

Global Weak Roman Domination in Graphs

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ABSTRACT. A *Roman dominating function* (RDF) of a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A vertex u with $f(u) = 0$ is said to be *undefended* if it is not adjacent to a vertex with $f(v) > 0$. For a graph G , a function $f: V(G) \rightarrow \{0, 1, 2\}$ is said to be a *weak Roman dominating function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f': V(G) \rightarrow \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weight of f is defined to be the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a weak Roman dominating function of a graph G is called the *weak Roman domination number* of G and is denoted by $\gamma_r(G)$. A WRDF with weight $\gamma_r(G)$ is called a $\gamma_r(G)$ -function. A set $S \subseteq V$ is a *global dominating set* if S dominates both G and its complement $\bar{G}$. The *global domination number* $\gamma_g(G)$ of a graph G is the minimum cardinality of a *global dominating set* S . We extend the idea of *global domination* to *weak Roman domination* as follows: For a graph G , the function $f: V(G) \rightarrow \{0, 1, 2\}$ is a *global weak Roman dominating function* (GWRDF) if f is a WRDF for both G and its complement $\bar{G}$. The weight of a *global weak Roman dominating function* is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a *global weak Roman*

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dominating function of a graph G is called the *global weak Roman domination number* of G and is denoted by $\gamma_{gr}(G)$. In this paper, we initiate a study of this parameter.

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1. INTRODUCTION

Cockayne *et al.* [1] defined a *Roman dominating function* (RDF) in a graph G to be a function $f: V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a *Roman dominating function* is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. Roman domination in graphs has been studied in [5, 6, 7, 11]. This definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart titled “Defend the Roman Empire” [13].

Henning *et al.* [4] defined the *weak Roman dominating function* as follows: For a graph G , let $f: V(G) \rightarrow \{0, 1, 2\}$ be a function. A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex v with the positive weight. The function $f: V(G) \rightarrow \{0, 1, 2\}$ is said to be a *weak Roman dominating function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f': V(G) \rightarrow \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. We say that v *defends* u . The weight of f is defined to be $\sum_{u \in V} f(u)$. The minimum weight of a weak Roman dominating function of a graph G is called the *weak Roman domination number* of G and is denoted by $\gamma_r(G)$. A WRDF with weight $\gamma_r(G)$ is called a $\gamma_r(G)$ -function. Much works related to Weak Roman domination in graphs has been done in [8, 9, 10]. A *weak Roman dominating function* f can also be written as $f = \{V_0, V_1, V_2\}$ where $V_i = \{v/f(v) = i\}$, $i = 0, 1, 2$. Notice that in a WRDF, every vertex in V_0 is dominated by at least one vertex in $V_1 \cup V_2$, while in an RDF every vertex in V_0 is dominated by atleast one vertex in V_2 . Furthermore, in a WRDF every vertex in V_0 can be defended without creating an undefended vertex.

A set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to a vertex of S and the minimum cardinality of a dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. E.Sampath Kumar [12], defined a *global dominating set* as follows: A set $S \subseteq V$ is a global dominating set if S dominates both G and its complement $\overline{G}$. The *global domination*

number $\gamma_g(G)$ is the minimum cardinality of a global dominating set in G and a global dominating set of minimum cardinality is called a γ_g -set. We extend the idea of *global domination* to *weak Roman domination* as follows: On a graph $G = (V, E)$, a function $f: V(G) \rightarrow \{0, 1, 2\}$ is said to be a *global weak Roman dominating function* (GWRDF) if f is a WRDF for both G and its complement $\overline{G}$. The weight of a *global weak Roman dominating function* is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a global weak Roman dominating function of a graph G is called the *global weak Roman domination number* of G and is denoted by $\gamma_{gr}(G)$. One can easily observe that for any graph G , $\gamma_g(G) \leq \gamma_{gr}(G)$ and $\gamma_r(G) \leq \gamma_{gr}(G)$. In this paper, we initiate a study of this parameter.

