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Coefficient Estimates for a General Subclass of m-fold Symmetric Bi-univalent Functions by Using Faber Polynomials

Safa Salehian[a*], Ahmad Motamednezhad[b], Nanjundan Magesh[c]

[a]Department of Mathematics, Gorgan Branch, Islamic Azad University, Gorgan, Iran
[b]Faculty of Mathematical Sciences, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

[c]Post-Graduate and Research Department of Mathematics, Government Arts College for Men, Krishnagiri 635001, Tamilnadu, India

E-mail: s.salehian84@gmail.com
E-mail: a.motamedne@gmail.com
E-mail: nmagi_2000@yahoo.co.in

Abstract. In the present paper, we introduce a new subclass $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$ of the m-fold symmetric bi-univalent functions. Also, we find the estimates of the Taylor-Maclaurin initial coefficients $|a_{m+1}|, |a_{2m+1}|$ and general coefficients $|a_{mk+1}| (k \geq 2)$ for functions in this new subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

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*Corresponding Author

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ showing in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in Δ (see [5, 7, 9]). Every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ, if both f and f^{-1} are univalent in Δ (see [21]).

We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in Δ given by (1.1). For a brief history and interesting examples of functions in the class $\sigma_{\mathcal{B}}$, see [21]. In fact that this widely-cited work by Srivastava et al. [21] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [7, 8, 17, 18, 19, 20, 22, 23, 24, 28, 29] and others [30, 31].

Some of the important and well-investigated subclasses $\sigma_{\mathcal{B}}$ of bi-univalent function include (for example) the subclass $\mathcal{H}_{\Sigma}(\beta)$ $(0 \leq \beta < 1)$ and the subclass $\mathcal{S}^*(\beta)$ bi-starlike functions of order $\beta(0 \leq \beta < 1)$(see [6, 21]). By definition, we have

$$\mathcal{H}_{\Sigma}(\beta) = \{f \in \sigma_{\mathcal{B}} : \ Re(f'(z)) > \beta \ , \ Re(g'(w)) > \beta \ (z, w \in \Delta)\}$$

and

$$\mathcal{S}^*(\beta) = \left\{ f \in \sigma_{\mathcal{B}} : \ Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ , \ Re\left\{\frac{wg'(w)}{g(w)}\right\} > \beta \ (z, w \in \Delta)\right\},$$

where the function $g = f^{-1}$ given by (1.2).

For each function $f \in \mathcal{S}$ function

$$h(z) = \sqrt[m]{f(z^m)} \tag{1.3}$$

is univalent and maps unit disk Δ into a region with m-fold symmetry. A function f is said to be m-fold symmetric (see [13, 14]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \ (z \in \Delta, m \in \mathbb{N}). \tag{1.4}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in Δ, which are normalized by the series expansion (1.4). In fact, the functions in class $\mathcal{S}$ are one-fold symmetric.

In [22] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \sigma_{\mathcal{B}}$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.4), they obtained the series expansion for f^{-1} as follows:

$$f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1} \tag{1.5}$$

$$= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \big[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\big]w^{3m+1} + \cdots . \tag{1.6}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in Δ. For $m = 1$, formula (1.6) coincides with formula (1.2) of the class $\sigma_{\mathcal{B}}$. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right] \ \textit{and} \ [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [3, 10, 11, 12, 15, 16, 18, 24, 25, 26]).

The aim of the this paper is to introduce new subclass of Σ_m and obtain estimates on initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and general coefficients $|a_{mk+1}| (k \geq 2)$ for functions in the subclass and improve some recent works of many authors.

2. Preliminary results

In the present paper by using the Faber polynomial expansions we obtain estimates of general coefficients $|a_{mk+1}|$ where $k \geq 2$, of functions in the subclass $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$ of Σ_m. For this purpose we need the following lemmas.

Lemma 2.1. [1, 2] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *be univalent function in* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. *Then we can write,*

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} F_k(a_2, a_3, ..., a_{k+1}) z^k, \tag{2.1}$$

where $F_k(a_2, a_3, ..., a_{k+1})$ *is a Faber polynomial of degree* k,

$$F_k(a_2, a_3, ..., a_{k+1}) = \sum_{i_1+2i_2+...+ki_k=k} A_{(i_1,i_2,...,i_k)} a_2^{i_1} a_3^{i_2} \cdots a_{k+1}^{i_k} \tag{2.2}$$

and

$$A_{(i_1,i_2,...,i_k)} := (-1)^{k+2i_1+3i_2+...+(k+1)i_k} \frac{(i_1 + i_2 + ... + i_k - 1)!k}{i_1!i_2!...i_k!}.$$

The first Faber polynomials $F_k(a_2, a_3, ..., a_{k+1})$ *are given by:*

$F_1(a_2) = -a_2,\ F_2(a_2, a_3) = a_2^2 - 2a_3$ *and* $F_3(a_2, a_3, a_4) = -a_2^3 + 3a_2a_3 - 3a_4$.

