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# Spaceability on Morrey Spaces

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Abstract. In this paper, as a main result for Morrey spaces, we prove that the set $\mathcal{M}^p_q(\mathbb{R}^n)\setminus\bigcup_{q<r\le p}\mathcal{M}^p_r(\mathbb{R}^n)$ is spaceable in $\mathcal{M}^p_q(\mathbb{R}^n)$, where $0<q<p<\infty$.

**Keywords:** Spaceability, Morrey spaces, Banach spaces.

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## 1. Introduction

Let $0<q\le p<\infty$. Define the *Morrey (quasi-)norm* $\|\cdot\|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q}\equiv\sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(Q)}\ :\ Q \text{ is a cube in } \mathbb{R}^n\right\}$$

for a measurable function $f$, where a cube is defined to be the set of the form $\{a+y\,:\,a\in\mathbb{R}^n, y\in[0,t]^n\}$ for some $a\in\mathbb{R}^n$ and $t>0$. The *Morrey space* $\mathcal{M}^p_q$ is the set of all measurable functions $f$ for which $\|f\|_{\mathcal{M}^p_q}$ is finite. Morrey spaces date back to 1938, when C.B. Morrey considered elliptic differential equations and discussed continuity of the solutions using a lemma [7]. His lemma was refined by Peetre [8]. Morrey's lemma gave rise to the theory of function spaces;

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see [8] and [10]. The function spaces dealt with are called Morrey spaces. Let $0 < q < p < \infty$. For each $r \in (q,p]$ the set $\mathcal{M}^p_r$ is a proper subset of $\mathcal{M}^p_q$ [4, 9]. In this paper, we show that the difference of these two sets is large enough. Precisely, by a techmical lemma we prove:

**Theorem 1.1.** *If $0 < q < p$, then $\mathcal{M}^p_q \setminus \bigcup_{q<r\le p} \mathcal{M}^p_r$ is a spaceable subset of $\mathcal{M}^p_q$.*

Another different proof will also be given via [6, Theorem 3.3]. We recall that a subset $S$ of a topological vector space $X$ is called *spaceable* if $S \cup \{0\}$ contains a closed infinite-dimensional linear subspace of $X$. The concepts spaceability and lineability were introduced by the paper [1] and then have been studied on different kinds of function or sequence spaces (see [5, 6]). In particular, for spaceability of the difference of Lebesgue spaces (see [2, 3, 11]).

## 2. Main Result

Let $p > q > 0$ and $R > 1$ be fixed so that

$$(R+1)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}}. \tag{2.1}$$

For a vector $\varepsilon \in \{0,1\}^n$, we define an affine transformation $T_\varepsilon$ by

$$T_\varepsilon(x) \equiv \frac{1}{R+1}x + \frac{R}{R+1}\varepsilon \quad (x \in \mathbb{R}^n).$$

Let $E_0 := [0,1]^n$. Suppose that we have defined $E_0, E_1, E_2, \ldots, E_j$. Define

$$E_{j+1} := \bigcup_{\varepsilon\in\{0,1\}^n} T_\varepsilon(E_j)$$

and

$$E_{j,0} := [0,(1+R)^{-j}]^n.$$

The following technical lemma is proved in [10, Proposition 4.1]. Here, for the sake of convenience of readers, we reproduce the proof.

**Lemma 2.1.** *Under above notations we have*

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{-j\frac{n}{p}} = \|\chi_{E_{j,0}}\|_{\mathcal{M}^p_q} = \|\chi_{E_{j,0}}\|_p = \|\chi_{E_j}\|_q, \tag{2.2}$$

*where the implicit constants in $\sim$ does not depend on $j$ but can depend on $p$ and $q$ and $\|\cdot\|_{\mathcal{M}^p_q}$ is the Morrey norm.*

*Proof.* A direct calculation shows that

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \ge \|\chi_{E_{j,0}}\|_{\mathcal{M}^p_q} = (1+R)^{-j\frac{n}{p}} = \|\chi_{E_{j,0}}\|_p = \|\chi_{E_j}\|_q,$$