2. NOTATION

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges with $|V(G)| = n$. For notation and graph theoretic terminology, we in general follow [2, 3]. The *degree* of a vertex v in a graph G is the number of edges incident with v and is denoted by $\deg(v)$. A vertex of degree zero is called an *isolated* vertex, while a vertex of degree one is called a *leaf* vertex or a *pendant* vertex of G . We call a vertex v to be a *full degree vertex* if $\deg(v) = n - 1$. In any connected graph G , the *distance* between two vertices u and v is the number of edges in the shortest path joining u and v if any and is denoted by $d(u, v)$. The *eccentricity* of a vertex v is $\text{ecc}(v) = \max \{d(v, w) : w \in V\}$. The *radius* of a graph G is $\text{rad}(G) = \min \{\text{ecc}(v) : v \in V\}$ and the *diameter* of the graph G is $\text{diam}(G) = \max \{\text{ecc}(v) : v \in V\}$. For $n \geq 4$, the *wheel* W_n is defined to be the graph $K_1 + C_{n-1}$ where C_{n-1} is a cycle on $(n - 1)$ vertices. A graph in which any two distinct vertices are adjacent is called a complete graph, denoted by K_n . A complete bipartite graph of the form $K_{1,n}$ is called a *star* graph. We call the vertex of degree n , the *head* vertex. A maximal complete subgraph of a graph G is called a *clique*. A connected graph having no cycle is called a *tree*. A set S of vertices is called *independent* if no two vertices in S are adjacent. For any set $S \subseteq V$, the *induced subgraph* S is the maximal subgraph of G with vertex set S and is denoted by $G[S]$. A *support* is a vertex which is adjacent to at least one leaf vertex. A *weak support* is a vertex which is adjacent to exactly one leaf vertex. A *strong support* is a vertex which is adjacent to at least two leaf vertices. A set $S \subseteq V$ is said to be a *2-packing* if $d(u, v) \geq 3$ for every pair of vertices u, v in S .

3. PATHS, CYCLES AND WHEELS

In this section, we investigate the global weak Roman domination number of paths, cycles and wheels. One can easily observe that for complete graphs K_n , $\gamma_{gr}(K_n) = n$. We omit the proof of the following theorem as it is straightforward.

Theorem 3.1. For $n \geq 4$, $\gamma_{gr}(C_n) = \gamma_{gr}(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.

One can easily observe that for wheels W_n , $\gamma_r(W_n) = 2$. In the following theorem, we determine the γ_{gr} -values for wheels.

Theorem 3.2. For wheels $G = W_n$, $n \geq 4$

$$\gamma_{gr}(G) = \begin{cases} 4, & n = 4, \\ 3, & n = 5, \\ 4, & n \geq 6. \end{cases}$$

Proof. Let v be the vertex in W_n such that $\deg(v) = n - 1$. Let v_1, v_2 be two adjacent vertices on the cycle C_{n-1} .

When $n = 4$, clearly $G = K_4$. Therefore, $\gamma_{gr}(G) = 4$.

When $n = 5$, we define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(v) = f(v_1) = f(v_2) = 1$ and $f(v_3) = f(v_4) = 0$.

When $n = 6$, we define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(v) = f(v_2) = 1$, $f(v_1) = 2$ and $f(w) = 0$ for every $w \in V(W_n) \setminus \{v, v_1, v_2\}$.

When $n \geq 7$, we define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(v) = 2$, $f(v_1) = f(v_2) = 1$ and $f(w) = 0$ for every $w \in V(W_n) \setminus \{v, v_1, v_2\}$. In all the cases, clearly, f is a $\gamma_{gr}(G)$ -function. Hence the proof. $\square$

4. GRAPHS WITH GLOBAL WEAK ROMAN DOMINATION NUMBER n AND $n - 1$

In this section we characterize graphs with global weak Roman domination number n and $n - 1$ respectively.

Theorem 4.1. For any graph G , $\gamma_{gr}(G) = n$ if and only if G or $\overline{G} = K_n$.

Proof. Suppose that $G = K_n$, then clearly $\gamma_{gr}(G) = n$.

Conversely, suppose that $\gamma_{gr}(G) = n$. We claim that either $G = K_n$ or $\overline{G} = K_n$. Suppose to the contrary that neither G nor $\overline{G}$ is K_n . Then there exists two vertices x and y in G such that $xy \in E(G)$. Choose x such that $\deg(x) < n - 1$. We define $f : V(G) \rightarrow \{0, 1, 2\}$ by $f(x) = 0$, $f(u) = 1$ for all $u \in V(G) \setminus \{x\}$. This implies $f(V) \leq n - 1$, a contradiction. Hence $\gamma_{gr}(G) = n$. $\square$

Theorem 4.2. For any graph G , $\gamma_{gr}(G) = n - 1$ if and only if either G or $\overline{G}$ is one of the graphs $K_{1,3}$, $K_n - e$ or $K_n - 2e$ where the two edges are incident.

