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# On the Independence Graph of Hamming Graph

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Abstract. The independence graph $Ind(G)$ of a graph $G$ is the graph with vertices as maximum independent sets of $G$ and two vertices are adjacent, if and only if the corresponding maximum independent sets are disjoint. In this work, we find the independence graph of Cartesian product of $d$ copies of complete graphs $K_q$, which is known as the Hamming graph $H(d, q)$. Greenwell and Lovasz [7] found that the independence number of direct product of $d$ copies of $K_q$ as $q^{d-1}$. We prove that the independence number of Hamming graph $H(d, q)$, which is cartesian product of $d$ copies of $K_q$, is also $q^{d-1}$. As an application of our findings, we find answers for rook problem in higher dimensional square chess board.

**Keywords:** Hamming graph, Cartesian product, Independent set, Independence Graph, Rook problem.

**2000 Mathematics subject classification:** 05C62, 05C69.

## 1. Introduction

For given integers $d > 1, q > 1$, the Hamming graph [4], denoted by $H(d, q)$ is defined as follows. Let $S = \{1, 2, \cdots, q\}$. The Hamming graph $H(d, q)$ has vertex set $S_d$, the set of all sequences of length $d$ from $S$. Two vertices are

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adjacent if they differ in precisely one coordinate. The Hamming graph $H(d,q)$ is, equivalently, the cartesian product of $d$ complete graphs $K_q$. Particular values of $d$ and $q$ yield wide class of graphs. For $q = 2$, the Hamming graph is also known as the Hypercube, denoted by $Q_d$. For $d = 2$, the graph $H(2,q)$ is known as Lattice graphs, denoted by $L_2(q)$. We note that the diameter of Hamming graph $H(d,q)$ is $d$. Hamming graphs are known for their applications in coding theory. For more on Hamming graphs, we refer [4].

A subset $I$ of vertices of a graph $G$, is said to be *independent set*, if every pair of vertices in $I$ are not adjacent in $G$. The maximum cardinality of independent sets in $G$, is known as independence number of $G$, denoted by $\alpha(G)$. Such independent sets of maximum cardinality are called as *maximum independent sets* or $\alpha$-sets of $G$. The *independence graph*, $Ind(G)$ of a graph $G$ is the graph with vertices as $\alpha$-sets of $G$ and two vertices are adjacent, if and only if the corresponding $\alpha$-sets are disjoint. Independence graphs were defined and studied by P. Hell, et all in [8] and X. Zhu [15]. They are further studied by B. Bresar and B. Zmazek in [3].

In 1974, Greenwell and Lovasz [7] found that the independence number of direct product of $d$ copies of $K_q$ as $q^{d-1}$. Independence number of cartesian product of graphs is well studied [1, 8, 9, 12]. In this article we find that the independence number of Hamming graph $H(d,q)$, which is cartesian product of $d$ copies of $K_q$, is also $q^{d-1}$. Moreover we give a way to construct the $\alpha$-sets of Hamming graph $H(d,q)$ and hence find it's independence graph. Works on finding the number of maximum independent sets are seen in [6, 10]. Also, we provide an application of this work in rook problem in higher dimensional chess boards.

## 2. Independence graph of hamming graph

**Theorem 2.1.** *The independence number of $H(d,q)$ is $q^{d-1}$.*

*Proof.* Clearly for every graph $G$, $\alpha(G \,\Box\, K_q) \leq |V(G)|$. Thus we can get $\alpha(H(d,q)) = \alpha(\Box_{i=1}^{d} K_q) \leq q^{d-1}$. Since for any graphs $G$ and $H$, the chromatic number of their Cartesian product, $\chi(G \,\Box\, H)$ is $max\,\{\chi(G), \chi(H)\}$, we have $\chi(H(d,q)) = \chi(\Box_{i=1}^{d} K_q) = q$. Also, $\frac{|V|}{\alpha(G)} \leq \chi(G)$. Hence we have, for any Hamming graph $H(d,q)$, $\frac{q^d}{\alpha(H(d,q))} \leq q$, which implies $\alpha(H(d,q)) \geq q^{d-1}$. Thus the result follows. $\square$

The *Covering number* $\beta(G)$ of a graph $G$ is the minimum size of any vertex cover in $G$. By lemma 3.1.21 of [14], $\alpha(G) + \beta(G) = |V(G)|$. Hence we obtain the following result.