Thus, we need to show

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \lesssim \|\chi_{E_{j,0}}\|_p.$$

Let us calculate

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim \sup_{S\in\mathcal{Q}} |S|^{\frac{1}{p}-\frac{1}{q}}|S\cap E_j|^{\frac{1}{q}},$$

where $\mathcal{Q}$ denotes the set of all cubes. Fix $j \in \mathbb{N}$. Let us temporally say that $Q \in \mathcal{Q}$ is wasteful, if there exists a cube $S \in \mathcal{Q}$ such that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |S|^{\frac{1}{p}-\frac{1}{q}}|S \cap E_j|^{\frac{1}{q}}.$$

Thus by definition, if $\ell(Q) := |Q|^{\frac{1}{n}} > 1$, then

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |E_j|^{\frac{1}{q}} = |[0,1]^n|^{\frac{1}{p}-\frac{1}{q}}|[0,1]^n \cap E_j|^{\frac{1}{q}}.$$

In addition, if the side-length of a cube $Q$ is less than $(R+1)^{-j}$, then it is wasteful. Indeed, then the equalities

$$\begin{aligned}
&\sup\left\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} \,:\, Q \in \mathcal{Q}, |Q| \le (R+1)^{-jn}\right\}\\
&= \sup\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} \,:\, Q \in \mathcal{Q}, Q \subset E_j\}\\
&= \sup\left\{|Q|^{\frac{1}{p}} \,:\, Q \in \mathcal{Q},\, Q \subset E_j\right\}\\
&= |E_{j,0}|^{\frac{1}{p}}
\end{aligned}$$

hold. Here, we obtain the first equality by translating $Q$ so that $Q$ which is included in $E_{j,0}$. This calculation shows that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} < |E_{j,0}|^{\frac{1}{p}}$$

for any such cube. Thus, if the cube $Q$ is not wasteful, then $(R+1)^{-j} \le \ell(Q) \le 1$. So, there exists $k \in \{1, 2, \ldots, n\}$ such that $(R+1)^{-kn} \le |Q| \le (R+1)^{-(k-1)n}$. In this case, since any connected component $P$ of $E_k$ satisfies $(R+1)^{-kn} = |P| \le |Q| \le (R+1)^{-(k-1)n}$, $3Q$ contains a connected component of $E_k$. Hence, it follows that

$$\begin{aligned}
&\|\chi_{E_j}\|_{\mathcal{M}^p_q}\\
&\sim \sup\{|Q|^{\frac{1}{p}-\frac{1}{q}}|Q \cap E_j|^{\frac{1}{q}} \,:\, Q \text{ contains a connected component of } E_j\ \}.
\end{aligned}$$

Let $S$ be a cube which contains a connected component of $E_j$ and is not wasteful. By symmetry, we may assume $S = I \times I \times \cdots \times I$ for some interval $I$. We define

$$S^* := \mathrm{co}\left(\bigcup\{W \,:\, W \text{ is a connected component of } E_j \text{ intersecting } S\}\right),$$

where $\mathrm{co}(A)$ stands for the smallest convex set containing a set $A$. Then a geometric observation shows that $S^*$ engulfs $k^n$ connected component of $E_j$ for some $1 \le k \le 2^j$. Take an integer $l$ such that $2^{l-1} \le k \le 2^l$. Then we have

$$|S^* \cap E_j| = k^n(1+R)^{-jn}, \quad |S^*| \sim (1+R)^{-jn+ln}.$$

Consequently, from (2.1) we have

$$|S^*|^{\frac{1}{p}-\frac{1}{q}}|S^*\cap E_j|^{\frac{1}{q}} \sim 2^{\frac{ln}{q}}(1+R)^{-\frac{jn}{q}}(1+R)^{(-j+l)\left(\frac{n}{p}-\frac{n}{q}\right)}$$
$$= 2^{\frac{ln}{q}}(1+R)^{-j\frac{n}{p}+ln\left(\frac{1}{p}-\frac{1}{q}\right)}$$
$$= 2^{\frac{ln}{q}}(1+R)^{ln\left(\frac{1}{p}-\frac{1}{q}\right)}(1+R)^{-j\frac{n}{p}} = (1+R)^{-j\frac{n}{p}}.$$

Therefore, (2.2) is obtained. □

So, we can say that the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q}$ reflects local regularity of the functions more precisely than the Lebesgue norm $\|\cdot\|_p$.