Lemma 2.2. *Let* $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}$, *be univalent function in* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. *Then we can write,*

$$\frac{zf'(z)}{f(z)} = 1 - F_m(a_{m+1})z^m - F_{2m}(a_{m+1}, a_{2m+1})z^{2m} - \cdots - F_{mk}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1})z^{mk} - \cdots, \tag{2.3}$$

where

$$F_m(a_{m+1}) = F_m(0, \cdots, 0, a_{m+1}),$$

$$F_{2m}(a_{m+1}, a_{2m+1}) = F_{2m}(0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}), \cdots,$$

$$F_{mk}(a_{m+1}, a_{2m+1}, \cdots, a_{mk+1}) =$$

$$F_{mk}(0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, \cdots, 0, \cdots, 0, a_{(k-1)m+1}, 0, \cdots, 0, a_{mk+1}), \cdots.$$

Proof. By using Lemma 2.1 for function $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots \in \Sigma_m$, we have

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k\geq 1} F_k(a_2, a_3, \cdots, a_{k+1})z^k.$$

Noting that (2.2), for $t \in \mathbb{N}$, we have

$$F_{tm}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}}_{tm}) =$$

$$\sum_{mi_m+2mi_{2m}+\cdots+tmi_{tm}=tm} A_{(i_1,i_2,...,i_{tm})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

Also, for $1 \leq j \leq m-1$ and $t \in \mathbb{N}$, the equation

$$mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm + j$$

doesn't have positive integer solution, so

$$F_{tm+j}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, 0, \cdots, 0}_{tm+j}) =$$

$$\sum_{mi_m+2mi_{2m}+\cdots+tmi_{tm}=tm+j} A_{(i_1,i_2,...,i_{tm+j})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}} = 0.$$

□

By applying Lemma 2.2 for the function $zf'(z) = z+\sum_{k=1}^{\infty}(mk+1)a_{mk+1}z^{mk+1}$, we can obtain the following lemma.

Lemma 2.3. *Let* $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}$, *be univalent function in* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. *Then we can write,*

$$1 + \frac{zf''(z)}{f'(z)} = \frac{z(zf'(z))'}{zf'(z)} = 1 - F_m\Big((m+1)a_{m+1}\Big)z^m - F_{2m}\Big((m+1)a_{m+1}, (2m+1)a_{2m+1}\Big)z^{2m} - \cdots - F_{mk}\Big((m+1)a_{m+1}, (2m+1)a_{2m+1}, \cdots, (mk+1)a_{mk+1}\Big)z^{mk} - \cdots \tag{2.4}$$

Lemma 2.4. [14] *If* $h \in \mathcal{P}$, *then* $|c_k| \leq 2$ *for each* k, *where* $\mathcal{P}$ *is the family of all functions* h *analytic in* $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ *for which* $Re(h(z)) > 0$ *where* $h(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$.

3. The subclass $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$

In this section, we introduce the general subclass $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$.

Definition 3.1. For $0 \leq \beta < 1$ and $\lambda \geq 0$, a function $f \in \Sigma_m$ given by (1.4) is said to be in the subclass $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$, if the following two conditions are satisfied:

$$Re\left\{(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta \ (z \in \Delta) \tag{3.1}$$

and

$$Re\left\{(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right\} > \beta \ (w \in \Delta), \tag{3.2}$$

where $g = f^{-1}$ given by (1.5).

Remark 3.2. By setting $\lambda = 0$, the subclass $\mathcal{H}_{\Sigma_m}(\beta, \lambda)$ reduces to the subclass $\mathcal{S}^{\beta}_{\Sigma_m}$ considered by Altinkaya and Yalcin [4].

