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A Study of Metric Spaces of Interval Numbers in n-Sequences Defined by Orlicz Function

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Abstract. In recent years, a variety of work has been done in the field of single, double and triple sequences. Study on n-tuple sequence is new in this field. The main interest of this paper is to explore the idea of n-tuple sequences $x = (x_{i_1,i_2,\ldots,i_n})$ in metric spaces. We introduce the concept of n-sequence space of interval number and discussed its arithmetic properties. Furthermore, we combined the concept of Orlicz function, statistical convergence, interval number and n-sequence to construct some new n-sequence spaces and discussed their properties. Some suitable examples for these spaces have been constructed.

Keywords: n-sequence, Statistical convergence, Interval number, Orlicz function.

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1. Introduction

The Banach space gave birth to many useful concepts in mathematics, Orlicz space is no different. After the development of Lebesgue theory of integration, Z. W Birnbaum and W. Orlicz [2] introduced Orlicz space as the generalization of L^p spaces, $1 \leq p < \infty$. In the definition of L^p space, they replaced x^p by

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a more general convex function ϕ. Later Orlicz used this idea to construct the space L^M. A comprehensive study of Orlicz space was done by Lindenstrauss and Tzafriri [10] as they construct the sequence space l^M,

$$l^M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \text{ for some } \rho > 0 \right\},$$

and proved that it contains a subspace isomorphic to l^p $(1 \leq p < \infty)$. Many others had introduced different classes of sequence spaces defined by Orlicz function, (see [11, 15, 17]). The sequence space $M(\phi)$ was introduced by Sargent [18] giving its relationship with l^P. Some other work related to sequence spaces can be seen in [1, 16, 24, 23].

Statistical convergent was introduced by Fast [6] and Steinhaus [21] in 1951 and later developed by Schoenberg [20]. Over the last few decades several authors have explored statistical convergence in various directions using sequence spaces (see [4, 7, 8, 9] and many more).

A single sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to a number L, if for a given $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where the vertical bar indicates the number of elements.

In 1951, P.S. Dwyer [3] suggested the arithmetic interval, whose proper structure was provided by Moore [12] in 1959. Later, Moore and Yang [13] provided its computational methods. Further work related to interval numbers has been done in [5] and [19].

An interval number is a set consisting of a closed interval of real numbers x, where $a \leq x \leq b$; $a, b \in \mathbb{R}$. [19] investigated some properties of interval numbers. The set of all real valued closed interval is denoted by $\mathbb{IR}$. i.e., $\mathbb{IR} = \{\bar{x} : \bar{x} \text{ is a closed interval}\}$. Thus an interval number is a closed subset of real numbers.

Let x_f and x_l are first and last points of $\bar{x}$, respectively. For all $\bar{x}_1, \bar{x}_2 \in \mathbb{IR}$ following properties satisfies:

$$\begin{aligned}
\bar{x}_1 &= \bar{x}_2 \Leftrightarrow x_{f_1} = x_{f_2} \text{ and } x_{l_1} = x_{l_2};\\
\bar{x}_1 &+ \bar{x}_2 = \{x \in \mathbb{R} : x_{f_1} + x_{f_2} \leq x \leq x_{l_1} + x_{l_2}\};\\
\text{if } \alpha &\geq 0, \text{ then } \alpha\bar{x} = \{x \in \mathbb{R} : \alpha x_{f_1} \leq x \leq \alpha x_{l_1}\} \text{ and}\\
\text{if } \alpha &< 0, \text{ then } \alpha\bar{x} = \{x \in \mathbb{R} : \alpha x_{l_1} \leq x \leq \alpha x_{f_1}\};\\
\bar{x}_1\bar{x}_2 &= \{\, x \in \mathbb{R} : \min\{x_{f_1}.x_{f_2}, x_{f_1}.x_{l_2}, x_{l_1}.x_{f_2}, x_{l_1}.x_{l_2}\}\\
&\qquad \leq x \leq \max\{x_{f_1}.x_{f_2}, x_{f_1}.x_{l_2}, x_{l_1}.x_{f_2}, x_{l_1}.x_{l_2}\}\}.
\end{aligned}$$

$\mathbb{IR}$ is a complete metric space with respect to the metric given by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|x_{f_1} - x_{f_2}|, |x_{l_1} - x_{l_2}|\}.$$

The sequence of interval numbers is a transformation f from $\mathbb{N}$ to $\mathbb{IR}$ defined by $f(m) = \bar{x} = \bar{x}_m$, where $\bar{x}_m$ is the m^{th} term of the sequence $(\bar{x}_m)$. The set of all such sequences is denoted by w^i. [5] defined the sequence of double interval numbers and discuss its properties.

