Iranian Journal of Mathematical Sciences and Informatics
Vol. 15, No. 1 (2020), pp 125-133
DOI: 10.21859/IJMSI.15.1.125

# On Identities with Additive Mappings in Rings

Abu Zaid Ansari

Department of Mathematics, Faculty of Science, Islamic University of Madinah, K.S.A

E-mail: ansari.abuzaid@gmail.com

Abstract. If $F, D : R \to R$ are additive mappings which satisfy $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. Then, $F$ is a generalized left derivation with associated Jordan left derivation $D$ on $R$. Similar type of result has been done for the other identity forcing to generalized derivation and at last an example has given in support of the theorems.

**Keywords:** Prime (Semiprime) ring, Additive mappings, Generalized (Jordan) left derivations, Generalized (Jordan) derivations, (Jordan)centralizers.

**2010 Mathematics Subject Classification:** 16D90, 16W25, 16N60

## 1. Introduction

In this paper $R$ is an associative ring with identity. A ring $R$ is $n$-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. A ring $R$ is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. An additive mapping $f : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $D : R \to R$ is said to be a left derivation (resp. Jordan left derivation) if $D(xy) = xD(y) + yD(x)$ (resp. $D(x^2) = 2xD(x)$) holds for all $x, y \in R$. An additive mapping $D : R \to R$ is said to be a right derivation (resp.

Received 22 March 2017; Accepted 26 Jun 2017

Jordan right derivation) if $D(xy) = D(x)y + D(y)x$ (resp. $D(x^2) = 2D(x)x$) holds for all $x, y \in R$. If $D$ is both left as well as right derivation, then $D$ is a derivation. Clearly, every left (resp. right) derivation on a ring $R$ is a Jordan left (resp. Jordan right) derivation but the converse need not be true in general. Following Zalar [14], an additive mapping $T : R \to R$ is called left (resp. right) centralizer of $R$ if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. In particular $T$ is Jordan left (resp. Jordan right) centralizer of $R$ if $x = y$. Obviously, every centralizer is a Jordan centralizer on $R$ but the converse is not true in general. Zalar in [14], proved: Every Jordan left centralizer on a 2-torsion free semiprime ring is a left centralizer. Following Ashraf et. al. [3], an additive mapping $F : R \to R$ is said to be a generalized left derivation (resp. generalized Jordan left derivation) if there exists a Jordan left deviation $D : R \to R$ such that $F(xy) = xF(y) + yD(x)$ ( resp. $F(x^2) = xF(x) + xD(x)$) for all $x, y \in R$. $F$ is a generalized left derivation if and only if $F$ is of the form $F = T + D$, where $T$ right centralizer of $R$ and $D$ is a left derivation. The concept of generalized left derivations cover the concept of left derivation and if $D = 0$, $F$ includes the concept of right centralizer. In 2013 [4], Ashraf et. al had proved that additive mappings $F, D : R \to R$ satisfying the properties $F(x^{n+1}) = x^nF(x) + nx^nD(x)$ for all $x \in R$, and show that if $R$ is a $(n + 1)$!-torsion free ring with identity, then $D$ is Jordan left derivation and $F$ is generalized Jordan left derivation on $R$. Similar type of result has been done in [2, 5]. In view of [2, 4, 5], we extend the results in the following setting.

## 2. Main Results

**Theorem 2.1.** *Let $n > 1$ be a fixed integer and $R$ be any $n$-torsion free ring. If $F, D : R \to R$ are additive mappings satisfying $F(x^ny^n) = x^nF(y^n) + y^nD(x^n)$ for all $x, y \in R$. Then, $F$ is generalized left derivation with associated Jordan left derivation $D$ on $R$.*

*Proof.* We have

$$F(x^ny^n) = x^nF(y^n) + y^nD(x^n) \text{ for all } x, y \in R. \tag{2.1}$$

Replacing $x$ by $e$ in the above equation, we get $D(e) = 0$. Again, replace $x$ by $x + e$ in the above equation to get

$$\begin{aligned}\binom{n}{0}[F(x^ny^n) &- x^nF(y^n) - y^nD(x^n)] + \binom{n}{1}[F(x^{n-1}y^n) - x^{n-1}F(y^n)\\ &- y^nD(x^{n-1})] + \binom{n}{2}[F(x^{n-2}y^n) - x^{n-2}F(y^n)\\ &- y^nD(x^{n-2})] + ... + \binom{n}{n-1}[F(xy^n) - xF(y^n)\\ &- y^nD(x)] + \binom{n}{n}[F(y^n) - F(y^n) - y^nD(e)] = 0.\end{aligned}$$