Remark 3.3. For one-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{H}_{\Sigma_1}(\lambda, \beta) = \mathcal{H}_{\Sigma}(\lambda, \beta)$. Special cases of this subclass illustrated below:

- The subclass $\mathcal{H}_{\Sigma}(\lambda, \beta)$ is the same the subclass $B_{\Sigma}(\beta, \lambda)$ studied by Li and Wang [27].
- By putting $\lambda = 0$, then the subclass $\mathcal{H}_{\Sigma}(\lambda, \beta)$ reduces to the subclass $\mathcal{S}_{\sigma_B}(\beta)$ of bi-starlike functions of order $\beta (0 \leq \beta < 1)$.
- By putting $\lambda = 1$, then the subclass $\mathcal{H}_{\Sigma}(\lambda, \beta)$ reduces to the subclass $\mathcal{K}_{\sigma_B}(\beta)$ of bi-convex functions of order $\beta (0 \leq \beta < 1)$.

Theorem 3.4. *Let f given by* (1.4) *be in the subclass* $\mathcal{H}_{\Sigma_m}(\lambda,\beta)$ $(0 \le \beta < 1$, $\lambda \ge 0)$. *If* $a_{mt+1} = 0,\ 1 \le t \le k-1$, *then*

$$|a_{mk+1}| \le \frac{2(1-\beta)}{mk(1+\lambda mk)}\ ,\ k \ge 2.$$

Proof. By applying Lemmas 2.2 and 2.3 for function $f \in \mathcal{H}_{\Sigma_m}(\lambda,\beta)$ of the form (1.4), we can write

$$\begin{aligned}(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big) &= 1-\Big[(1-\lambda)F_m(a_{m+1})+\lambda F_m\Big((m+1)a_{m+1}\Big)\Big]z^m\\ &-\Big[(1-\lambda)F_{2m}(a_{m+1},a_{2m+1})+\lambda F_{2m}\Big((m+1)a_{m+1},(2m+1)a_{2m+1}\Big)\Big]z^{2m}-\cdots\\ &-\Big[(1-\lambda)F_{mk}(a_{m+1},\cdots,a_{mk+1})+\lambda F_{mk}\Big((m+1)a_{m+1},\cdots,(mk+1)a_{mk+1}\Big)\Big]z^{mk}\\ &-\cdots \end{aligned} \tag{3.3}$$

and similarly for $g(w) = f^{-1}(w) = w + \sum\limits_{k=1}^{\infty} A_{mk+1}w^{mk+1}$, we have

$$\begin{aligned}(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big) &= 1-\Big[(1-\lambda)F_m(A_{m+1})+\lambda F_m\Big((m+1)A_{m+1}\Big)\Big]w^m\\ &-\Big[(1-\lambda)F_{2m}(A_{m+1},A_{2m+1})+\lambda F_{2m}\Big((m+1)A_{m+1},(2m+1)A_{2m+1}\Big)\Big]w^{2m}-\cdots\\ &-\Big[(1-\lambda)F_{mk}(A_{m+1},\cdots,A_{mk+1})+\lambda F_{mk}\Big((m+1)A_{m+1},\cdots,(mk+1)A_{mk+1}\Big)\Big]w^{mk},\\ &-\cdots. \end{aligned} \tag{3.4}$$

On the other hand, since $f \in \mathcal{H}_{\Sigma_m}(\lambda,\beta)$ by definition, there exist two positive real part functions $p(z) = 1 + \sum\limits_{k\ge1} p_{mk}z^{mk}$ and $q(w) = 1 + \sum\limits_{k\ge1} q_{mk}w^{mk}$ where $Re\,(p(z)) > 0$ and $Re\,(q(w)) > 0$ in Δ. So that

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1+\frac{wg''(w)}{g'(w)}\right) = \beta + (1-\beta)q(w). \tag{3.6}$$

Comparing the corresponding coefficients of (3.3) and (3.5),we get

$$\begin{aligned}-\Big[(1-\lambda)F_{mk}(a_{m+1},\cdots,a_{mk+1})+\lambda F_{mk}\Big((m+1)a_{m+1},\cdots,(mk+1)a_{mk+1}\Big)\Big]\\ = (1-\beta)p_{mk}\ (k\ge1).\end{aligned} \tag{3.7}$$

Comparing the corresponding coefficients of (3.4) and (3.6),we get

$$\begin{aligned}-\Big[(1-\lambda)F_{mk}(A_{m+1},\cdots,A_{mk+1})+\lambda F_{mk}\Big((m+1)A_{m+1},\cdots,(mk+1)A_{mk+1}\Big)\Big]\\ = (1-\beta)q_{mk}\ (k\ge1).\end{aligned} \tag{3.8}$$

