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On the 2-absorbing Submodules

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Abstract. Let R be a commutative ring and M be an R-module. In this paper, we investigate some properties of 2-absorbing submodules of M. It is shown that N is a 2-absorbing submodule of M if and only if whenever $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$. Also, if N is a 2-absorbing submodule of M and M/N is Noetherian, then a chain of 2-absorbing submodules of M is constructed. Furthermore, the annihilation of $E(R/\mathfrak{p})$ is studied whenever 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of R and $E(R/\mathfrak{p})$ is an injective envelope of $R/\mathfrak{p}$.

Keywords: 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.

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1. Introduction

Throughout this paper R is a commutative ring with non-zero identity and M is an unitary R-module. We defined a submodule N of M is 2-absorbing whenever $abm \in N$ for some $a, b \in R$, $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$, see for instance [1, 3, 4, 6, 7, 9, 10]. It is well known that, a submodule N of M is prime if and only if $IL \subseteq N$ for an ideal I of R and

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a submodule L of M, then either $L \subseteq N$ or $I \subseteq N :_R M$. This statement persuaded us to prove that, a submodule N of M is 2-absorbing if and only if $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$. As a corollary of this theorem, it is shown that $L = \{m \in M : \mathfrak{p} \subseteq r(N : m)\}$ is a 2-absorbing submodule of M, where N is a 2-absorbing submodule of M with $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$ for some prime ideals $\mathfrak{p}, \mathfrak{q}$ of R. Also, it is shown that if M/N is Noetherian, then there exists a chain of 2-absorbing submodules of M that begins with N. Assume that $E(R/\mathfrak{p})$ is an injective envelope of $R/\mathfrak{p}$, it is shown that if 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, then $r(0 :_R E(R/\mathfrak{p})) = \mathfrak{p}$ and $0 :_R x$ is determined for all nonzero element x of $E(R/\mathfrak{p})$.

Now, we define the concepts that we will use later. For a submodule L of M let $L :_R M$ denote the ideal $\{r \in R : rM \subseteq L\}$. Similarly, for an element $m \in M$ let $L :_R m$ denote the ideal $\{r \in R : rm \in L\}$. If I is an ideal of R, then $r(I)$ denotes the radical of I. We say that $\mathfrak{p} \in \mathrm{Spec}(R)$ is an associated prime ideal of M if there exists $m \in M$ with $0 :_R m = \mathfrak{p}$. The set of associated prime ideals of M is denoted by $\mathrm{Ass}_R(M)$, the set of integers is denoted by $\mathbb{Z}$.

2. 2-absorbing Submodules

Let N be a proper submodule of M. We say that N is a *2-absorbing* submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$.

Lemma 2.1. *Let I be an ideal of R and N be a 2-absorbing submodule of M. If $a \in R$, $m \in M$ and $Iam \subseteq N$, then $am \in N$ or $Im \subseteq N$ or $Ia \subseteq N :_R M$.*

Proof. Let $am \notin N$ and $Ia \nsubseteq N :_R M$. Then there exists $b \in I$ such that $ba \notin N :_R M$. Now, $bam \in N$ implies that $bm \in N$, since N is a 2-absorbing submodule of M. We have to show that $Im \subseteq N$. Let c be an arbitrary element of I. Thus $(b + c)am \in N$. Hence, either $(b + c)m \in N$ or $(b + c)a \in N :_R M$. If $(b+c)m \in N$, then by $bm \in N$ it follows that $cm \in N$. If $(b+c)a \in N :_R M$, then $ca \notin N :_R M$, but $cam \in N$. Thus $cm \in N$. Hence, we conclude that $Im \subseteq N$. □

Lemma 2.2. *Let I, J be ideals of R and N be a 2-absorbing submodule of M. If $m \in M$ and $IJm \subseteq N$, then $Im \subseteq N$ or $Jm \subseteq N$ or $IJ \subseteq N :_R M$.*