**Corollary 2.2.** *Covering number $\beta(H(d,q)) = q^{d-1}(q-1)$.*

We call a partition of vertex set into $\alpha$-sets as $\alpha$-partition. It is easy to see that Hamming graph $H(d,q)$ admits an $\alpha$-partition, as the chromatic number is $q$ and every colour class is an independent set. But, it is hard to find the number of $\alpha$-sets of $H(d,q)$. In the next theorem, we give a way to construct $\alpha$-partition of $H(d,q)$ with the help of a cycle permutation, using induction on $d$, and prove that the number of $\alpha$-sets in $H(d,q)$ is $q[(q-1)!]^{d-1}$.

**Theorem 2.3.** *The number of $\alpha$-sets in $H(d,q)$ is $q[(q-1)!]^{d-1}$.*

*Proof.* First we construct an $\alpha$-partition $\{I_l^d\}_{l=1}^q$of $H(d,q)$ using induction on $d$. To prove $\{I_l^d\}_{l=1}^q$ is an $\alpha$-partition, we need to prove for every $l$, $I_l^d$ is an independent set of size $q^{d-1}$ and $I_{l_1}^d$ and $I_{l_2}^d$ are disjoint for distinct $l_1, l_2$.

Now for $d=1$, $H(1,q)$ has $I_1^1=\{1\}, I_2^1=\{2\}, \cdots, I_q^1=\{q\}$ as $q$ independent sets, each of size $q^{d-1}=1$. For $d>1$, we construct the independent sets of $H(d,q)$ from the independent sets of $H(d-1,q)$ recursively, as follows:

- Consider the cycle $\sigma=(1,2,...,n)$. Let $\sigma^l$ denote the right circular $l-$shift of $\sigma$ and $\sigma^l(i)$ denote the element in the $i^{th}$ position in $\sigma^l$. We take $\sigma^0=\sigma$.
- By Induction assumption, we have $H(d-1,q)$ has $q$ pairwise disjoint independent sets each of size $q^{d-2}$.
- Independent sets of $H(d,q)$ are defined recursively as,

$$I_l^d = \cup_{i=1}^q \left(\{\sigma^{l-1}(i)\} \times I_i^{d-1}\right), l=1,2,\cdots,q.$$

- By $\{\sigma^{l-1}(i)\} \times I_i^{d-1}$ we mean, with every element of $I_i^{d-1}$, we join $\sigma^{l-1}(i)$ as the first coordinate and make it as the vertex of $H(d,q)$.

By construction, $I_l^d$ is of size $q^{d-1}$ for all $l$. Take $1 \le l_1, l_2 \le q$ and $l_1 \ne l_2$. we claim $I_{l_1}^d$ and $I_{l_2}^d$ are disjoint.

Let $u \in I_{l_1}^d$ and $v \in I_{l_2}^d$, then $u \in \{\sigma^{l_1-1}(i)\} \times I_i^{d-1}$ for some $i$ and $v \in \{\sigma^{l_2-1}(j)\} \times I_j^{d-1}$ for some $j$. Therefore, $u=\left(\sigma^{l_1-1}(i), u'\right)$ where, $u' \in I_i^{d-1}$ and $v=\left(\sigma^{l_2-1}(i), v'\right)$ where, $v' \in I_j^{d-1}$. If $\sigma^{l_1-1}(i) \ne \sigma^{l_2-1}(j)$, then $u \ne v$. If not $\sigma^{l_1-1}(i) = \sigma^{l_2-1}(j)$, then $i \ne j$ as $l_1 \ne l_2$. By induction assumption $I_i^{d-1} \cap I_j^{d-1} = \emptyset$ implies $u' \ne v'$, which again implies $u \ne v$. Therefore $I_{l_1}^d \cap I_{l_2}^d = \emptyset$ for distinct $l_1, l_2$.

Now we claim $I_l^d$ is independent for every $l=1$ to $q$. As for fixed $i$, $\{\sigma^{l-1}(i)\} \times I_i^{d-1}$ is equivalent to the independent set $I_i^{d-1}$ of $H(d-1,q)$, it is enough to prove that, for distinct $i$ and $j$, $\{\sigma^{l-1}(i)\} \times I_i^{d-1} \cup \{\sigma^{l-1}(j)\} \times I_j^{d-1}$ is independent. Let $u \in \{\sigma^{l-1}(i)\} \times I_i^{d-1}$ and $v \in \{\sigma^{l-1}(j)\} \times I_j^{d-1}$. Now $u=\left(\sigma^{l-1}(i), u'\right)$ and $v=\left(\sigma^{l-1}(j), v'\right)$ where $u' \in I_i^{d-1}$ and $v' \in I_j^{d-1}$. Clearly, $\sigma^{l-1}(i)$ and $\sigma^{l-1}(j)$ are distinct as $i$ and $j$ are distinct. But by induction assumption, $I_i^{d-1} \cap I_j^{d-1} = \emptyset$ and that implies $u' \ne v'$. Consequently $u$ and $v$ differ atleast in two places. Hence $u$ and $v$ are not adjacent.