Note that for $a_{mt+1} = 0$ $(1 \le t \le k-1)$, we get

$$A_{mk+1} = -a_{mk+1},\ F_{mk}(0, \cdots, 0, a_{mk+1}) = -mka_{mk+1}$$
$$\textit{and } F_{mk}\Big(0, \cdots, 0, (mk+1)a_{mk+1}\Big) = -mk(mk+1)a_{mk+1}. \tag{3.9}$$

So from (3.7), (3.8) and (3.9), we have

$$mk(1+\lambda mk)a_{mk+1} = (1-\beta)p_{mk},$$
$$mk(1+\lambda mk)A_{mk+1} = -mk(1+\lambda mk)a_{mk+1} = (1-\beta)q_{mk}.$$

Taking the absolute values of the above equalities and using Lemma 2.4, we gain

$$|a_{mk+1}| = \frac{(1-\beta)|p_{mk}|}{mk(1+\lambda mk)} = \frac{(1-\beta)|q_{mk}|}{mk(1+\lambda mk)} \le \frac{2(1-\beta)}{mk(1+\lambda mk)}.$$

So completes the proof. □

Theorem 3.5. *Let f given by* (1.4) *be in the subclass $\mathcal{H}_{\Sigma_m}(\lambda, \beta)$* $(0 \le \beta < 1$, $\lambda \ge 0)$. *Then*

$$|a_{m+1}| \le \min\left\{\frac{2(1-\beta)}{m(1+\lambda m)}, \sqrt{\frac{2(1-\beta)}{m^2(1+\lambda m)}}\right\}$$

and

$$|a_{2m+1}| \le \min\left\{\frac{1-\beta}{m(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{m^2(1+\lambda m)^2}, \frac{(1+m)(1-\beta)}{m^2(1+\lambda m)}\right\}.$$

Proof. By putting $k = 1, 2$ in (3.7), we get:

$$m(1+\lambda m)a_{m+1} = (1-\beta)p_m, \tag{3.10}$$
$$2m(1+2\lambda m)a_{2m+1} - m(1+2\lambda m+\lambda m^2)a_{m+1}^2 = (1-\beta)p_{2m}. \tag{3.11}$$

Similarly, by putting $k = 1, 2$ in (3.8), we get:

$$-m(1+\lambda m)a_{m+1} = (1-\beta)q_m, \tag{3.12}$$
$$-2m(1+2\lambda m)a_{2m+1} + m(1+2m+2\lambda m+3\lambda m^2)a_{m+1}^2 = (1-\beta)q_{2m}. \tag{3.13}$$

From (3.10) and (3.12), we get

$$p_m = -q_m \tag{3.14}$$

and

$$a_{m+1}^2 = \frac{(1-\beta)^2(p_m^2+q_m^2)}{2m^2(1+\lambda m)^2}. \tag{3.15}$$

Adding (3.11) and (3.13), we get

$$a_{m+1}^2 = \frac{(1-\beta)(p_{2m}+q_{2m})}{2m^2(1+\lambda m)}. \tag{3.16}$$

From the equations (3.15), (3.16) and by using Lemma 2.4, we get:

$$|a_{m+1}| \le \frac{2(1-\beta)}{m(1+\lambda m)} \textit{ and } |a_{m+1}| \le \sqrt{\frac{2(1-\beta)}{m^2(1+\lambda m)}},$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (3.13) from (3.11), we get

$$a_{2m+1} = \frac{(1-\beta)(p_{2m}-q_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)}{2}a_{m+1}^2. \tag{3.17}$$

Therefore, we find from (3.15) and (3.17) that

$$a_{2m+1} = \frac{(1-\beta)(p_{2m}-q_{2m})}{4m(1+2\lambda m)} + \frac{(m+1)(1-\beta)^2(p_m^2+q_m^2)}{4m^2(1+\lambda m)^2}. \tag{3.18}$$

Also, from (3.16) and (3.17), we have

$$a_{2m+1} = \frac{[1+2m+2\lambda m+3\lambda m^2]p_{2m} + [1+2\lambda m+\lambda m^2]q_{2m}}{4m^2(1+\lambda m)(1+2\lambda m)}(1-\beta). \tag{3.19}$$