A chain of equalities in (2.2) is the motivation of choosing $R$ in (2.1).

*Remark* 2.2. In Lemma 2.1, if one defines $F_j := \{x\in\mathbb{R}^n : (R+1)^{-j}x\in E_j\}$, then $\{F_j\}_{j=1}^\infty$ is an increasing sequence of sets and each $F_j$ is made up of disjoint union of cubes of length 1, $\|\chi_{F_j}\|_{\mathcal{M}^p_q}\sim 1$, and each component of $F_j$ is a cube of size 1.

**Proposition 2.3.** *Let $0<q<r\le p$. Then $\|\chi_{F_j}\|_{\mathcal{M}^p_r}\ge\left(\frac{1+R}{2}\right)^{(\frac{n}{q}-\frac{n}{r})j}$, and so that $\lim\limits_{j\to\infty}\|\chi_{F_j}\|_{\mathcal{M}^p_r}=\infty$.*

*Proof.* Simply use

$$\|\chi_{F_j}\|_{\mathcal{M}^p_r}\ge(1+R)^{\frac{jn}{p}-\frac{jn}{r}}|F_j|^{\frac{1}{r}}=(1+R)^{\frac{jn}{q}-\frac{jn}{r}}|F_j|^{\frac{1}{r}-\frac{1}{q}}=\left(\frac{1+R}{2}\right)^{(\frac{n}{q}-\frac{n}{r})j}.$$

□

Now, we prove the main result of this paper.

*Proof of Theorem 1.1.* Under above notations, put

$$F:=\bigcup_{j=1}^\infty F_j. \tag{2.3}$$

Then $\chi_F\in\mathcal{M}^p_q$. For each $j=1,2,3,\dots$ let us define

$$g_j(x):=\chi_F((R+1)^{-j-1}x)-\chi_F((R+1)^{-j}x) \qquad (x\in\mathbb{R}^n),$$

and

$$h_j(x):=\chi_F(R^{-j-1}x) \quad (x\in\mathbb{R}^n).$$

Then by Lemma 2.1 we have $\|g_j\|_{\mathcal{M}^p_q}\sim\|h_j\|_{\mathcal{M}^p_q}<\infty$, while $\|g_j\|_{\mathcal{M}^p_r}\sim\|h_j\|_{\mathcal{M}^p_r}=\infty$ for all $q<r\le p$. We set

$$V:=\left\{\sum_{j=1}^\infty\lambda_j g_j \,:\, \text{for all } j,\lambda_j\in\mathbb{C}\right\}\bigcap\mathcal{M}^p_q.$$

Since the convergence with norm of $\mathcal{M}^p_q$ is stronger than the almost eveywhere convergence, $V$ is closed in $\mathcal{M}^p_q$. Also, note that

$$V \cap \bigcup_{q<r\leq p} \mathcal{M}^p_r = \{0\},$$

since the charactristic function $\chi_F$ belongs to the set $\mathcal{M}^p_q \setminus \bigcup_{q<r\leq p} \mathcal{M}^p_r$. Therefore, $\mathcal{M}^p_q \setminus \bigcup_{q<r\leq p} \mathcal{M}^p_r$ is spaceable in $\mathcal{M}^p_q$. □

Here is an alternative proof of Thoerem 1.1 with $q \geq 1$ using a result by Kitson and Timoney [6]. We invoke this result from [6].