Proof. Let $I \nsubseteq N :_R m$ and $J \nsubseteq N :_R m$. We have to show that $IJ \subseteq N :_R M$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $am \notin N$ but $aJm \subseteq N$. Now, Lemma 2.1 shows that $aJ \subseteq N :_R M$ and so $(I \setminus N :_R m)J \subseteq N :_R M$, similarly there exists $b \in J \setminus N :_R m$ such that $Ib \subseteq N :_R M$ and also $I(J \setminus N :_R m) \subseteq N :_R M$. Thus we have $ab \in N :_R M$, $ad \in N :_R M$ and $cb \in N :_R M$. By $a + c \in I$ and $b + d \in J$ it follows that $(a + c)(b + d)m \in N$. Therefore, $(a + c)m \in N$ or $(b + d)m \in N$ or

$(a+c)(b+d) \in N :_R M$. If $(a+c)m \in N$, then $cm \notin N$ hence, $c \in I \setminus N :_R m$ which implies that $cd \in N :_R M$. Similarly by $(b+d)m \in N$, we can deduce that $cd \in N :_R M$. If $(a+c)(b+d) \in N :_R M$, then $ab+ad+cb+cd \in N :_R M$ and so $cd \in N :_R M$. Therefore, $IJ \subseteq N :_R M$. $\square$

Theorem 2.3. *Let N be a proper submodule of M. The following statement are equivalent:*

(i) *N is a 2-absorbing submodule of M;*
(ii) *If $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$.*

Proof. $(ii) \Rightarrow (i)$ is obvious. To prove $(i) \Rightarrow (ii)$, assume that $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M and $IJ \nsubseteq N :_R M$. Then by Lemma 2.2 for all $m \in L$ either $Im \subseteq N$ or $Jm \subseteq N$. If $Im \subseteq N$, for all $m \in L$ we are done. Similarly if $Jm \subseteq N$, for all $m \in L$ we are done. Suppose that $m, m' \in L$ are such that $Im \nsubseteq N$ and $Jm' \nsubseteq N$. Thus $Jm \subseteq N$ and $Im' \subseteq N$. Since $IJ(m+m') \subseteq N$ we have either $I(m+m') \subseteq N$ or $J(m+m') \subseteq N$. By $I(m+m') \subseteq N$, it follows that $Im \subseteq N$ which is a contradiction, similarly by $J(m+m') \subseteq N$ we get a contradiction. Therefore either $IL \subseteq N$ or $JL \subseteq N$. $\square$

A submodule N of M is called strongly 2-absorbing if it satisfies in condition (ii), see [5]. Therefore, Theorem 2.3 shows that N is a 2-absorbing submodule of M if and only if N is a strongly 2-absorbing submodule of M.

Corollary 2.4. *Let M be an R-module and N be a 2-absorbing submodule of M. Then $N :_M I = \{m \in M : Im \subseteq N\}$ is a 2-absorbing submodules of M for all ideal I of R. Furthermore $N :_M I^n = N :_M I^{n+1}$, for all $n \geq 2$.*

Proof. Let I be an ideal of R, $a, b \in R$, $m \in M$ and $abm \in N :_M I$. Thus $Iabm \subseteq N$. Hence, $Im \subseteq N$ or $Iab \subseteq N :_R M$ or $abm \in N$, by Lemma 2.2. If $Im \subseteq N$ we are done. If $Iab \subseteq N :_R M$, then $ab \in (N :_R M) :_R I = (N :_M I) :_R M$. If $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$. Thus $Iam \subseteq N$ or $Ibm \subseteq N$ or $Iab \subseteq N :_R M$ which complete the proof.

