of order pq.

We denote by $\pi(G)$ the set of prime divisors of $|G|$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let π_1, π_2 , $\dots$, $\pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$ (see [16] and [22]).

We can express $|G|$ as a product of integers m_1, m_2, $\dots$, $m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. The numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd component of G. Write $OC(G)$ for the set $\{m_1 , m_2, \dots, m_{t(G)}\}$ of order components of G and $T(G)$ for the set of connected components of G. According to the classification theorem of finite simple groups and [20, 23, 18], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-4 in [12]. All further unexplained notation is standard and we refer to [15], for example.

2. Preliminary Results

Definition 2.1. A Frobenius group is a transitive permutation group in which the stabilizer of any two points is trivial.

Definition 2.2. A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

We quote some known results about Frobenius group and 2-Frobenius group, which are useful in the sequel.

Lemma 2.3. [9] *Let G be a 2-Frobenius group of even order, i.e., G is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*

(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;

(b) G/K *and* K/H *are cyclic,* $|G/K| \mid (|K/H|-1)$*,* $(|G/K|, |K/H|) = 1$ *and* $G/K \lesssim \mathrm{Aut}(K/H)$*.*

Lemma 2.4. [9] *Suppose that* G *is a Frobenius group of even order and* H*,* K *are the Frobenius kernel and the Frobenius complement of* G*, respectively. Then* $t(G) = 2$*,* $T(G) = \{\pi(H), \pi(K)\}$*.*

Lemma 2.5. [23] *If* G *is a finite group such that* $t(G) \geq 2$*, then* G *has one of the following structures:*
(a) G *is a Frobenius group or a* 2*-Frobenius group;*
(b) G *has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ *and* K/H *is a non-abelian simple group. In particular,* H *is nilpotent,* $G/K \lesssim \mathrm{Out}(K/H)$ *and the odd order components of* G *are the odd order components of* K/H*.*

3. Proof of the Main Theorem

By [16, Corollary 2.11], PSL$(5, q)$ has one conjugacy class size $\frac{|\mathrm{GL}(5,q)|}{(q^5-1)}$. Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

By hypothesis, there exists an element x of order p in G such that $C_G(x) = \langle x \rangle$ and $C_G(x)$ is a Sylow p-subgroup of G. By the Sylow theorem, we have that $C_G(y) = \langle y \rangle$ for any element y in G of order p. So, $\{p\}$ is a prime graph component of G and $t(G) \geq 2$. In addition, p is the maximal prime divisor of $|G|$ and an odd order component of G.

If $t(G) = 2$, then $OC(G) = OC(\mathrm{PSL}(5, q))$. By [17], $G \cong \mathrm{PSL}(5, q)$.

If $t(G) \geq 3$, then we will show that there is no such group.

Since $t(G) \geq 3$, Lemma 2.3(a) and 2.4 show that G is neither a Frobenius group nor a 2-Frobenius group. By Lemma 2.5, G has normal series

$$1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$$

such that K/H is a non-abelian simple group and p is an odd order component of K/H. Moreover, $t(K/H) \geq 3$.

According to the classification theorem of finite simple groups and the results in Tables 1–4 in [12], K/H is an alternating group, sporadic group or simple group of Lie type.

Let $K/H \cong \mathrm{Alt}_r$, where r and $r-2$ are prime. Since

$$\frac{q^4 + q^3 + q^2 + q + 1}{(5, q-1)} = p \in \pi(K/H)$$

and $q \geq 2$ is a prime power, we have $p \geq 31$. It follows that $r = p = 31$ and $q = 2$. So, $|G| = |\mathrm{PSL}(5, 2)| = 2^{10} \times 3^2 \times 5 \times 7 \times 31$. Since $|\mathrm{Alt}_r|$ divides $|G|$, we get a contradiction.

Let K/H be isomorphic to one of the sporadic simple groups, ${}^2A_3(2)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, or ${}^2F_4(2)'$, we must have $\frac{q^4+q^3+q^2+q+1}{(5,q-1)} = p = 3$, 5, 7, 9, 11, 13, 17, 19, 23, 29, 31, 37 , 41, 43, 47, 59, 67, 71, 73, 127, 757, or 1093. The equation contains a solution only when $\frac{q^4+q^3+q^2+q+1}{(5,q-1)} = 31$, and in this case $q = 2$. Thus, K/H can be isomorphic to J_4, ON , Ly, $F_2 = B$, or $F_3 = Th$. But in all cases $11 \in \pi(K/H)$ and $|G| = |\mathrm{PSL}(5,2)| = 2^{10} \times 3^2 \times 5 \times 7 \times 31$, which is a contradiction because $|K/H| \mid |G|$.

Let K/H be a simple group of Lie type except for ${}^2A_3(2)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, or ${}^2F_4(2)^{'}$. If $t(K/H) = 3$, then $p \in \{OC_2(K/H), OC_3(K/H)\}$, and if $t(K/H) \in \{4, 5\}$, then

$$p \in \{OC_2(K/H),\ OC_3(K/H),\ OC_4(K/H),\ OC_5(K/H)\}.$$

By Tables 1-4 in [12], all of possibilities for K/H are $\mathrm{PSL}(2,q')$, where $4 \mid q'$, $\mathrm{PSL}(2,q')$, where $4 \mid q'-1$, $\mathrm{PSU}(6,2)$, $\mathrm{PSL}(3,2)$,${}^2D_t(3)$, where $t = 2^u + 1 \geq 5$, ${}^2D_{t+1}(2)$, where $t = 2^n - 1$ and $n \geq 2$, $G_2(q')$, where $q' \equiv 0 \pmod 3$, ${}^2G_2(q')$, where $q'^{(2t+1)} > 3$, $F_4(q')$, where q' is even, ${}^2F_4(q')$, where $q'^{(2t+1)} \geq 2$, $\mathrm{PSL}(3,4)$, ${}^2B_2(q')$, where q'^{2t+1} and $t \geq 1$, $E_8(q')$.