**Theorem 2.4.** [6, Theorem 3.3]. *Let $Z_m$ ($m \in \mathbb{N}$) be Banach spaces and $X$ a Fréchet space. Let $T_m : Z_m \to X$ be continuous linear operators and $Y$ the linear span of $\bigcup_{m=1}^{\infty} T_m(Z_m)$. If $Y$ is not closed in $X$, then the complement $X \setminus Y$ is spaceable.*

For a different proof of Theorem 1.1, simply apply Theorem 2.4 with

$$X = \mathcal{M}^p_q, \quad Z_m = \mathcal{M}^p_{q+\frac{p-q}{m}}, \quad T_m = \text{inclusion mapping from } Z_m \text{ to } \mathcal{M}^p_q.$$

We will check the following property of the Morrey space $\mathcal{M}^p_q$ to see that Theorem 2.4 is applicable.

**Proposition 2.5.** *Let $0 < q < p < \infty$. Then $\displaystyle\bigcup_{q<r\leq p} \mathcal{M}^p_r$ is not closed in $\mathcal{M}^p_q$.*

We torelate the case $0 < q < 1$ here. Once this is shown, Theorem 1.1 will be proved with the help of the aforementioned theorem.

*Proof.* Let $F_j$ be as in Remark 2.2. For each $k \in \mathbb{N}$ define

$$f_k := \sum_{j=1}^{k} \frac{1}{j^2 \|\chi_{F_j}\|_{\mathcal{M}^p_q}} \chi_{F_j}.$$

Then $(f_k)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{M}^p_q$. So, $(f_k)_{k=1}^{\infty}$ converges to a function $f$ in $\mathcal{M}^p_q$ since $\mathcal{M}^p_q$ is a Banach space. Note that each $f_k \in L^p \subset \mathcal{M}^p_r$ for all $q < r \leq p$. If $f$ is a member in $\mathcal{M}^p_r$ for some $r \in (q, p]$, then

$$\left\{ \frac{1}{j^2 \|\chi_{F_j}\|_{\mathcal{M}^p_q}} \chi_{F_j} \right\}_{j=1}^{\infty}$$

would form a bounded set in $\mathcal{M}^p_r$ since $\mathcal{M}^p_r$ enjoys the lattice property. This is a contradiction to Proposition 2.3. □

## 3. Appendix

The situation is different from the case of Lebesgue spaces.

**Proposition 3.1.** *Let* $0 < q < p < \infty$. *Then* $\bigcup_{q<r\le p} \mathcal{M}^p_r$ *is not dense in* $\mathcal{M}^p_q$.

*Proof.* Let $F$ be as in (2.3). We prove that $2\chi_F$ is not in the closure of $\bigcup_{q<r\le p} \mathcal{M}^p_r$ by showing that for every $q < r \le p$, $f \notin \mathcal{M}^p_r$ if $f \in \mathcal{M}^p_q$ satisfies $\|2\chi_F - f\|_{\mathcal{M}^p_q} < 1$. Indeed, if $K$ is one of the connected components of $F$, then $\|f\|_{L^r(K)} \ge \|f\|_{L^q(K)} > c_q = (2^{\min(1,q)} - 1)^{\frac{1}{\min(1,q)}}$ since

$$\begin{aligned} 1 &> (\|2\chi_F - f\|_{\mathcal{M}^p_q})^{\min(1,q)} \\ &\ge (\|2 - f\|_{L^q(K)})^{\min(1,q)} \\ &\ge 2^{\min(1,q)} - (\|f\|_{L^q(K)})^{\min(1,q)}. \end{aligned}$$

Thus,

$$\|f\|_{\mathcal{M}^p_r} \ge |[0,R^j]^n|^{\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r([0,R^j]^n)} \ge c_q R^{\frac{jn}{p}-\frac{jn}{r}} |E_j|^{\frac{1}{r}} = c_q 2^{\frac{jn}{r}-\frac{jn}{q}} R^{\frac{jn}{q}-\frac{jn}{r}}$$

for all $j \in \mathbb{N}$. Hence, $\|f\|_{\mathcal{M}^p_r} = \infty$, or equivalently $f \notin \mathcal{M}^p_r$. □

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