For the second statement, it is enough to show that $N :_M I^2 = N :_M I^3$. It is clear that $N :_M I^2 \subseteq N :_M I^3$. Let $m \in N :_M I^3$. Then $I^3 m \subseteq N$. Now, by Lemma 2.2, we have $I^2 m \subseteq N$ or $Im \subseteq N$ or $I^3 \subseteq N :_R M$. If $I^2 m \subseteq N$ or $Im \subseteq N$, we are done. If $I^3 \subseteq N :_R M$, then $I^2 \subseteq N :_R M$ since $N :_R M$ is a 2-absorbing ideal of R by [9, Theorem 2.3]. $\square$

It is clear that, $n\mathbb{Z}$ is a 2-absorbing ideal of $\mathbb{Z}$ if and only if $n = 0, p, p^2, pq$, where p, q are distinct prime integers. It is easy to see that $4\mathbb{Z} :_{\mathbb{Z}} 6\mathbb{Z} = 2\mathbb{Z}$ but $4\mathbb{Z} :_{\mathbb{Z}} 36\mathbb{Z} = \mathbb{Z}$. Hence, the equality mentioned in the Corollary 2.4, is not necessarily true for $n = 1$.

Theorem 2.5. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$ where $\mathfrak{p}$ and $\mathfrak{q}$ are the only distinct prime ideals of R that are minimal over $N :_R M$. Then $L = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$ is a 2-absorbing submodule of M containing N. Also, either $r(L :_R M) = \mathfrak{q}$ or $r(L :_R M) = \mathfrak{p}' \cap \mathfrak{q}$, where $\mathfrak{p}'$ is a prime ideal of R containing $\mathfrak{p}$.*

Proof. It is clear that L is a submodule of M containing N. Assume that $a, b \in R$, $m \in M$ and $abm \in L$. We have to show that $am \in L$ or $bm \in L$ or $ab \in L :_R M$. Since $\mathfrak{p} \subseteq r(N :_R abm)$, thus $\mathfrak{p}^2 abm \subseteq N$, by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have $abm \in N$ or $\mathfrak{p}^2 m \subseteq N$ or $\mathfrak{p}^2 ab \subseteq N :_R M$. If $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$ which implies that $am \in L$ or $bm \in L$ or $ab \in L : M$. If $\mathfrak{p}^2 m \subseteq N$, then $\mathfrak{p}^2 \subseteq N :_R m$ and so $\mathfrak{p} \subseteq r(N :_R m)$ thus $m \in L$ and we are done. If $\mathfrak{p}^2 ab \subseteq N :_R M$, then by [2, Theorem 2.13], we have $\mathfrak{p}^2 a \subseteq N :_R M$ or $\mathfrak{p}^2 b \subseteq N :_R M$ or $ab \in N :_R M$. In the first case we conclude that $\mathfrak{p}^2 \subseteq N :_R am$ and so $am \in L$. By a similar argument in the second case we can deduced that $bm \in L$. If $ab \in N :_R M$, then $ab \in L :_R M$. Therefore, the result follows.

For the second statement, first we show that $r(N :_R M) = r(L :_R M) \cap \mathfrak{p}$. It is clear $r(N :_R M) \subseteq r(L :_R M) \cap \mathfrak{p}$. Assume that $a \in (L :_R M) \cap \mathfrak{p}$. Thus $aM \subseteq L$ and so, by definition of L, $\mathfrak{p} \subseteq r(N :_R am)$, for all $m \in M$. Hence, [2, Theorem 2.4] shows that $\mathfrak{p}^2 \subseteq N :_R am$, for all $m \in M$. Therefore, $a^3 \in N :_R m$, for all $m \in M$. So that $a^3 \in N :_R M$ and then $a \in r(N :_R M)$. Thus $r(L :_R M) \cap \mathfrak{p} \subseteq r(N :_R M)$. Now, $L :_R M$ is a 2-absorbing ideal of R, therefore either $r(L :_R M) = \mathfrak{p}'$ or $r(L :_R M) = \mathfrak{p}' \cap \mathfrak{q}'$, for some prime ideals $\mathfrak{p}', \mathfrak{q}'$ of R. In the first case we have $r(N :_R M) = \mathfrak{p} \cap \mathfrak{p}'$ which implies that $\mathfrak{p}' = \mathfrak{q}$ and in the second case we have $r(N :_R M) = \mathfrak{p} \cap \mathfrak{p}' \cap \mathfrak{q}'$ which implies that either $\mathfrak{p}' = \mathfrak{q}$ or $\mathfrak{q}' = \mathfrak{q}$. □