Proof. Suppose that $\gamma_{gr}(G) = n - 1$. Clearly, G is not complete. If $n = 3$, one can easily verify that $\gamma_{gr}(G) = 2$ implies that $G = K_3 - e$ or $G = K_3 - 2e$. If $n = 4$, then we claim that $G = K_4 - e$ or $G = K_4 - 2e$ where both the edges are incident or $K_4 - 3e$ where the three edges either form a triangle in K_4 or they are incident in K_4 . Suppose to the contrary that $G = K_4 - 2e$ where the two edges are not incident in K_4 or $K_4 - 3e$ where the three edges form a P_4 in K_4 . In the former case, $G = C_4$ and $\gamma_{gr}(G) = 2$ and

in the latter case $G = P_4$ and $\gamma_{gr}(G) = 2$, which is a contradiction. Thus, if $G = K_4 - 3e$ and the three edges form a triangle in K_4 , then $G = K_{1,3}$. If $G = K_4 - 3e$ and the three edges are incident, then $G = K_3 \cup K_1$ and thus $\overline{G} = K_{1,3}$.

If $n \geq 5$, we claim that $G = K_n - e$ or $K_n - 2e$, where the two edges are incident in $\overline{G}$. Assume that $n \geq 5$. Since $\gamma_{gr}(G) = n - 1$, there exists a γ_{gr} -function of G say $f = (V_0, V_1, V_2)$ such that one of the following holds.

- i) $|V_2| = 0, |V_1| = n - 1, |V_0| = 0$.
- ii) $|V_2| = 1, |V_1| = n - 3, |V_0| = 2$.

Suppose that (i) holds. Let $x \in V_0$. Clearly, x is adjacent to some vertex in V_1 say y . Now, V_1 induces either a K_{n-1} or a $(n-1)K_1$ in G . If V_1 induces a K_{n-1} , we claim that G is either $K_n - e$ or $K_n - 2e$, where the two edges are incident in $\overline{G}$. That is, we claim that x is adjacent to at least $n - 3$ vertices of V_1 . Suppose to the contrary that x is not adjacent to at least three vertices in V_1 . Then, define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(v) = 2$, if $v = x$, $g(v) = 0$, if $v \notin N(x)$ and $g(v) = 1$, otherwise. Clearly, g is a GWRDF with $g(V) = n - 2$, a contradiction. Hence, x is adjacent to at least $n - 3$ vertices in V_1 . If x is adjacent to $n - 2$ vertices in V_1 , then $G = K_n - e$. If x is adjacent to $n - 3$ vertices in V_1 , then $G = K_n - 2e$, where both the edges are incident with x in $\overline{G}$. If V_1 induces $(n - 1)K_1$, then we claim that G is either $P_2 \cup (n - 2)K_1$ or $P_3 \cup (n - 3)K_1$. That is, we claim that x is adjacent to at most two vertices in V_1 . Suppose to the contrary that x is adjacent to at least three vertices in V_1 . Then, define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(v) = 2$, if $v = x$, $g(v) = 0$, if $v \in N(x)$ and $g(v) = 1$, otherwise. Clearly, g is a GWRDF with $g(V) = n - 2$, a contradiction. Hence, x is adjacent to at most two vertices in V_1 . If x is adjacent to exactly two vertices in V_1 , then $G = P_3 \cup (n - 3)K_1$. That is, $\overline{G} = K_n - 2e$ where the two edges are incident in G . If x is adjacent to only y in V_1 , then $G = P_2 \cup (n - 2)K_1$. That is, $\overline{G} = K_n - e$.

Suppose that (ii) holds. Let $V_0 = \{x, y\}$ and $V_2 = \{z\}$ (x and y are not adjacent). Now, V_1 induces either a K_{n-3} or a $(n - 3)K_1$. Suppose that V_1 induces a K_{n-3} . First, we claim that z is adjacent to every vertex of K_{n-3} . Suppose that z is not adjacent to a vertex say w in K_{n-3} . Then, define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(v) = 1$, if $v \in \{y, z, w\}$ and $g(v) = 0$, otherwise. Clearly, g is a GWRDF with $g(V) = n - 2$, a contradiction. Hence z is adjacent to every vertex in K_{n-3} . Next, we claim that either x or y is not adjacent to exactly one vertex of K_{n-3} . Suppose to the contrary that x is not adjacent to at least two vertices say, a, b in K_{n-3} . If y is not adjacent to a and b , then define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(v) = 1$, if $v \in \{y, z, b\}$ and $g(v) = 0$, otherwise. If y is adjacent to a and y is not adjacent to b , then define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(v) = 1$, if $v \in \{z, a, b\}$ and $g(v) = 0$, otherwise. If y is not adjacent to a and b , then define $g : V(G) \rightarrow \{0, 1, 2\}$ by $g(v) = 2$, if $v = z$, $g(v) = 1$, if $v = x$

and $g(v) = 0$, otherwise. Hence, in the above three cases, we see that g is a GWRDF of G with $g(V) = n - 2$, a contradiction. Hence, x is not adjacent to at most one vertex of K_{n-3} . Similarly, y is not adjacent to at most one vertex of K_{n-3} .