Thus our idea is to consider an $\alpha$-partition of $H(d-1,q)$ and to place different elements of the partition in different layers of $H(d,q)$ using the cycle $\sigma$, so that the resultant set is an $\alpha$-set of $H(d,q)$. Rotation of $\sigma$ produces disjoint $\alpha$-sets of $H(d,q)$, and hence an $\alpha$-partition of $H(d,q)$. Conversely, every $\alpha$-set of $H(d,q)$ arises in this way, as it's projection on $H(d-1,q)$ is an $\alpha$-partition of $H(d-1,q)$.

Infact, the construction will work out for any cycle on $\{1,2,...q\}$. So, every $\alpha$-partition of $H(d-1,q)$ produces $(q-1)!$ $\alpha$-partitions of $H(d,q)$, as there are $(q-1)!$ different cycles on $\{1,2,\cdots,q\}$. Now, in $H(1,q)$, we have only one $\alpha$-partition, consequently $H(2,q)$ has $(q-1)!$ $\alpha$-partitions and $H(3,q)$ has $[(q-1)!]^2$ $\alpha$-partitions. Proceeding like this, $H(d,q)$ has $[(q-1)!]^{d-1}$ $\alpha$-partitions. Hence there are $q[(q-1)!]^{d-1}$ $\alpha$-sets in $H(d,q)$. □

The construction of $\alpha$-sets and $\alpha$-partitions given in the above theorem is illustrated in the following example.

Example 2.4. When $d=1$, $H(1,3)$ has $I_1^1=\{1\}, I_2^1=\{2\}$, and $I_3^1=\{3\}$ as $\alpha$-sets.

When $d=2$, $H(2,3)$ has $I_1^2=\{\mathbf{1}1,\mathbf{2}2,\mathbf{3}3\}, I_2^2=\{\mathbf{3}1,\mathbf{1}2,\mathbf{2}3\}, I_3^2=\{\mathbf{2}1,\mathbf{3}2,\mathbf{1}3\}$, $I_4^2=\{\mathbf{1}1,\mathbf{3}2,\mathbf{2}3\}, I_5^2=\{\mathbf{2}1,\mathbf{1}2,\mathbf{3}3\}$, and $I_6^2=\{\mathbf{3}1,\mathbf{2}2,\mathbf{1}3\}$ as $\alpha$-sets and two $\alpha$-partitions are $\{I_1^2,I_2^2,I_3^2\}$ and $\{I_4^2,I_5^2,I_6^2\}$.

When $d=3$, Considering the $\alpha$-partition $\{I_1^2,I_2^2,I_3^2\}$ of $H(2,3)$ and the cycle $\sigma=(123)$, we have the $\alpha$-sets of $H(3,3)$ as

$I_1^3=\{\mathbf{1}11,\mathbf{1}22,\mathbf{1}33,\mathbf{2}31,\mathbf{2}12,\mathbf{2}23,\mathbf{3}21,\mathbf{3}32,\mathbf{3}13\}$,

$I_2^3=\{\mathbf{3}11,\mathbf{3}22,\mathbf{3}33,\mathbf{1}31,\mathbf{1}12,\mathbf{1}23,\mathbf{2}21,\mathbf{2}32,\mathbf{2}13\}$,

$I_3^3=\{\mathbf{2}11,\mathbf{2}22,\mathbf{2}33,\mathbf{3}31,\mathbf{3}12,\mathbf{3}23,\mathbf{1}21,\mathbf{1}32,\mathbf{1}13\}$. Similarly we can produce nine more $\alpha$-sets of $H(3,3)$ which form four $\alpha$-partitions of $H(3,3)$, together with $\{I_1^3,I_2^3,I_3^3\}$.