Corollary 2.6. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$ where $\mathfrak{p}$ and $\mathfrak{q}$ are the only distinct prime ideals of R that are minimal over $N :_R M$. If M/N is a Noetherian R-module, then*

(i) *there exists a chain $N = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$ of 2-absorbing submodules of M. Furthermore, $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M/L_{n-1}) \cup \mathrm{Ass}(L_{n-1}/L_{n-2}) \cup \mathrm{Ass}(L_{n-2}/L_{n-3}) \cup \cdots \cup \mathrm{Ass}(L_1/N)$, where $\mathrm{Ass}(L_i/N)$ is the union of at most two totally ordered set, for all i .*

(ii) *there exists a chain $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$ of submodules of M such that L_i is a 2-absorbing submodule of L_{i+1}, for all $0 \leq i \leq n-1$.*

Proof. (i) Let $L_1 = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$. Then by Corollary 2.4, L_1 is a 2-absorbing submodule of M and so either $r(L_1 :_R M) = \mathfrak{q}$ or $r(L_1 :_R M) = \mathfrak{p}_1 \cap \mathfrak{q}$, where $\mathfrak{p}_1$ is a prime ideal of R containing $\mathfrak{p}$. If $r(L_1 :_R M) = \mathfrak{q}$, then choose $L_2 = \{m \in M : \mathfrak{q} \subseteq r(L_1 :_R m)\} = M$. Hence, $N \subseteq L_1 \subseteq L_2 = M$ is requested chain. If $r(L_1 :_R M) = \mathfrak{p}_1 \cap \mathfrak{q}$, set $L_2 = \{m \in M : \mathfrak{p}_1 \subseteq r(L_1 :_R m)\}$

and so either $r(L_2 :_R M) = \mathfrak{q}$ or $r(L_2 :_R M) = \mathfrak{p}_2 \cap \mathfrak{q}$, where $\mathfrak{p}_2$ is a prime ideal of R containing $\mathfrak{p}_1$. Proceeding in this way, we can achieve $N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$ of 2-absorbing submodules of M. The last statement is obvious, by [9, Theorem 2.6].

(ii) Let $L_1 = \{m \in M : \mathfrak{p} \subseteq r(N :_R m)\}$. Then N is a 2-absorbing submodule of L_1. So that either $r(N :_R L_1) = \mathfrak{p}_1$ or $r(N :_R L_1) = \mathfrak{p}_1 \cap \mathfrak{q}_1$, for some prime ideals $\mathfrak{p}_1, \mathfrak{q}_1$ of R. If $r(N :_R L_1) = \mathfrak{p}_1$, then choose $L_2 = \{x \in L_1 : \mathfrak{p}_1 \subseteq r(N :_R x)\} = N$. Hence, in this case $N \subseteq L_1 \subseteq L_0 = M$ is the requested chain. If $r(N :_R L_1) = \mathfrak{p}_1 \cap \mathfrak{q}_1$, then set $L_2 = \{x \in L_1 : \mathfrak{p}_1 \subseteq r(N :_R x)\}$ and continue the same way to achieve the chain $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$ of 2-absorbing submodules of M. □

Theorem 2.7. *Let N be a 2-absorbing submodule of M. Then $N :_R M$ is a prime ideal of R if and only if $N :_R m$ is a prime ideal of R for all $m \in M \setminus N$.*

Proof. Assume that $a, b \in R$, $m \in M \setminus N$ and $ab \in N :_R m$. Then $abm \subseteq N$. We have $am \in N$ or $bm \in N$ or $ab \in N :_R M$ since N is a 2-absorbing submodule of M. If $am \in N$ or $bm \in N$ we are done. If $ab \in N :_R M$, then by assumption either $a \in N :_R M$ or $b \in N :_R M$. Thus either $a \in N :_R m$ or $b \in N :_R m$. So $N :_R m$ is a prime ideal.