If x and y are not adjacent to a and b ($a \neq b$) then by assigning 0 to a and b and 1 elsewhere, we get a GWRDF of weight $n - 2$, a contradiction. If x and y are not adjacent to the same vertex say a , then by assigning 0 to both y and a and 0 elsewhere, we get a GWRDF of weight $n - 2$, a contradiction. Hence, exactly one of x or y is not adjacent to at most one vertex of K_{n-2} . Without loss of generality, let x be not adjacent to at most one vertex of K_{n-3} and y be adjacent to every vertex of K_{n-3} . If x is not adjacent to exactly one vertex of K_{n-3} , then $G = K_n - 2e$, where the two edges are incident at x in $\overline{G}$. If x is adjacent to every vertex of K_{n-3} , then $G = K_n - e$ where $e = xy$.

Suppose that V_1 induces a $(n-3)K_1$, then we claim that $G = P_3 \cup (n-3)K_1$. Since, $n \geq 5$, at least two vertices are in K_{n-3} say a and b . Now, we claim that $\deg(a) = \deg(b) = 0$. Suppose to the contrary that either $\deg(a) > 0$ or $\deg(b) > 0$. If $\{x, y, z, a, b\}$ induces a $K_{1,3} \cup K_1$, then by assigning 2 to z , the head vertex of $K_{1,3}$, 0 to the leaf vertices of $K_{1,3}$ and 1 elsewhere, we get a GWRDF of weight $n - 2$, a contradiction. Otherwise by assigning 0 to a and y (where a is adjacent to x) and 1 otherwise we get a GWRDF of weight $n - 2$, a contradiction. Hence, $G = P_3 \cup (n-3)K_1$. That is, $\overline{G} = K_n - 2e$, where the two edges are xz and yz . Thus, G or $\overline{G}$ is one of the graphs $K_{1,3}$, $K_n - e$, $K_n - 2e$.

Conversely, if $G = K_{1,3}$ then, $\overline{G} = K_1 \cup K_3$. Therefore, $\gamma_{gr}(G) = 3$. Suppose that $G = K_n - e$. Then, G contains $n - 2$ full degree vertices. Hence, $\overline{G} = (n - 2)K_1 \cup K_2$. Therefore $\gamma_{gr}(G) = n - 1$. If $G = K_n - 2e$, where the two edges are incident, then G contains $n - 3$ full degree vertices. Hence, $\overline{G} = (n - 3)K_1 \cup P_3$. Therefore $\gamma_{gr}(G) = n - 1$. $\square$

5. TREES WITH $\gamma_{gr}(T) = \gamma_r(T) + 1$

In this section we prove that for any tree T , $\gamma_{gr}(T)$ is bounded above by $\gamma_r(T) + 1$ and bounded below by $\gamma_r(T)$. Further we characterize the class of all trees for which $\gamma_{gr}(T) = \gamma_r(T) + 1$.

Theorem 5.1. *For any tree T , $\gamma_r(T) \leq \gamma_{gr}(T) \leq \gamma_r(T) + 1$.*

Proof. Let $f = \{V_0, V_1, V_2\}$ be a $\gamma_r(T)$ -function.

Case (i). $V_2 \neq \emptyset$.

If there are two vertices $v, w \in V_2$ such that $d(v, w) \geq 3$ or $d(v, w) = 1$ then $\{v, w\}$ dominates $\overline{T}$. Hence f is a $\gamma_r(\overline{T})$ -function. Therefore $\gamma_r(T) = \gamma_{gr}(T)$.

If for every $v, w \in V_2$, $d(v, w) = 2$, then let $z \in N(v) \cap N(w)$. Now $g: V(G) \rightarrow \{0, 1, 2\}$ defined by $g(z) = 1$ and $f(x) = g(x)$ for every $x \in V(T) \setminus \{z\}$ is a WRDF of $\overline{T}$. Hence $\gamma_{gr}(T) \leq \gamma_r(T) + 1$.

If $|V_2| = 1$, then let $V_2 = \{v\}$. If T is a star with $n \geq 4$, clearly $\gamma_{gr}(T) = 3$ and $\gamma_r(T) = 2$. Thus, $\gamma_{gr}(T) = \gamma_r(T) + 1$. When $T = P_3$, $\gamma_{gr}(T) = \gamma_r(T)$. If T is not a star, then $|V_1| \geq 1$. Suppose that there is a vertex $y \in V_1$ such that $d(v, y) \neq 2$. (Otherwise choose f such that $d(v, y) \neq 2$ for some $y \in V_1$). Now $\{v\}$ dominates $V(T) \setminus N(v)$ in $\bar{T}$ and $N(v) \cup \{y\}$ induces a clique in $\bar{T}$. Hence f is a $\gamma_r(\bar{T})$ -function. Therefore $\gamma_r(T) = \gamma_{gr}(T)$.