So, from the equations (3.18), (3.19) and applying Lemma 2.4, we get

$$|a_{2m+1}| \le \frac{1-\beta}{m(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{m^2(1+\lambda m)^2}$$

and

$$|a_{2m+1}| \le \frac{(1+m)(1-\beta)}{m^2(1+\lambda m)}.$$

□

Theorem 3.6. *Let f given by* (1.4) *be in the subclass $\mathcal{H}_{\Sigma_m}(\lambda,\beta)$* $(0 \le \beta < 1$, $\lambda \ge 0)$. *Also let ρ be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \le \begin{cases} \frac{(1-\beta)}{m(1+2\lambda m)} \ ; \ |T(\rho)| \le 1 \\ \\ \frac{(1-\beta)|T(\rho)|}{m(1+2\lambda m)} \ ; \ |T(\rho)| \ge 1 \end{cases}$$

where

$$T(\rho) = \frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}.$$

Proof. From the equation (3.17), we get

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{m-2\rho+1}{2}a_{m+1}^2 + \frac{(1-\beta)(p_{2m}-q_{2m})}{4m(1+2\lambda m)}. \tag{3.20}$$

From the equation (3.16) and (3.20), we have

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{(1-\beta)}{4m(1+2\lambda m)}\left\{\left[\frac{(m-2\rho+1)(1+2\lambda m)}{m\,(1+\lambda m)} + 1\right]p_{2m} + \left[\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)} - 1\right]q_{2m}\right\}.$$

Next, taking the absolute values we obtain

$$|a_{2m+1} - \rho a_{m+1}^2| \le \frac{(1-\beta)}{4m(1+2\lambda m)}\left\{\left|\frac{(m-2\rho+1)(1+2\lambda m)}{m\,(1+\lambda m)} + 1\right||p_{2m}|\right.$$

$$+\left|\frac{(m-2\rho+1)(1+2\lambda m)}{m(1+\lambda m)}-1\right||q_{2m}|\Bigg\}.$$

Then, by using Lemma 2.4, we conclude that

$$|a_{2m+1}-\rho a_{m+1}^2|\leq\begin{cases}\frac{(1-\beta)}{m(1+2\lambda m)};\ |T(\rho)|\leq 1\\ \\ \frac{(1-\beta)|T(\rho)|}{m(1+2\lambda m)};\ |T(\rho)|\geq 1.\end{cases}$$

□

4. Corollaries and Consequences

By setting $\lambda=0$ in Theorem 3.4, we conclude the following result.

Corollary 4.1. *Let f given by* (1.4) *be in the subclass $\mathcal{S}_{\Sigma_m}^{\beta}(0\leq\beta<1)$. If $a_{mt+1}=0,\ 1\leq t\leq k-1$, then*

$$|a_{mk+1}|\leq\frac{2(1-\beta)}{mk}\ ,\ k\geq 2.$$

By setting $\lambda=0$ in Theorem 3.5, we conclude the following result.

Corollary 4.2. *Let f given by* (1.4) *be in the subclass $\mathcal{S}_{\Sigma_m}^{\beta}(0\leq\beta<1)$. Then*

$$|a_{m+1}|\leq\begin{cases}\frac{\sqrt{2(1-\beta)}}{m};\ 0\leq\beta\leq\frac{1}{2}\\ \\ \frac{2(1-\beta)}{m}\ ;\ \frac{1}{2}\leq\beta<1\end{cases}$$

and

$$|a_{2m+1}|\leq\begin{cases}\frac{(m+1)(1-\beta)}{m^2}\ ;\ 0\leq\beta\leq\frac{1+2m}{2(1+m)}\\ \\ \frac{2(m+1)(1-\beta)^2}{m^2}+\frac{1-\beta}{m};\ \frac{1+2m}{2(1+m)}\leq\beta<1.\end{cases}$$

Remark 4.3. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.2 are better than those given in [4, Corolary 7].

By setting $\lambda=0$ in Theorem 3.6, we conclude the following result.

Corollary 4.4. *Let f given by* (1.4) *be in the subclass $\mathcal{S}_{\Sigma_m}^{\beta}(0\leq\beta<1)$. Also let ρ be real number. Then*

$$|a_{2m+1}-\rho a_{m+1}^2|\leq\begin{cases}\frac{(1-\beta)}{m}\ ;\ |m-2\rho+1|\leq m\\ \\ \frac{(1-\beta)|m-2\rho+1|}{m^2}\ ;\ |m-2\rho+1|\geq m.\end{cases}$$

By setting $m=1$ in Theorem 3.5, we conclude the following result.

