Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 2 (2022), pp 139-146
DOI: 10.52547/ijmsi.17.2.139

The Auslander-Reiten Conjecture for Group Rings

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Abstract. This paper studies the vanishing of Ext modules over group rings. Let R be a commutative noetherian ring and Γ a group. We provide a criterion under which the vanishing of self extensions of a finitely generated $R\Gamma$-module M forces it to be projective. Using this result, it is shown that $R\Gamma$ satisfies the Auslander-Reiten conjecture, whenever R has finite global dimension and Γ is a finite acyclic group.

Keywords: Auslander-Reiten conjecture, Group rings, Periodic cohomology.

2000 Mathematics subject classification: 20J05, 13D22, 16S34.

1. Introduction

The celebrated *Auslander-Reiten conjecture* [5] says that, over an Artin algebra Λ, if M is a finitely generated Λ-module such that $\mathrm{Ext}^i_\Lambda(M, M \oplus \Lambda) = 0$ for all $i > 0$, then M is projective. This long-standing conjecture, which is rooted in a conjecture of Nakayama [22], is known to be true for several classes of algebras including algebras of finite representation type [5] and symmetric artin algebras with radical cube zero [19]. This conjecture is also closely related to other important conjectures such as *the Tachikawa conjecture* [24] and *the finitistic dimension conjecture* [17]. We should point out that the Auslander-Reiten conjecture is also called *Gorenstein projective conjecture*, if the module M (in the formulation of the conjecture) is assumed in addition to be Gorenstein projective over Λ; see [21]. It is proved in [26] that the Gorenstein projective conjecture holds for algebras of finite Cohen-Macaulay

Received 2 October 2018; Accepted 2 March 2021

type. However the validity of the Auslander-Reiten conjecture for algebras of finite Cohen-Macaulay type remains unknown. Recall that an Artin algebra Λ is said to be of finite Cohen-Macaulay type provided that there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective Λ-modules. The Auslander-Reiten conjecture actually makes sense for any Noetherian ring. In fact, there are already some results in the study of the Auslander-Reiten conjecture for commutative algebras; see for instance [8, 11, 13, 20]. In particular, Auslander, Ding and Solberg in [6], studied the following condition for commutative Noetherian rings, not necessarily for Artin algebras.

(ARC) Let R be a commutative Noetherian ring and M a finitely generated R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, then M is projective.

Commutative Noetherian local rings known to satisfy (ARC) include Gorenstein local rings of codimension at most four [23] and excellent Cohen-Macaulay normal domains containing the rational numbers [20]. The main result of Araya indicates that the validity of (ARC) for the class of commutative Gorenstein rings depends on its validity for such rings of dimension at most one; see [1, Theorem 3]. This result has been extended in [7] to left Gorenstein R-algebras Λ, whenever R is a commutative Gorenstein ring.

Our aim in this note, is to study the Auslander-Reiten conjecture over group rings. To be precise, assume that R is a commutative noetherian ring and Γ is an arbitrary group. It will turn out that if $R\Gamma$ satisfies (ARC), then the same is true for $R\Gamma'$, whenever Γ' is a subgroup of Γ of finite index; see Proposition 3.2. Moreover, assume that Γ is an abelian group and Γ' is a subgroup of Γ of finite index. Then it is proved that any self-orthogonal $R\Gamma$-module which is projective over $R\Gamma'$, will be a projective $R\Gamma$-module; see Theorem 3.5. This result enables us to deduce that the group ring $R\Gamma$ satisfies the Auslander-Reiten conjecture, whenever R has finite global dimension and Γ is a finite acyclic group; see Corollary 3.8.

2. Basic definitions

This section is devoted to stating the definitions and basic properties of notions which we will freely use in the later sections. Let us start with our convention.

Convention 2.1. Throughout the paper, R is a commutative noetherian ring with identity, Γ is a group and $R\Gamma$ is the corresponding group ring. We assume that all modules are finitely generated left modules, unless otherwise specified.