Case (ii). $V_2 = \emptyset$.

If $T = P_2$, then clearly $\gamma_{gr}(T) = \gamma_r(T) + 1$. Let $T \neq P_2$. Then $|V_1| \geq 2$. If $|V_1| = 2$, then $T = P_4$ or P_3 . In both the cases $\gamma_r(T) = \gamma_{gr}(T)$. Let $|V_1| \geq 3$. Let $v, w \in V_1$. In $\bar{T}$, both v and w are adjacent to every vertex of $V(T) \setminus \{(N_T(v) \cup N_T(w))\}$. Further, $A = \{v\} \cup (N_T(w) \setminus \{y\})$ and $B = \{w\} \cup (N_T(v) \setminus \{y\})$, where $y \in N(v) \cap N(w)$ induces a clique in $\bar{T}$ respectively. (The possibility that $N(v) \cap N(w) = \emptyset$ is not ruled out). Also, since $|V_1| \geq 3$, there exists a $z \in V_1$, such that either $N_T(w) \cup \{z\}$ or $N_T(v) \cup \{z\}$ induces a clique in $\bar{T}$. Without loss of generality, let $N_T(w) \cup \{z\}$ induce a clique in $\bar{T}$. Now, for every $u \in V_0 \cap A$ and $u \in V_0 \cap B$, $f' : V(\bar{T}) \rightarrow \{0, 1, 2\}$ is defined by $f'(z) = f(z) - 1$, $f'(u) = f(u) + 1$ and $f'(x) = f(x)$ and $f'(w) = f(w) - 1$, $f'(u) = f(u) + 1$, $f'(x) = f(x)$ respectively for every $x \in V(T) \setminus \{u, z, w\}$ leaves no vertex undefended in $\bar{T}$. Hence f is a $\gamma_r(\bar{T})$ - function. Therefore, $\gamma_r(T) = \gamma_{gr}(T)$. $\square$

Theorem 5.2. *For any tree T , $\gamma_{gr}(T) = \gamma_r(T) + 1$ if and only if each support vertex of T is adjacent to at least three leaf vertices and the distance between any two support vertices is exactly two.*

Proof. Suppose that $\gamma_{gr}(T) = \gamma_r(T) + 1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_r(T)$ -function. First we claim that the distance between any two support vertices is exactly two. Suppose that there are two support vertices x and y such that $d(x, y) \neq 2$. Now we consider the following cases. **Case (i).** $x \in V_1 \cup V_2$, $y \in V_2$.

If $x, y \in V_2$, then clearly, $\{x, y\}$ is a dominating set in $\bar{T}$. Thus f is a $\gamma_{gr}(T)$ -function, a contradiction.

If $x \in V_1$ and $y \in V_2$, then $\{y\}$ dominates $V(T) \setminus N(y)$ in $\bar{T}$ and $N(y) \cup \{x\}$ induces a clique in $\bar{T}$. Hence f is a $\gamma_{gr}(T)$ -function. Therefore $\gamma_{gr}(T) = \gamma_r(T)$, a contradiction.

Case (ii). $x, y \in V_1$.

Clearly $|V_1 \cup V_2| \geq 3$. Hence there exists a vertex $z \in V_2 \cup V_1$. Suppose that $z \in N_T(x) \cup N_T(y)$. Without loss of generality, let $z \in N_T(x)$. Then both $N_T(x) \cup \{z\}$ and $N_T(y) \cup \{x\}$ induce cliques in $\bar{T}$ respectively. Further, for each $u \in V_0 \cap (N(x) \cap N(y))$ in $\bar{T}$ the function $f' : V(T) \rightarrow \{0, 1, 2\}$ defined by $f'(u) = f(u) + 1$, $f'(y) = f(y) - 1$, $f'(w) = f(w)$, for every $w \in V(T) \setminus \{u, y\}$ leaves no vertex undefended. Hence $\gamma_{gr}(T) = \gamma_r(T)$, a contradiction.