Using (2.1) together with the fact that $D(e) = 0$, we have

$$\begin{aligned}\binom{n}{1}[F(x^{n-1}y^n) &- x^{n-1}F(y^n) - y^nD(x^{n-1})] + \binom{n}{2}[F(x^{n-2}y^n)\\ &- x^{n-2}F(y^n) - y^nD(x^{n-2})] + ... + \binom{n}{n-1}[F(xy^n)\\ &- xF(y^n) - y^nD(x)] = 0.\end{aligned}$$

Replacing $x$ by $kx$, we obtain

$$\begin{aligned}\binom{n}{1}k^{n-1}[F(x^{n-1}y^n) &- x^{n-1}F(y^n) - y^nD(x^{n-1})] + \binom{n}{2}k^{n-2}[F(x^{n-2}y^n)\\ &- x^{n-2}F(y^n) - y^nD(x^{n-2})]... + \binom{n}{n-1}k[F(xy^n)\\ &- xF(y^n) - y^nD(x)] = 0.\end{aligned}$$

We can write the above equation as

$$\sum_{r=1}^{n-1}\binom{n}{r}k^{n-r}[F(x^{n-r}y^n) - x^{n-r}F(y^n) - y^nD(x^{n-r})] = 0 \text{ for all } x, y \in R.$$

Replace $k$ by $1, 2, \cdots, n-1$ in turn and consider the resulting system of $n-1$ homogeneous equations to get that the matrix of the system is a Van der Monde matrix

$$\mathbb{V} = \begin{pmatrix} 1 & 1 & ... & 1\\ 2 & 2^2 & ... & 2^n\\ . & . & ... & .\\ . & . & ... & .\\ . & . & ... & .\\ n-1 & (n-1)^2 & ... & (n-1)^n \end{pmatrix}.$$

Here $|\mathbb{V}| = \textit{product of positive integer}$, each of which is less than $n-1$, it implies that

$$\binom{n}{r}k^{n-r}[F(x^{n-r}y^n) - x^{n-r}F(y^n) - y^nD(x^{n-r})] = 0 \text{ for all } x, y \in R,$$

$r = 1, 2, \cdots, n-1$. In particular take $r = n-1$, we obtain

$$\binom{n}{n-1}[F(xy^n) - xF(y^n) - y^nD(x)] = 0 \text{ for all } x, y \in R.$$

This yields

$$n[F(xy^n) - xF(y^n) - y^nD(x)] = 0 \text{ for all } x, y \in R.$$

Since $R$ is $n$-torsion free, we find that

$$F(xy^n) = xF(y^n) + y^nD(x) \text{ for all } x, y \in R. \tag{2.2}$$

Again, replacing $y$ by $y + e$, we obtain

$$\begin{aligned}\binom{n}{0}[F(xy^n) &- xF(y^n) - y^nD(x)] + \binom{n}{1}[F(xy^{n-1}) - xF(y^{n-1}) \\ &- y^{n-1}D(x)] + \binom{n}{2}[F(xy^{n-2}) - xF(y^{n-2}) - y^{n-2}D(x)] \\ &+ ... + \binom{n}{n-1}[F(xy) - xF(y) - yD(x)] \\ &+ \binom{n}{n}[F(x) - xF(e) - D(x)] = 0.\end{aligned}$$

Taking $y = e$ in (2.2), we have

$$F(x) = xF(e) + D(x) \text{ for all } x \in R. \tag{2.3}$$

Using (2.2) and (2.3) in the above relation, we have

$$\begin{aligned}\binom{n}{1}[F(xy^{n-1}) &- xF(y^{n-1}) - y^{n-1}D(x)] \\ &+ \binom{n}{2}[F(xy^{n-2}) - xF(y^{n-2}) - y^{n-2}D(x)] \\ &+ ... + \binom{n}{n-1}[F(xy) - xF(y) - yD(x)] = 0.\end{aligned}$$