2.2. **Gorenstein modules.** An $R\Gamma$-module M is said to be Gorenstein projective if it is a syzygy of some exact sequence of projective $R\Gamma$-modules

$$\mathbf{T}_\bullet : \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow T_{-1} \longrightarrow \cdots ,$$

which remains exact after applying the functor $\mathsf{Hom}_{R\Gamma}(-, P)$, for any projective $R\Gamma$-module P. The exact sequence $\mathbf{T}_\bullet$ is called a totally acyclic complex of projectives. The reader is advised to look at [16] for the basic properties of these modules.

Remark 2.3. Gorenstein projective modules, which are a refinement of projective modules, were defined by Enochs and Jenda in [15]. This concept even goes back to Auslander and Bridger [4], who introduced the G-dimension of a finitely generated module M over a two-sided noetherian ring; and then Avramov, Martisinkovsky and Reiten proved that M is Gorenstein projective if and only if the G-dimension of M is zero (see also the remark following Theorem (4.2.6) in [12], for the historical information).

Definition 2.4. A complete resolution of an $R\Gamma$-module M is a diagram $\mathbf{T}_\bullet \xrightarrow{u} \mathbf{P}_\bullet \xrightarrow{\pi} M$, where $\mathbf{P}_\bullet \xrightarrow{\pi} M$ is a projective resolution of M, $\mathbf{T}_\bullet$ is a totally acyclic complex, u is a morphism of complexes and u_n is bijective for all $n \gg 0$. It is known that M admits a complete resolution if and only if it has finite Gorenstein projective dimension; see [2, Proposition 2.8].

Remark 2.5. Assume that M is an $R\Gamma$-module of finite Gorenstein projective dimension, say m. So there is a complete projective resolution $\mathbf{T}_\bullet \xrightarrow{u} \mathbf{P}_\bullet \xrightarrow{\pi} M$ such that for any $i \geq m$, u_i is bijective. Then for each $R\Gamma$-module N and any integer $n \in \mathbb{Z}$, the usual *Tate cohomomoy module* $\widehat{\mathsf{Ext}}^n_{R\Gamma}(M, N)$ is defined by the equality $\widehat{\mathsf{Ext}}^n_{R\Gamma}(M, N) = \mathsf{H}^n(\mathsf{Hom}_{R\Gamma}(\mathbf{T}_\bullet, N)$. It is not hard to see that, for any integer $n \geq 0$, there is a comparison morphism $\varepsilon_n : \mathrm{Ext}^n_{R\Gamma}(M, N) \longrightarrow \widehat{\mathsf{Ext}}^n_{R\Gamma}(M, N)$, where $\varepsilon_n = \mathsf{H}^n(\mathsf{Hom}(u, N))$. In particular, ε_n is bijective for all $n > m = \mathrm{Gpd}_{R\Gamma} M$, the Gorenstein projective dimension of M; see [2, Proposition 3.8].

Definition 2.6. Let M be an $R\Gamma$-module and let $\cdots \longrightarrow P_k \xrightarrow{d_k} P_{k-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a projective resolution of M. We preserve R_i for any $i \geq 0$ to denote the i-th syzygy of M, i.e, $R_i = \ker(P_{i-1} \longrightarrow P_{i-2})$, where $P_{-1} = M$ and $P_{-2} = 0$.
We say that an $R\Gamma$-module M has a q-periodic projective resolution after k steps, where $k \geq 0$ and $q > 1$ are integers, if it admits a projective resolution such that $R_k \cong R_{k+q}$. Equivalently, the functors $\mathrm{Ext}^n_{R\Gamma}(M, -)$ and $\mathrm{Ext}^{n+q}_{R\Gamma}(M, -)$ are naturally isomorphic for all $n \geq k+1$; see [25, 1]. If $k = 0$, then M is said to have a q-periodic resolution.
A group Γ is said to have q-periodic resolution after k steps, if the trivial $R\Gamma$-module R has a q-periodic resolution after k steps; see [25].