If $z \notin N_T(x) \cup N_T(y)$, then either $d(z, y) \geq 3$ or $d(z, x) \geq 3$. Without loss of generality let $d(z, y) \geq 3$ then both $\{z\} \cup N_T(y)$ and $\{y\} \cup N_T(x)$ induce cliques in $\bar{T}$ respectively. Further for each $u \in V_0 \cap (N(x) \cap N(y))$ in $\bar{T}$, the function $f' : V(T) \rightarrow \{0, 1, 2\}$ defined by $f'(u) = f(u) + 1$, $f'(x) = f(x) - 1$, $f'(w) = f(w)$, for every $w \in V(T) \setminus \{u, x\}$ leaves no vertex undefended. Hence $\gamma_{gr}(T) = \gamma_r(T)$, a contradiction. Therefore $d(x, y) = 2$. Hence we conclude that the distance between any two support vertices is exactly two.

Finally, we claim that each support vertex is adjacent to at least three leaf vertices. Suppose to the contrary, that there is a support vertex say x , which is adjacent to exactly two leaf vertices. Let u_1 and u_2 be the leaf neighbors of x . If there is a strong support vertex $w \neq x$, then define $g : V(T) \rightarrow \{0, 1, 2\}$ such that

$$g(v) = \begin{cases} 2, & \text{if } v \text{ is a strong support and } v \neq x, \\ 1, & \text{if } v = x \text{ or } v = u_1 \text{ or } v \text{ is a weak support} \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\{w\}$ dominates $V(T) \setminus N(w)$, and $\{u_1\}$ induces a clique with $N(w)$. Therefore, g is a $\gamma_{gr}(T)$ -function and a $\gamma_r(T)$ -function as well. Thus, $\gamma_{gr}(T) = \gamma_r(T)$, a contradiction.

If T does not have a strong support vertex other than x , then we define $g : V(T) \rightarrow \{0, 1, 2\}$ as follows:

$$g(v) = \begin{cases} 2, & \text{if } v = x, \\ 1, & \text{if } v \text{ is a leaf vertex and } v \notin N(x) \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\{x\}$ dominates $V(T) \setminus N(x)$ and $N(x) \cup \{z\}$ induces a clique, where z is a leaf vertex which is not adjacent to x . Therefore, g is a $\gamma_{gr}(T)$ -function and a $\gamma_r(T)$ -function as well. Thus $\gamma_{gr}(T) = \gamma_r(T)$, a contradiction.

If there is no strong support vertex in T , then we define $g : V(T) \rightarrow \{0, 1, 2\}$ as follows:

$$g(v) = \begin{cases} 0, & \text{if } v \text{ is a leaf vertex,} \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, g is a $\gamma_{gr}(T)$ -function and a γ_r function as well. Hence, each support vertex is adjacent to at least three leaf vertices. Thus $\gamma_{gr}(T) = \gamma_r(T)$, a contradiction. Hence the proof.

Conversely, if T is of the given type, then clearly $\gamma_{gr}(T) = 2s + 1$ and $\gamma_r(T) = 2s$, where s is the number of support vertices.

Hence $\gamma_{gr}(T) = \gamma_r(T) + 1$. □

6. TREES WITH $\gamma_{gr}(T) = \gamma_g(T)$

In this section we characterize trees T with $\gamma_{gr}(T) = \gamma_g(T)$. We now define a family of trees $\mathfrak{S}$ as follows:

A tree $T \in \mathfrak{S}$ if the following holds.

- (i) T has no strong support vertex.
- (ii) Every non-support vertex w which is adjacent to a support vertex has at least one neighbor, which is neither a support vertex nor adjacent to a support vertex.
- (iii) For every vertex w which is neither a support vertex nor adjacent to a support vertex, each member of $N(w)$ is adjacent to a support vertex. If more than one such vertex exists then the set of all such vertices form a *2-packing*.

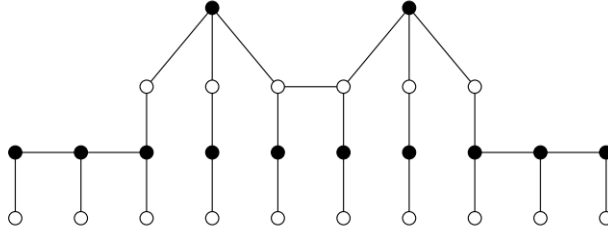


FIGURE 1. A tree $T \in \mathfrak{S}$. Dark circles indicate a γ_g -set of T .