Replacing $y$ by $ky$, we get

$$\sum_{r=1}^{n-1}\binom{n}{r}k^{n-r}[F(xy^{n-r}) - xF(y^{n-r}) - y^{n-r}D(x)] = 0 \text{ for all } x, y \in R.$$

Using the same steps as we did before equation (2.2), we arrive at

$$F(xy) = xF(y) + yD(x) \text{ for all } x, y \in R. \tag{2.4}$$

Replace $x$ by $x^2$ in (2.3) to obtain $F(x^2) = x^2F(e) + D(x^2)$ for all $x \in R$. Again, replacing $y$ by $x$ in (2.4), we get $F(x^2) = xF(x) + xD(x)$ for all $x, y \in R$. Comparing the previous two relations, we get $x^2F(e) + D(x^2) = xF(x) +$

$xD(x))$ for all $x \in R$. This implies that $D(x^2) = x(F(x) - xF(e)) + xD(x))$ for all $x \in R$. Again, using (2.3) in the previous relation, we get $D(x^2) = 2xD(x)$ for all $x \in R$. Therefore, $D$ is a Jordan left derivation. Hence, $F$ is a generalized left derivation associated with $D$.

□

The Following corollary is a consequence of the above theorem:

**Corollary 2.2.** *Let $n > 1$ be a fixed integer and $R$ be any $n$-torsion free semiprime ring. If $F, D : R \to R$ are additive mappings satisfying $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. Then*

1. *$D$ is a derivation on $R$ and $[D(x), y] = 0$ for all $x, y \in R$,*
2. *$D$ is a derivation which maps $R$ into $Z(R)$,*
3. *$R$ is commutative or $D = 0$,*
4. *$F(x) = xq$ for all $x \in R$ and some $q \in Q_l(R_C)$, where $Q_l(R_C)$ is left Martindale ring of quotients,*
5. *$F$ is a generalized derivation on $R$.*

*Proof.* (1) Given that $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. Then, by Theorem 2.1, $F$ is generalized left derivation and $D$ is Jordan left derivation. Hence, by [1, Theorem 3.1], $D$ is derivation and $[D(x), y] = 0$ for all $x, y \in R$.

(2) Since $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. Then by Theorem 2.1, $F$ is generalized left derivation and $D$ is Jordan left derivation on $R$. Hence, by [13, Theorem 2], we get the required result.

(3) Assume that $D \neq 0$. We have $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. Then, by Theorem 2.1, $F$ is generalized left derivation and $D$ is Jordan left derivation. Hence, using (1) we find that $D$ is a derivation and $[D(x), y] = 0$ for all $x, y \in R$. hence, in particular $[D(x), x] = 0$ for all $x \in R$. Since $R$ is prime and $D$ is nonzero derivation, $R$ is commutative by [9, Theorem 2]

(4) We have $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. Thus by Theorem 2.1, $F$ is generalized left derivation on $R$. Since $R$ is a noncommutative 2-torsion free prime ring and $F$ is a generalized left derivation on $R$. In view of (3), we have $D = 0$. Thus we obtain $F(xy) = xF(y)$ for all $x, y \in R$. That is $F$ is a right centralizer on $R$. Hence, there exists $q \in Q_l(R_C)$ such that $F(x) = xq$ for all $x \in R$ by Proposition 2.10 of [1].

(5) Since $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in R$. In view of Theorem 2.1 and (3), $D$ is a derivation and $R$ is commutative. Since $R$ is a 2-torsion free prime ring and $F$ is a generalized left derivation, we find that $F(yx) = F(xy) = xF(y) + yD(x) = F(y)x + xD(y)$ for all

$x, y \in R$. This implies that $F(yx) = F(y)x + yD(x)$ for all $x, y \in R$. Hence, $F$ is generalized derivation on $R$.