Remark 2.7. It is worth pointing out that if Γ is a finite group having period q after k steps, then it has periodic cohomology with period q in the classical sense, i.e. $k = 0$. On the other hand, if an infinite group has period q after k steps, then k has to be greater than or equal to one; see [25].

3. Results

In this section, we state and prove the results of this note. It is shown that if a group Γ is nice enough, then vanishing of self-extensions of any $R\Gamma$-module M, forces M to be projective. This result enables us to show that the Auslander-Reiten conjecture holds for the class of $R\Gamma$-modules whose restriction to R is projective, whenever Γ is a finite acyclic group.

The result below says that satisfying the Auslander-Reiten conjecture descends from a group to its subgroups of finite index. First we state a remark.

Remark 3.1. Let Γ be a group and Γ' be a subgroup of Γ of finite index. Let M be a left $R\Gamma'$-module. Then a verbatim pursuit of the proof of [10, Proposition III 5.9], gives rise to an isomorphism $R\Gamma \otimes_{R\Gamma'} M \cong \mathsf{Hom}_{R\Gamma'}(R\Gamma, M)$, as left $R\Gamma$-modules. One should note that the left hand side is a left $R\Gamma$-module, since $R\Gamma$ is an $R\Gamma$-$R\Gamma'$-bimodule. However, the case for the right hand side follows from the $R\Gamma'$-$R\Gamma$-bimodule structure of $R\Gamma$.

Proposition 3.2. *Let Γ' be a subgroup of Γ of finite index. If $R\Gamma$ satisfies the Auslander-Reiten conjecture, then so does $R\Gamma'$.*

Proof. Assume that M is an $R\Gamma'$-module such that $\mathrm{Ext}^i_{R\Gamma'}(M, M \oplus R\Gamma') = 0$ for all $i > 0$. We must show that M is a projective module. To do this, consider the following isomorphisms;

$$\begin{aligned}&\mathrm{Ext}^i_{R\Gamma} R\Gamma \otimes_{R\Gamma} (R\Gamma \otimes_{R\Gamma'} M), M \oplus R\Gamma')\\ &\cong \mathrm{Ext}^i_{R\Gamma'}(R\Gamma \otimes_{R\Gamma'} M, M \oplus R\Gamma')\\ &\cong \oplus_{i=1}^n \mathrm{Ext}^i_{R\Gamma'}(M, M \oplus R\Gamma'),\end{aligned}$$

where the first isomorphism follows from Remark 3.1 and the second isomorphism comes from the adjointness of Hom and $\otimes$. The third isomorphism holds trivially and the last one follows from the $R\Gamma'$-isomorphism $R\Gamma \cong \oplus_{i=1}^n R\Gamma'$, where n is the index of Γ' in Γ. Thus $\mathrm{Ext}^i_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} M, (R\Gamma \otimes_{R\Gamma'} M) \oplus R\Gamma) = 0$ for all $i > 0$. Consequently, by invoking our hypothesis, one may infer that $R\Gamma \otimes_{R\Gamma'} M$ is a projective $R\Gamma$-module, and so it will be projective over $R\Gamma'$. Now by making use of the fact that M is an $R\Gamma'$-direct summand of $R\Gamma \otimes_{R\Gamma'} M$, we conclude that M is a projective $R\Gamma'$-module, as desired. □

Proposition 3.3. *Let Γ be an abelian group and let Γ' be a subgroup of Γ of finite index. Then $R\Gamma$ satisfies the Auslander-Reiten conjecture if and only if so does $R\Gamma'$.*