Conversely, suppose that $ab \in N :_R M$ for some $a, b \in R$ and assume that there exist $m, m' \in M$ such that $am \notin N$ and $bm' \notin N$. By $abm, abm' \in N$ it follows that $bm \in N$ and $am' \in N$ since $N :_R m$ and $N :_R m'$ are prime ideals of R. If $m+m' \in N$, then $am \in N$ which is a contradiction. Thus $m+m' \notin N$. Now by $ab(m' + m'') \in N$ we have $a(m' + m'') \in N$ or $b(m' + m'') \in N$ which is a contradiction. Thus $aM \subseteq N$ or $bM \subseteq N$ which implies that $N :_R M$ is prime. □

Corollary 2.8. *Let N be a 2-absorbing submodule of M. Then $N :_R M$ is a prime ideal of R if and only if $N :_R K$ is a prime ideal of R for all submodules K of M containing N.*

Proof. By Theorem 2.7 and [9, Theorem 2.6] it follows that $\{N :_R x : x \in K \setminus N\}$ is a totally ordered set of prime ideals of R. Hence, $N :_R K = \cap_{x \in K} N :_R x$ is a prime ideal of R. □

Theorem 2.9. *Let $\mathfrak{p}$ be a prime ideal of R and $E(R/\mathfrak{p})$ be an injective envelop of $R/\mathfrak{p}$. If 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, then*

(i) $\mathfrak{p}^2 \subseteq 0 :_R E(R/\mathfrak{p}) \subseteq \mathfrak{p}$ *so that* $r(0 :_R E(R/\mathfrak{p})) = \mathfrak{p}$.

(ii) $\mathfrak{p}^2 \subseteq 0 :_R x \subset 0 :_R ax = \mathfrak{p}$, *for all non-zero element x of $E(R/\mathfrak{p})$ and all $a \in \mathfrak{p} \setminus 0 :_R x$.*

(iii) $\mathfrak{p}^2 \subseteq 0 :_R x = 0 :_R a^n x \subseteq \mathfrak{p}$, *for all $a \notin \mathfrak{p}$.*

Proof. (i) We have $r(0 :_R x) = \mathfrak{p}$ for all non-zero element x of $E(R/\mathfrak{p})$, by [8, Theorem 18.4]. Also it is obvious $0 :_R E(R/\mathfrak{p}) \subseteq 0 :_R x$. Thus $0 :_R E(R/\mathfrak{p}) \subseteq \mathfrak{p}$.

Now, assume that $a \in \mathfrak{p}^2$ and x is a non-zero element of $E(R/\mathfrak{p})$. Since 0 is a 2-absorbing submodule of M, $0 :_R x$ is a 2-absorbing ideal of R, by [9, Theorem 2.4]. Therefore we have $\mathfrak{p}^2$ is a subset of $0 :_R x$, by [2, Theorem 2.4]. Hence, $ax = 0$ and therefore $aE(R/\mathfrak{p}) = 0$ and $\mathfrak{p}^2 \subseteq 0 :_R E(R/\mathfrak{p})$.

(ii) Let x be a non-zero element of $E(R/\mathfrak{p})$. Then we have $\mathfrak{p}^2 \subseteq 0 :_R x \subseteq \mathfrak{p}$. Assume that $a \in \mathfrak{p} \setminus 0 :_R x$. Thus $ax \neq 0$ but $a^2x = 0$ which shows that $0 :_R x \subset 0 :_R ax$. If $b \in \mathfrak{p}$, then $ab \in \mathfrak{p}^2$ and $abx = 0$. Thus $b \in 0 :_R ax$ and so $\mathfrak{p} \subseteq 0 :_R ax$.