Corollary 4.5. *Let f given by* (1.1) *be in the subclass* $B_\Sigma(\beta,\lambda)$ $(0 \le \beta < 1$, $\lambda \ge 0)$. *Then*

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}} \ ; \ \lambda + 2\beta \le 1 \\ \\ \frac{2(1-\beta)}{1+\lambda} \ ; \ \lambda + 2\beta \ge 1 \end{cases}$$

and

$$|a_3| \le \begin{cases} \frac{2(1-\beta)}{1+\lambda} \ ; \ 0 \le \beta \le \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \\ \\ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2} \ ; \ \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \le \beta < 1. \end{cases}$$

Remark 4.6. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 4.5 are better than those given in [27, Theorem 3.2].

By setting $\lambda = 0$ in Corollary 4.5, we conclude the following result.

Corollary 4.7. *Let f given by* (1.1) *be in the subclass* $\mathcal{S}_{\sigma_{\mathrm{B}}}(\beta)$ *of bi-starlike functions of order* $\beta(0 \le \beta < 1)$. *Then*

$$|a_2| \le \begin{cases} \sqrt{2(1-\beta)}; \ 0 \le \beta \le \frac{1}{2} \\ \\ 2(1-\beta) \ ; \ \frac{1}{2} \le \beta < 1 \end{cases}$$

and

$$|a_3| \le \begin{cases} 2(1-\beta) \ ; \ 0 \le \beta \le \frac{3}{4} \\ \\ (1-\beta) + 4(1-\beta)^2 \ ; \ \frac{3}{4} \le \beta < 1. \end{cases}$$

Remark 4.8. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 4.7 are better than those given in [27, Corollary 3.3].

By setting $\lambda = 1$ in Corollary 4.5, we conclude the following result.

Corollary 4.9. *Let f given by* (1.1) *be in the subclass* $\mathcal{K}_{\sigma_{\mathrm{B}}}(\beta)$ *of bi-convex functions of order* $\beta(0 \le \beta < 1)$. *Then*

$$|a_2| \le 1 - \beta$$

and

$$|a_3| \le \begin{cases} 1-\beta \ ; \ 0 \le \beta \le \frac{1}{3} \\ \\ \frac{1-\beta}{3} + (1-\beta)^2 \ ; \ \frac{1}{3} \le \beta < 1. \end{cases}$$