For the case $t(K/H) = 3$, we only consider $F_4(q')$, where q' is even. We can do the other cases similarly.

If $K/H \cong F_4(q')$ with q' is even, then

$$q'^4 + 1 = \frac{q^4+q^3+q^2+q+1}{(5,q-1)},$$

or

$$q'^4 - q'^2 + 1 = (q^4+q^3+q^2+q+1)/(5,q-1).$$

So,

$$p^6 = (q'^4+1)^6 < (q'^5)^6 = q'^{30}$$

and

$$q'^{24}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^2-1) \mid \frac{q^{10}(q^2-1)(q^3-1)(q^4-1)(q^5-1)}{(5,q-1)}.$$

Therefore, $q'^{36} \leq \frac{q^{10}(q^2-1)(q^3-1)(q^4-1)(q^5-1)}{(5,q-1)} < p^6 < q'^{30}$, which is a contradiction.

For the case $t(K/H) > 3$, we only consider ${}^2B_2(q')$, where $q^{'} = 2^{(2t+1)}$, $t \geq 1$ and $E_8(q')$. The other cases are similarly.

If $K/H \cong^2 B_2(q')$, where $q^{'} = 2^{(2t+1)}$ and $t \geq 1$, then $p = q' - 1$ or $p = q' \pm \sqrt{2q'} + 1$.

Let $q' - 1 = p$. If $(5, q-1) = 1$, then we can see that $2(3 \times 2^{2t} - 1) = q(q^3+q^2+q+1)$. If $|q|_2 = 2$, then $q^3+q^2+q+1 = 15$ and $t = 2$, Therefore, $31 \nmid |G|$ and $31 \mid |K/H|$, a contradiction. Thus $|q^3+q^2+q+1|_2 = 2$, and so $|p-1|_2 = 2$. Then $2^{2(2t+1)} \leq |K/H|_2 \leq |G|_2 \leq 2^5$, a contradiction. If $(5,q-1) = 5$, then $2(5 \times 2^{2t} - 3) = q(q^3+q^2+q+1)$. Thus, $|q^3+q^2+q+1|_2 = 2$, and so $|p-1|_2 = 2$. Then $2^{2(2t+1)} \leq |K/H|_2 \leq |G|_2 \leq 2^5$, a contradiction.

Let $q' + \sqrt{2q'} + 1 = p$. If $(5, q-1) = 5$, then $\frac{q^4+q^3+q^2+q+1}{5} = 2^{t+1}(2^t+1)$. Hence,

$$(q-1)(q^3+2q^2+3q+4) = 5 \times 2^{t+1}(2^t+1).$$

Since $5 \mid q-1$, we have $q-1=5k$ for some positive integer k. Thus $5k(k+1)=2^{t+1}(2^t+1)$ and so, $k(k+1)=2^{t+1}(\frac{2^t+1}{5})$. Now, if $2^{t+1} \mid k$, then $k+1 \leq \frac{2^t+1}{5}$ and if $2^{t+1} \mid k+1$, then $k \leq \frac{2^t+1}{5}$, which are impossible. If $(5,q-1)=1$, then $q^4+q^3+q^2+q+1=2^{t+1}(2^t+1)$. Because $q^4+q^3+q^2+q+1$ is an odd number and $2^{t+1}(2^t+1)$ is an even number, we get a contradiction.

Similarly, we can rules out the case when $p=q'-\sqrt{2q'}+1$.

Let $K/H \cong E_8(q')$. Then

$$p=\frac{q'^{10}+q'^5+1}{q'^2-q'+1}=q'^8-q'^7+q'^5-q'^4+q'^3-q'+1,$$

or

$$\frac{q'^{10}-q'^5+1}{q'^2-q'+1}=q'^8+q'^7-q'^5-q'^4-q'^3+q'+1,$$

and or

$$\frac{q'^{10}+1}{q'^2+1}=q'^8-q'^6+q'^4-q'^2+1, q'^8-q'^4+1\}.$$

Thus, $p<q'^9$. On the other hand, $p^{10}<q'^{90}$ and

$$|G|=\frac{q^{10}(q^2-1)(q^3-1)(q^4-1)(q^5-1)}{(5,q-1)}<p^{10}.$$

Since $q'^{120} \mid |K/H|$ and $|K/H| \mid |G|$, we get a contradiction. Now the main theorem is proved.

Corollary 3.1. *Let q be prime power. Then Thompson's conjecture holds for the simple groups* PSL$(5,q)$, *where* $\frac{q^5+q^4+q^3+q^2+q+1}{(5,q-1)}$ *is a prime number.*

Proof. Let G be a group with trivial center and $N(G)=N(\mathrm{PSL}(5,q))$. Then it is proved in [10, Lemma 1.4] that $|G|=|\mathrm{PSL}(5,q)|$. Hence, the corollary follows from the main theorem. □

By [5, 4] PSL$(n,2)$, where 2^n-1 is a prime number and PSL$(3,q)$, where $\frac{q^3-1}{(q-1)(3,q-1)}$ is a prime number characterizable by their order and one conjugacy class size. So, we bring the following question:

Question. Let $\frac{q^p-1}{(q-1)(p,q-1)}$ and p be prime numbers. Is group PSL(p,q) characterizable by its order and one conjugacy class size $\frac{|\mathrm{GL}(p,q)|}{(q^p-1)}$?

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