Proof. According to Proposition 3.2, we only need to show the 'if' part. So assume that M is an $R\Gamma$-module such that $\mathrm{Ext}^i_{R\Gamma}(M, M \oplus R\Gamma) = 0$ for all $i > 0$. We intend to show that M is a projective $R\Gamma$-module. To this end, one should note that, since $R\Gamma$ is a commutative ring and $R\Gamma \cong \oplus_{i=1}^n R\Gamma'$, where n is the index of Γ' in Γ, we obtain the $R\Gamma$-isomorphism $R\Gamma \otimes_{R\Gamma'} M \cong \oplus_{i=1}^n M$. So, in order to obtain the desired result, it suffices to show that $R\Gamma \otimes_{R\Gamma'} M$ is a projective $R\Gamma$-module. The vanishing hypothesis imposed on M, yields that $\mathrm{Ext}^i_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} M, (R\Gamma \otimes_{R\Gamma'} M) \oplus R\Gamma) = 0$ for all $i > 0$. Therefore, by using the adjointness of Hom and $\otimes$, we may conclude that $\mathrm{Ext}^i_{R\Gamma'}(M, (R\Gamma \otimes_{R\Gamma'} M) \oplus R\Gamma) = 0$ for all $i > 0$, implying that $\mathrm{Ext}^i_{R\Gamma'}(M, M \oplus R\Gamma') = 0$ for all $i > 0$. Hence, our assumption leads us to infer that M is projective over $R\Gamma'$, and then $R\Gamma \otimes_{R\Gamma'} M$ will be a projective $R\Gamma$-module, as needed. □

Remark 3.4. Assume that Γ is a group and Γ' is a subgroup of Γ of finite index such that its index is invertible in R. Assume that M is an $R\Gamma$-module with $\mathrm{Ext}^i_{R\Gamma}(M, M \oplus R\Gamma) = 0$ for all $i > 0$. It is worth noting that although in this case M is an $R\Gamma$-direct summand of $R\Gamma \otimes_{R\Gamma'} M$, but $\oplus_{i=1}^n M \cong R\Gamma \otimes_{R\Gamma'} M$ only as $R\Gamma'$-modules, and so the same vanishing hypothesis does not hold for $R\Gamma \otimes_{R\Gamma'} M$. We would like to thank the referee for mentioning this point.

The next theorem is the main result of this paper. This result is effective in the sense that we can specify how many Ext functors must vanish to give the conclusion of the Auslander-Reiten conjecture.

Theorem 3.5. *Let Γ be an abelian group and let Γ' be a subgroup of finite index such that Γ/Γ' has periodic cohomology with period n. Assume that M is an $R\Gamma$-module, whose restriction to $R\Gamma'$ is projective. If $\mathrm{Ext}^n_{R\Gamma}(M, M) = 0$, then M is projective as an $R\Gamma$-module.*

Proof. Since Γ/Γ' is a finite group, as we have mentioned in Remark 2.7, it has periodic cohomology in the classical sense. By the hypothesis, the trivial $R(\Gamma/\Gamma')$-module R admits a projective resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{n-1} \stackrel{d_{n-1}}{\longrightarrow} \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0,$$

such that $\ker d_{n-1} \cong R$. It should be noted that this is also an exact sequence of $R\Gamma$-modules. Since this sequence splits over R, applying the functor $- \otimes_R M$ gives rise to the following exact sequence of $R\Gamma$-modules;

$$\cdots \to F_1 \otimes_R M \to F_0 \otimes_R M \to F_{n-1} \otimes_R M \stackrel{d_{n-1}}{\to} \cdots \to F_1 \otimes_R M \to F_0 \otimes_R M \to M \to 0.$$