Theorem 6.1. *For any tree T , $\gamma_{gr}(T) = \gamma_g(T)$ if and only if $T \in \mathfrak{S}$.*

Proof. Suppose that $\gamma_{gr}(T) = \gamma_g(T)$. We claim that $T \in \mathfrak{S}$. Since $\gamma_{gr}(T) = \gamma_g(T)$, for every γ_r -function $f = (V_0, V_1, V_2)$ of T , $V_2 = \emptyset$ and V_1 is a $\gamma_g(T)$ -set. First, we claim that T has no strong support vertex. Suppose that, w is a strong support vertex. Then, there is a γ_r -function g of T such that $g(w) = 2$, which implies that $V_2 \neq \emptyset$, a contradiction. Hence, no vertex is a strong support vertex. Hence (i) holds. Let w be a non-support vertex which is adjacent to a support vertex. Now, we claim that w has at least one neighbor which is neither a support vertex nor adjacent to a support vertex. Suppose to the contrary, that each member of $N(w)$ is either a support vertex or adjacent to a support vertex. Then, w is not defended by any vertex in V_1 , which implies that f is not a WRDF, a contradiction. Hence (ii) holds. Let w be a vertex which is neither a support vertex nor adjacent to a support vertex. We claim that each member of $N(w)$ is adjacent to a support vertex. Suppose that at least one member say w_1 in $N(w)$ is not adjacent to a support vertex. Let $w_2 \in N(w) \setminus \{w_1\}$. Let $V_1 = S$. Now, we consider the following cases.

Case (i). $w \in S$, $w_1, w_2 \notin S$.

Subcase (a). $w_1, w_2 \in D_G(w)$.

Now, since $w_1, w_2 \in D_G(w)$, there exists vertices $a_1 \in N(w_1) \setminus \{w\}$ and $b_1 \in N(w_2) \setminus \{w\}$ respectively such that $a_1, b_1 \in S$. By our assumption, a_1 is not a support.

If $D_G(a_1) = \emptyset$, then $(S \setminus \{a_1, w\} \cup \{w_1\})$ is a GDS of G , a contradiction to the minimality of S . If $D_G(a_1) \neq \emptyset$, let $a_2 \in D_G(a_1)$. Now, a_2 is not a leaf vertex. Hence, there exists a $a_3 \in N(a_2) \setminus \{a_1\}$. Now, if $a_3 \in S$, then $(S \setminus \{a_1, w\} \cup \{w_1\})$ is a GDS of G , a contradiction to the minimality of S . If $a_3 \notin S$, then there exists a $z \in N(a_3) \setminus \{a_2\}$ such that $z \in S$ and z defends a_3 . Now, $(S \setminus \{z, a_1, w\}) \cup \{w_1, a_3\}$ is a GDS of T , contradicting the minimality of S .

Subcase (b). $w_1 \in D_G(w)$, $w_2 \notin D_G(w)$.

Since $w_2 \notin D_G(w)$, there exists a vertex $b_1 \in N(w_2) \setminus \{w\}$ such that $b_1 \in S$ and b_1 defends w_2 . Now, $(S \setminus \{b_1, w\}) \cup \{w_2\}$ is a GDS of G , contradicting the minimality of S .

Subcase (c). $w_1 \notin D_G(w)$, $w_2 \in D_G(w)$.

Since $w_1 \notin D_G(w)$, there exists a vertex $a_1 \in N(w_1) \setminus \{w\}$ such that $a_1 \in S$ and a_1 defends w_1 . Now, $S \setminus \{a_1, w\} \cup \{w_1\}$ is a GDS of G , contradicting the minimality of S .

Subcase (d). $w_1, w_2 \notin D_G(w)$.

There exists a $a_1 \in N(w_1) \setminus \{w\}$ and $b_1 \in N(w_2) \setminus \{w\}$, such that $a_1, b_1 \in S$ and a_1 defends w_1 and a_2 defends w_2 . If $D_G(w) = \emptyset$, then $(S \setminus \{a_1, b_1, w\}) \cup \{w_1, w_2\}$ is a GDS of G , contradicting the minimality of S . If $D_G(w) \neq \emptyset$, then there exists a $w_3 \in (N(w) \setminus \{w_1, w_2\})$ such that $w_3 \in D_G(w)$. Now, $w_1 \notin D_G(w)$ and $w_3 \in D_G(w)$. Then, as discussed in subcases (b) and (c), we get a contradiction.

Case (ii). $w \in S$, $w_1 \in S$, $w_2 \notin S$.

Since $\deg w_1 > 1$, there exists a vertex $a_1 \in (N(w_1) \setminus \{w\})$. If $D_G(w_1) = \emptyset$, then $(S \setminus \{w_1\})$ is a GDS of G contradicting the minimality of S . If $D_G(w_1) \neq \emptyset$ without loss of generality, let $a_1 \in D_G(w_1)$. Since w_1 is not a support, there is a vertex $a_2 \in N(a_1) \setminus \{w_1\}$. If $a_2 \notin S$, then there is a vertex $a_3 \in N(a_2) \setminus \{a_1\}$, such that $a_3 \in S$ and $a_2 \in D_G(a_3)$. Then $(S \setminus \{w_1, a_3\}) \cup \{a_2\}$ is a GDS of G , contradicting the minimality of S .