(iii) Assume that $a \notin \mathfrak{p}$. It is obvious that $0 :_R x \subseteq 0 :_R a^n x$, for all $n \in \mathbb{N}$. Let $b \in \mathrm{Ann}_R(a^n x)$. Thus $ba^n x = 0$. But multiplication by a^n is an automorphism on $E(R/\mathfrak{p})$, so that $bx = 0$ and $b \in 0 :_R x$. Therefore, $0 :_R x = 0 :_R a^n x$.

□

Corollary 2.10. *Let R be a principal ideal domain and $\mathfrak{p}$ is a prime ideal of R. If 0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$, then for all non-zero element x of $E(R/\mathfrak{p})$ either $0 :_R x = \mathfrak{p}^2$ or $0 :_R x = \mathfrak{p}$.*

Proof. Let $\mathfrak{p} = (a)$. Then $\mathfrak{p}^2 = (a^2)$. Let x be a non-zero element of $E(R/\mathfrak{p})$. Then $\mathfrak{p}^2 \subseteq 0 :_R x = (b) \subseteq \mathfrak{p}$ by Theorem 2.9(ii). Thus $a^2 = bc$ and $b = ae$ for some $c, e \in R$. Hence, $a^2 = aec$. So $a = ec \in \mathfrak{p}$. Therefore, either $c \in \mathfrak{p}$ or $e \in \mathfrak{p}$. If $c \in \mathfrak{p}$, then $c = ac'$ and so $a = eac'$ which implies that $1 = ec'$ and $a = bc'$ thus $0 :_R x = \mathfrak{p}$. If $e \in \mathfrak{p}$, then $e = ae'$ and so $a = ae'c$ which implies that $1 = e'c$ and $b = a^2e'$ thus $0 :_R x = \mathfrak{p}^2$. □

The following example shows that the condition "0 is a 2-absorbing submodule of $E(R/\mathfrak{p})$" is essential. It is well-known that $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty) = \{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\}$, where p is a prime integer. But neither $p^2\mathbb{Z} = 0 :_{\mathbb{Z}} 1/p^3 + \mathbb{Z}$ nor $0 :_{\mathbb{Z}} 1/p^3 + \mathbb{Z} = p\mathbb{Z}$. Hence, 0 is not a 2-absorbing submodule of $E(\mathbb{Z}/p\mathbb{Z})$. Also, this example shows that if 0 is a 2-absorbing submodule of M, then it is not necessarily a 2-absorbing submodule of $E(M)$.

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References

1. D. F. Anderson, A. Badawi, On n-absorbing ideals of commutative rings, *Comm. Algebra*, **39**, (2011), 1646-1672.
2. A. Badawi, On 2-absorbing ideals of commutavie rings, *Bull. Austral. Math. Soc.*, **75**, (2007), 417-429.
3. A. Badawi, A. Y. Darani, On weakly 2-absorbing ideals of commutative rings, *Houston J. Math.*, **39** (2), (2013), 441-452.
4. A. Y. Darani, E. R. Puczylowski, On 2-absorbing commutative semigroups and their applications to rings, *Semigroup Forum*, **86**, (2013), 83-91.

5. A. Y. Darani, F. Soheilnia, On n-absorning submodules, *Math. Commun.*, **17**, (2012), 547-557.
6. A. Y. Darani, F. Soheilnia, 2-Absorning and weakly 2-Absorbing submodules, *Thai J. of Math.*, **9**, (2011), 577-584.
7. R. Jahani-Nezhad, The Dual of a Strongly Prime Ideal, *Iranian Journal of Mathematical Sciences and Informatics*, **5**, (2010), 19-26.
8. H. Matsumura, *Commutative Rings Theory*, Second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989.
9. Sh. Payrovi, S. Babaei, On 2-absorbing submodules, *Algebra Collq.*, **19**, (2012), 913-920.
10. Sh. Payrovi, S. Babaei, On the 2-absorbing ideals, *Int. Math. Forum*, **7**, (2012), 265-271.