**Proposition 2.5.** [3] *If $G$ is a graph such that $\alpha(G)\chi(G)=|V(G)|$, then the clique number of it's independence graph, $\omega(Ind(G))=\chi(G)$.*

**Theorem 2.6.** *The independence graph of $H(d,q)$ is $[(q-1)!]^{d-1}K_q$.*

*Proof.* Clearly, the $\alpha$-sets of $H(d,q)$, constructed from same $\alpha$-partition of $H(d-1,q)$ and rotation of a cycle $\sigma$, induces $K_q$ in $Ind(H(d,q))$. Suppose two $\alpha$-sets are produced by same $\alpha$-partition of $H(d-1,q)$, but, with different cycles, then as any two $q$-cycles must coincide in a place, say $i$, unless one is rotation of other, the $\alpha$-sets in $i^{th}$ layer of $H(d,q)$ must coincide. Consequently, they are not adjacent in $Ind(H(d,q))$. Also, we prove, if two $\alpha$-sets are produced by different $\alpha$-partitions of $H(d-1,q)$, they too are not adjacent in $Ind(H(d,q))$. Let $\alpha_1$ and $\alpha_2$ be two disjoint $\alpha$-sets of $H(d,q)$, produced from different $\alpha$-partitions $P_1$ and $P_2$ of $H(d-1,q)$ and cycles $\sigma$ and $\tau$, respectively, where $\sigma$ and $\tau$ need not be distinct. Then $\alpha_1$ is disjoint to the $\alpha$-sets produced by $P_2$ and rotation of $\tau$. Hence there is an $q+1$ clique in $Ind(H(d,q))$ which contradicts the previous proposition. Hence, it follows that $Ind(H(d,q))$ is $[(q-1)!]^{d-1}K_q$. □

*Remark* 2.1. In [3], the authors proved that for any graph $G$, there exists a graph $H$ such that $Ind(H) = G$, and provided a construction for the graph $H$. It should be noted that, there may exist many graphs $H$ satisfying $Ind(H) = G$ and Hamming graphs can not be obtained from that construction by taking $G$ as $[(q-1)!]^{d-1}K_q$. For example, taking $G$ as $K_2$, the construction in [3] gives $H$ as $K_{4,4}$. But, it can be easily verified that, for any $n$, $Ind(K_{n,n}) = K_2$ and by the previous theorem, for any $d$, Independence graph of Hypercube $Ind(Q_d) = K_2$.

## 3. An application

In this section, we provide an application of our findings in a kind of generalization of rook problem.

Rooks are pieces in chess that attack the pieces in it s row(rank) or it s column(file). Rook problem in chess, is finding the maximum number of non attacking rooks, that can be placed in a chessboard and the number of such placements. Rook problems are classic problems in combinatorial mathematics. Different variations and generalizations of rook problem have been carried out by many authors and rook problem has many applications[2, 5, 11, 13]. Here, we generalize the problem to higher dimensional square chess boards.

Now, Hamming graph $H(2,q)$ is known as Rook's graph as it's vertices represent the squares of a $q$ order chess board and two vertices are adjacent if rooks in the corresponding squares can attack each other. Using the independence theory of Hamming graph, we can find answer for a generalization of rook problem, that is rook problem in higher dimensional chess boards of $q$ order. For example, suppose we have a 3 dimensional chess board of order $q$ and a rook placed in the cell $(i,j,k)$ can attack the cells $(*,j,k),(i,*,k)$ and $(i,j,*)$. Then this chessboard can be identified with hamming graph $H(3,q)$. Hence, from Theorem 2.1 and Theorem 2.3 we get the maximum number of non attacking rooks, that can be placed in $d$ dimensional chess board of order $q$ is the independence number of $H(d,q)$, which is $q^{d-1}$ and, the number of such placements is the number of $\alpha$ sets in $H(d,q)$, which is $q[(q-1)!]^{d-1}$.

*Remark* 3.1. A similar generalization of rook problem is studied by Feryal Alayont and Nicholas Krywonos in [2], but, where the rooks attack in different manner to that of us.

## 4. Scope for future work

The Rook polynomial $R_B(x)$ of a board $B$ is a polynomial in $x$, where the co-efficients of $x^k$ is the number of ways to place $k$ non attacking rooks in the board $B$. In this article, we have found the number of $\alpha$-sets, that is, the number of independent sets of size $q^{d-1}$. Instead of this, one can try to find the number of independent sets of size $k$ and hence the rook polynomial of the $d$-dimensional $q$-order chess board.

Also, one of the generalizations of Hamming graph, $H(q_1, q_2, \cdots, q_d)$ is the Cartesian product of $K_{q_1}, K_{q_2}, \cdots, K_{q_d}$. Since the Cartesian product is commutative, we assume $q_1 \leq q_2 \leq \cdots \leq q_d$. From the observations, we arrive to the following conjecture about the independence number of $H(q_1, q_2, \cdots, q_d)$.

**Conjecture 4.1.** *The independence number of* $H(q_1, q_2, \cdots, q_d)$ *is* $\Pi_{i=1}^{d-1} q_i$.

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