□

**Lemma 2.3** ([8]). *Any linear derivation on a semisimple Banach Algebra is continuous.*

**Lemma 2.4** ([10]). *A continuous linear derivation on a commutative Banach Algebra maps algebra into its radical.*

Combining the above two results, Thomos proved the following:

**Lemma 2.5** ([11]). *There does not exist any nonzero linear derivations on commutative semisimple Banach algebras.*

In view of [8, 10, 11], the following consequence has been given:

**Theorem 2.6.** *Let $n > 1$ be any fixed integer and $A$ be a semisimple Banach algebra and let $F, D : A \to A$ be additive mappings satisfying $F(x^n y^n) = x^n F(y^n) + y^n D(x^n)$ for all $x, y \in A$. In this case $D = 0$.*

*Proof.* Since semisimple Banach algebra are semiprime, hence all the assumptions of Corollary 2.2 (1) are fulfilled. We have therefore a linear derivation on semisimple Banach algebra $A$. Hence, by Theorem 4 of [13], we get $D = 0$.

□

Now, come to the next theorem.

**Theorem 2.7.** *Let $n > 1$ be a fixed integer and $R$ be any $n$-torsion free semiprime ring. If $f, d : R \to R$ are additive mappings satisfying $f(x^n y^n) = f(x^n)y^n + x^n d(y^n)$ for all $x, y \in R$. Then, $f$ is generalized derivation with associated derivation $d$ on $R$.*

*Proof.* We have

$$f(x^n y^n) = f(x^n)y^n + x^n d(y^n) \text{ for all } x, y \in R. \tag{2.5}$$

Replacing $x$ by $e$ in the above equation, we get $d(e) = 0$. Again, replacing $x$ by $x + e$ in (2.5), we get

$$\begin{aligned} \binom{n}{0}[f(x^n y^n) &- f(x^n)y^n - x^n d(y^n)] + \binom{n}{1}[f(x^{n-1}y^n) - f(x^{n-1})y^n \\ &- x^{n-1}d(y^n)] + \binom{n}{2}[f(x^{n-2}y^n) - f(x^{n-2})y^n \\ &- x^{n-2}d(y^n)] + ... + \binom{n}{n-1}[f(xy^n) - f(x)y^n - xd(y^n)] \\ &+ \binom{n}{n}[f(y^n) - f(e)y^n - d(y^n)] = 0 \end{aligned}$$

Taking $x = e$ in (2.5), we get $f(y^n) = f(e)y^n + d(y^n)$ for all $x, y \in R$. Now using (2.5) together with the last relation, we have

$$\begin{aligned}\binom{n}{1}[f(x^{n-1}y^n) &- f(x^{n-1})y^n - x^{n-1}d(y^n)] \\ &+ \binom{n}{2}[f(x^{n-2}y^n) - f(x^{n-2})y^n - x^{n-2}d(y^n)] + ... \\ &+ \binom{n}{n-1}[f(xy^n) - f(x)y^n - xd(y^n)] = 0\end{aligned}$$

$$\sum_{r=1}^{n-1}\binom{n}{r}k^{n-r}[f(x^{n-r}y^n) - f(x^{n-r})y^n - x^{n-r}d(y^n)] = 0 \text{ for all } x, y \in R.$$

By the same logic the resulting matrix of the system is a Van der Monde matrix. Hence,

$$\binom{n}{r}k^{n-r}[f(x^{n-r}y^n) - f(x^{n-r})y^n - x^{n-r}d(y^n)] = 0 \text{ for all } x, y \in R,$$

$r = 1, 2, \cdots, n-1$. Now, in particular take $r = n-1$ and the fact that $R$ is $n$-torsion free, we get

$$f(xy^n) = f(x)y^n + xd(y^n) \text{ for all } x, y \in R. \tag{2.6}$$

Again replacing $y$ by $y + e$ and use the fact that $d(e) = 0$, we obtain

$$\begin{aligned}\binom{n}{1}[f(xy^{n-1}) &- f(x)y^{n-1} - xd(y^{n-1})] + \binom{n}{2}[f(xy^{n-2}) - f(x)y^{n-2} \\ &- xd(y^{n-2})] + ... + \binom{n}{n-1}[f(xy) - f(x)y - xd(y)]] = 0.\end{aligned}$$

Replace $y$ by $ky$ to get

$$\sum_{r=1}^{n-1}\binom{n}{r}k^{n-r}[f(xy^{n-r}) - f(x)y^{n-r} - xd(y^{n-r})] = 0 \text{ for all } x, y \in R.$$