Case (iii). $w \in S$, $w_1 \notin S$, $w_2 \in S$.

Without loss of generality, let $w_1 \in D_G(w)$. As in the Case 1, there exists vertices a_1, a_2 such that w_1, a_1, a_2 form a path in that order. If $a_1 \in S$, then $(S \setminus \{w\})$ is a GDS of G , contradicting the minimality of S . If $a_1 \notin S$, then without loss of generality, let $a_2 \in S$ and a_2 defends a_1 . Now $(S \setminus \{w, a_2\}) \cup \{a_1\}$ is a GDS of G , contradicting the minimality of S .

Case (iv). $w \notin S$, $w_1, w_2 \in S$.

Without loss of generality, let $w \in D_G(w_1)$. Then, there exists $a_1 \in N(w_1) \setminus \{w\}$. If $a_1 \in S$, then $S \setminus \{w_1\}$ is a GDS of G , contradicting the minimality of S .

If $a_1 \notin S$, then there exists $a_2 \in N(a_1) \setminus \{w_1\}$ such that $a_2 \in S$ and a_2 defends a_1 . Now, $(S \setminus \{w_1, a_2\}) \cup \{a_1\}$ is a GDS of G , contradicting the minimality of S .

Case (v). $w \notin S$, $w_1 \in S$, $w_2 \notin S$, $w \in D_G(w_1)$.

Since w_2 is not a support vertex, there exists vertices b_1, b_2 such that w_2, b_1, b_2 form a path in that order with $b_1 \in S$ and b_1 defends w_2 . Now, if $b_2 \in S$, then $(S \setminus \{b_1, w_1\}) \cup \{w\}$ is a GDS of G , contradicting the minimality of S . Hence, $b_2 \notin S$. If some vertex z in $N(b_2)$ other than b_1 can defend b_2 , then $(S \setminus \{b_1, w_1, z\}) \cup \{b_2, w\}$ is a GDS of G , contradicting the minimality of S . If some vertex z other than b_1 can defend w_2 , then $(S \setminus \{z, b_1\}) \cup \{w_2\}$ is a GDS of G , contradicting the minimality of S . Hence, $w_2, b_2 \in D_G(b_1)$. Next, we claim that $N(w_2) \cap S = \{b_1\}$. Suppose that, there exists a vertex $z \in (N(w_2) \setminus \{w, b_1\}) \cap S$. Note that z is not a support vertex and z cannot defend w_2 . Hence, there exists vertices z_1, z_2, z_3 such that z, z_1, z_2, z_3 form a path in that order with $z_3 \in S$, $z_1, z_2 \notin S$ and z_3 defends z_2 . Now, $(S \setminus \{z_3, z, b_1\}) \cup \{z_2, w_2\}$ is a GDS of G , contradicting the minimality of S . Now, when b_1 defends b_2 , w_2 is left undefended, a contradiction.

Let $W = \{v/v \text{ is neither a support vertex nor adjacent to a support vertex}\}$. We claim that W is a 2-packing. Suppose that, $v_1, v_2 \in W$ such that $d(v_1, v_2) \leq 2$. Then, clearly $v_1, v_2 \in S$. If $d(v_1, v_2) = 1$, then $(S \setminus \{v_1\})$ is a GDS of G , contradicting the minimality of S . If $d(v_1, v_2) = 2$, then $(S \setminus \{v_1, v_2\}) \cup \{x\}$, where $x \in N(v_1) \cap N(v_2)$ is a GDS of G , contradicting the minimality of S . Hence, W is a 2-packing. Hence $T \in \mathfrak{F}$.

Conversely suppose that $T \in \mathfrak{F}$. We claim that $\gamma_{gr}(T) = \gamma_g(T)$. Now let $W = \{v \in V(T) : v \text{ is neither a support vertex nor adjacent to a support vertex}\}$. Let $A =$ set of all support vertices. Now $W \cup A$ is a γ_g -set of T . Now define $f : V(T) \rightarrow \{0, 1, 2\}$ such that

$$f(v) = \begin{cases} 1, & \text{if } v \in W \cup A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, clearly f is a γ_{gr} -function of T and $\sum_{v \in V} f(v) = |W \cup A| = \gamma_g(T)$. Therefore $\gamma_{gr}(T) = \gamma_g(T)$. $\square$

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