According to [10, Proposition III 5.6], there is an $R\Gamma$-isomorphism $R\Gamma \otimes_{R\Gamma'} M \cong R(\Gamma/\Gamma') \otimes_R M$, implying that $R(\Gamma/\Gamma') \otimes_R M$ is a projective $R\Gamma$-module, because M is projective over $R\Gamma'$. Therefore, the latter exact sequence is a periodic projective resolution of the $R\Gamma$-module M. This, in turn, yields that $\mathrm{Ext}^i_{R\Gamma}(M, -) \cong \mathrm{Ext}^{i+n}_{R\Gamma}(M, -)$ for all $i > 0$. As M is a projective $R\Gamma'$-module, by [9, Construction 2.1], it will be Gorenstein projective over $R\Gamma$. Take a short exact sequence of $R\Gamma$-modules, $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$, where P is projective and L is Gorenstein projective. Consider the following isomorphisms;

$$\begin{aligned} \widehat{\mathsf{Ext}}^0_{R\Gamma}(M,M) &\cong \widehat{\mathsf{Ext}}^1_{R\Gamma}(M,L) \\ &\cong \mathrm{Ext}^1_{R\Gamma}(M,L) \\ &\cong \mathrm{Ext}^{n+1}_{R\Gamma}(M,L) \\ &\cong \mathrm{Ext}^n_{R\Gamma}(M,M). \end{aligned}$$

The first isomorphism follows from the fact that $\widehat{\mathsf{Ext}}^i_{R\Gamma}(M,P)=0$ for all $i \in \mathbb{Z}$, while the second isomorphism comes from Remark 2.5. The validity of the third isomorphism has been observed just above. Finally, the last isomorphism holds trivially. By the assumption, $\mathrm{Ext}^n_{R\Gamma}(M,M)=0$, and so $\widehat{\mathsf{Ext}}^0_{R\Gamma}(M,M)=0$. Hence, one may use [3, Theorem 5.9] and conclude that M has finite projective dimension. Consequently, [16, Proposition 10.2.3] (see also [18, Proposition 2.37]) forces M to be a projective $R\Gamma$-module. The proof is now finished. □

Corollary 3.6. *Let Γ be a finite acyclic group and let M be an $R\Gamma$-module which is projective over R. If $\mathrm{Ext}^2_{R\Gamma}(M,M)=0$, then M is a projective $R\Gamma$-module.*

Proof. According to [10, I. 6.3], Γ has periodic cohomology with period 2. Now letting Γ' be the trivial subgroup of Γ, the argument appearing in the proof of Theorem 3.5 yields that $\widehat{\mathsf{Ext}}^0_{R\Gamma}(M,M) \cong \mathrm{Ext}^2_{R\Gamma}(M,M)$. By the hypothesis, the right hand side vanishes and so the same is true for $\widehat{\mathsf{Ext}}^0_{R\Gamma}(M,M)$, implying that M has finite projective dimension. Moreover, since M is projective as an R-module, by [9, Construction 2.1] it will be a Gorensten projective $R\Gamma$-module. Thus M is a projective $R\Gamma$-module, as needed. □

As an immediate consequence of the above corollary, we record the next interesting result.

Corollary 3.7. *Let Γ be a finite acyclic group. Then the Auslander-Reiten conjecture holds for the class of all $R\Gamma$-modules which are projective over R.*

We end this paper with the following interesting result.

Corollary 3.8. *Let R be of finite global dimension and Γ a finite acyclic group. Then $R\Gamma$ satisfies the Auslander-Reiten conjecture.*

Proof. Pick an $R\Gamma$-module M with $\mathrm{Ext}^i_{R\Gamma}(M, M \oplus R\Gamma)=0$ for all $i>0$. As Γ is a finite group, combining Corollary 2.3 and Theorem 1.7 of [14], forces M to have finite Gorenstein projective dimension. So by making use of [18, Theorem 2.20], one may infer that M is indeed a Gorenstein projective module. Consequently, in view of [14, Proposition 1.1], M will be projective over R. Hence Corollary 3.6 implies that M is a projective $R\Gamma$-module, as well. So the proof is completed. □

Acknowledgments

The author would like to thank the referee for his/her careful reading and valuable comments. We also gratefully acknowledge the University of Gonbad-Kavous for financial support (grant number: 6.504).

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