Use the same technique to obtain

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R. \tag{2.7}$$

Replacing $y$ by $yz$ in the above relation, we obtain,

$$f(xyz) = f(x)yz + xd(yz) \text{ for all } x, y, z \in R.$$

Using (2.7), we arrive at

$$f(xy)z + xyd(z) = f(x)yz + xd(yz) \text{ for all } x, y, z \in R.$$

Again using (2.7), we obtain

$$f(x)yz + xd(y)z + xyd(z) = f(x)yz + xd(yz) \text{ for all } x, y, z \in R.$$

On simplifying, we have

$$x(d(yz) - d(y)z - yd(z)) = 0 \text{ for all } x, y, z \in R.$$

Multiplying both side by $d(yz) - d(y)z - yd(z)$, we find

$$(d(yz) - d(y)z - yd(z)x(d(yz) - d(y)z - yd(z)) = 0 \text{ for all } x, y, z \in R.$$

Using semiprimeness, we conclude that $d(yz) = d(y)z + yd(z)$ for all $y, z \in R$. Hence $d$ is a derivation. Therefore $f$ is a generalized derivation on $R$.

□

**Corollary 2.8.** *Let $n > 1$ be a fixed integer and $R$ be any $n$-torsion free semiprime ring. If $F : R \to R$ are additive mappings satisfying $F(x^n y^n) = F(x^n)y^n$ and $F(x^n y^n) = x^n F(y^n)$ for all $x, y \in R$. Then, $F$ is a centralizer on $R$.*

*Proof.* Taking $D = d = 0$ in Theorem 2.1 and 2.7, we get the required result.

□

The following example is in the favour of our theorems.

Example 2.9. Define $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in 2\mathbb{Z}_8 \right\}$, $\mathbb{Z}_8$ is the ring of addition and multiplication modulo 8. Define mappings $F, D, f, d : R \to R$ by $F\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, $D\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $d\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $F$ is not a generalized left derivation and $f$ is not a generalized derivation on $R$ but $F, D, f, d$ satisfy the identities $F(x^2y^2) = x^2F(y^2) + y^2D(x^2)$ and $f(x^2y^2) = f(x^2)y^2 + x^2d(y^2)$ for all $x, y \in R$.

## Acknowledgments

This research is supported by the project number 84/40, Deanship of Scientific research, Islamic University of Madinah, KSA.

## References

1. S. Ali, On generalized left derivations in rings and Banach algebras, *Aequat. Math.*, **81**, (2011), 209-226.
2. A. Z. Ansari, F. Shujat, Additive mappings satisfying algebraic conditions in rings *Rendiconti del Circolo Matematico di Palermo*, **63**(2), (2014), 211-219.
3. M. Ashraf, S. Ali, On generalized Jordan left derivations in rings, *Bull. Korean Math. Soc.* **45**(2), (2008), 253-261.

4. M. Ashraf, N. Rehman, and A. Z. Ansari, An additive mapping satisfying an algebraic condition in rings with identity, *Journal of Advanced Research in Pure Mathematics*, **5**(2), (2013), 38-45.
5. B. Dhara, R. K. Sharma, On addtive mappings in rings with identity elements, *Intere-national Mathematical Forum*, **4**(15), 2009, 727-732.
6. I. N. Herstein, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8**, (1957), 1104-1110.
7. I. N. Herstein, *Topics in ring theory*, Univ. Chicago Press, Chicago, 1969.
8. B. E. Johnson, and A.M. Sinklair, Continuity of derivations and a problem of Kaplansky, *Amer. J. Math.*, **90**, (1968), 1068-1073.
9. E. C. Posner, Derivations in prime rings, *Proc. amer. Math. Soc.*, (1957), 1093-1100.
10. I. M. Singer, and J. Wermer, Derivations on commutative normes spaces, *Math. Ann.*, **129**, (1995) 435-460.
11. M.P. Thomos, The image of a derivation is contained in the radical, *Annals of Math.*, **128**, (1988), 435-460.
12. Vukman, J. Jordan left derivations on semiprime rings, *Math. J. Okayama Univ.*, **39**, 1-6 (1997).
13. Vukman, J. On left Jordan derivations on rings and Banach algebras, *Aequationes Math*, **75**, (2008), 260-266.
14. Zalar, B. On centralizers of semiprime rings, Comment. *Math. Univ. carolin.*, **32**(4), (1991) 609-614.