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References

1. H. Airault, J. Ren, An Algebra of Differential Operators and Generating Functions on the Set of Univalent Functions, *Bull. Sci. Math.*, **126**(5), (2002), 343-367.
2. H. Airault, A. Bouali, Differential Calculus on the Faber Polynomials, *Bull. Sci. Math.*, **130**, (2006),179-222.
3. Ş. Altinkaya, S. Yalçin, On Some Subclass of m-fold Symmetric Bi-univalent Functions, *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.*, **67**(1), (2018), 29-36.
4. Ş. Altinkaya, S. Yalçin, Coefficient Bounds for Certain Subclasses of m-fold Symmetric bi-univalent Functions, *Journal of Mathematics*, **2015**, Article ID 241683, (2015).
5. D. A. Brannan, J. G. Clunie(Eds.), Aspect of Contemporary Complex Analysis, Advanced Study Institute Held at the University of Durham, Durham; July 1-20, (1979).
6. D. A. Brannan, T. S. Taha, On Some Classes of Bi-univalent Functions, *Studia Univ. Babes-Bolyai Math.*, **31**(2), (1986), 70-77.
7. D. Breaz, N. Breaz, H. M. Srivastava, An Extention of the Univalent Conditions for a Family of Integral Operators, *Appl. Math. Lett.*, **22**, (2009), 41-44.
8. M. Çağlar, E. Deniz, H. M. Srivastava, Second Hankel Determinant for Certain Subclasses of Bi-univalent Functions, *Turkish J. Math.*, **41**, (2017), 694-706.
9. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
10. S. S. Eker, Coefficient Bounds for Subclasses of m-fold Symmetric Bi-univalent Functions, *Turkish J. Math.*, **40**(3), (2016), 641-646.
11. S. S. Eker, Coefficient Estimates for New Subclasses of m-fold Symmetric Bi-univalent Functions, *Theory Appl. Math. Comput. Sci.*, **6**(2), (2016), 103-109.
12. J. M. Jahangiri, S. G. Hamidi, Advances on the Coefficient Bounds for m-fold Symmetric Bi-close-to-convex Functions, *Tbilisi Math. J.*, **9**(2), (2016), 75-82.
13. W. Koepf, Coefficients of Symmetric Functions of Bounded Boundary Rotation, *Proc. Amer. Math. Soc.*, **105**(2), (1989), 324-329.
14. Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.
15. F. M. Sakar, H. O. Güney, Faber Polynomial Coefficient Estimates for Subclasses of m-fold Symmetric Bi-univalent Functions Defined by Fractional Derivative, *Malays. J. Math. Sci.*, **11**(2), (2017), 275-287.
16. B. Senthil, B. S. Keerthi, Certain Subclass of m-fold Symmetric-sakaguchi Type Bi-univalent Functions, *Int. J. Pure Appl. Math.*, **109**(10), (2016), 29-37.
17. H. M. Srivastava, D. Bansal, Coefficient Estimates for a Subclass of Analytic and Bi-univalent Functions, *J. Egyptian Math. Soc.*, **23**, (2015), 242-246.
18. H. M. Srivastava, S. Gaboury, F. Ghanim, Initial Coefficient Estimates for Some Subclasses of m-fold Symmetric Bi-univalent Functions, *Acta Math. Sci. Ser. B*, **36**(3), (2016), 863-871.
19. H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient Estimates for Some Subclasses of m-fold Symmetric Bi-univalent Functions, *Acta Univ. Apulensis Math. Inform.*, **41**, (2015), 153-164.
20. H. M. Srivastava, B. S. Joshi, S. Joshi, H. Pawar, Coefficient Estimates for Certain Subclasses of Meromorphically Bi-univalent Functions, *Palest. J. Math.*, **5**, (2016), 250-258.
21. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain Subclasses of Analytic and Bi-univalent Functions, *Appl. Math. Lett.*, **23**, (2010), 1188-1192.
22. H. M. Srivastava, S. Sivasubramanian, R. Sivakumar, Initial Coefficient Bounds for a Subclass of m-fold Symmetric Bi-univalent Functions, *Tbilisi Math. J.*, **7**(2), (2014), 1-10.

23. H. M. Srivastava, S. Sumer Eker, M. Rosihan Ali, Coefficient Bounds for a Certain Class of Analytic and Bi-univalent Functions, *Filomat*, **29**, (2015), 1839-1845.
24. H. Tang, H. M. Srivastava, S. Sivasubramanian, P. Gurusamy, The Fekete-Szegö Functional Problems for Some Subclasses of m-fold Symmetric Bi-univalent Functions, *J. Math. Inequal*, **10**(4), (2016), 1063-1092.
25. Z. Tu, L. Xiong, Coefficient Problems for United Starlike and Convex Classes of m-fold Symmetric Bi-univalent Functions, *J. Math. Inequal*, **12**(4), (2018), 921-932.
26. A. K. Wanas, A. H. Majeed, Certain New Subclasses of Analytic and m-fold Symmetric Bi-univalent Functions, *Appl. Math. E-Notes*, **18**, (2018), 178-188.
27. X.-F. li, A.-P. Wang, Two New Subclasses of bi-univalent Functions, *International Mathematical Forum*, **7**(30), (2012), 1495-1504.
28. Q.-H. Xu, Y.-C. Gui, H. M. Srivastava, Coefficient Estimates for a Certain Subclass of Analytic and bi-univalent Functions, *Appl. Math. Lett.*, **25**, (2012), 990-994.
29. Q.-H. Xu, H.-G. Xiao, H. M. Srivastava, A Certain General Subclass of Analytic and bi-univalent Functions and Associated Coefficient Estimate Problems, *Appl. Math. Comput.*, **218**, (2012), 11461-11465.
30. A. Zireh, E. Analouei Adegani, S. Bulut, Faber Polynomial Coefficient Estimates for a Comprehensive Subclass of Analytic bi-univalent Functions Defined by Subordination, *Bull. Belg. Math. Soc. Simon Stevin.*, **23**, (2016), 487-504.
31. A. Zireh, S. Salehian, On the Certain Subclass of Analytic and Bi-univalent Functions Defined by Convolution, *Acta Univ. Apulensis Math. Inform.*, **44**, (2015), 9-19.