Moricz [14] and Tripathy et al. [22] found some interesting results related to n-sequence.

Definition 1.1. An n-sequence is a function whose domain is either $\mathbb{N}^n$ or subset of $\mathbb{N}^n$.

In the sequel, $\mathbb{N}^n$ stands for $\mathbb{N} \times \mathbb{N} \times ... \times \mathbb{N}_{(n\ times)}$. Throughout the article the set of all n-sequence will be denoted by w_n.

Take $\mathbb{N}^n$ as an ordered set, it can easily be proven by using the lexicographical order on $\mathbb{N}^n$, i.e. to compare component-wise. The reason is that one can only take limit over a monotonic set.

In this paper, we have combined the concept of Orlicz function, statistical convergence, interval number and n-sequences to construct the spaces ${}_n\bar{w}(M)$, ${}_n\bar{w}(M,p)$, ${}_n\bar{w}_0(M,p)$, ${}_n\bar{w}_\infty(M,p)$ and discuss their properties. Some examples have also been given to show that n-sequences are bounded, convergent, etc. with respect to the suitable metric.

2. Some sequence spaces of interval numbers defined by Orlicz function

In this section, we construct some sequence spaces defined by Orlicz function. We start with some basic concepts for n-sequences like boundedness and convergence with respect to the metric

$$d(x_{i_1,i_2,...,i_n}, y_{i_1,i_2,...,i_n}) = |x_{i_1,i_2,...,i_n} - y_{i_1,i_2,...,i_n}|.$$

Definition 2.1. An n-sequence $x = (x_{i_1,i_2,...,i_n})$ *such that* $i_1, i_2, ..., i_n \in \mathbb{N}$, is said to be bounded if $\sup\limits_{i_1,i_2,...,i_n} d(x_{i_1,i_2,...,i_n}, \theta) < \infty$. The space of all bounded n-sequence is denoted by ${}_nl_\infty$ and is a metric space with respect to the metric defined by

$$d(x_{i_1,i_2,...,i_n}, y_{i_1,i_2,...,i_n}) = \sup_{i_1,i_2,...,i_n} |x_{i_1,i_2,...,i_n} - y_{i_1,i_2,...,i_n}|.$$

Example 2.2. Consider an n-sequence $x = (x_{i_1,i_2,...,i_n})$ defined by

$$x_{i_1,i_2,...,i_n} = \begin{cases} 1, & \text{if all } i_j\text{'s are even;} \\ 2, & \text{if all } i_j\text{'s are odd;} \\ 3, & \textit{otherwise.} \end{cases}$$

Then $x = (x_{i_1,i_2,...,i_n}) \in {}_nl_\infty$ as $\sup\limits_{i_1,i_2,...,i_n} d(x_{i_1,i_2,...,i_n}, \theta) = 3$.

Definition 2.3. Consider an n-sequence $x = (x_{i_1,i_2,...,i_n})$ *such that* $i_1, i_2, ..., i_n \in \mathbb{N}$. If for a given $\epsilon > 0$, $\exists\ n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$d(x_{i_1,i_2,...,i_n}, x_0) < \epsilon,\ \forall\ i_1, i_2, ..., i_n > n_0,$$

then x_0 is called the limit of $(x_{i_1,i_2,...,i_n})$ in Pringsheim's sense and we say that n-sequence x is convergent in Pringshiem's sense to the limit x_0 and we write $P - \lim\limits_{i_1,i_2,...,i_n} x = x_0$.

The space of all convergent n-sequence and all null sequence in Pringsheim sense is denoted by c_n and ${}_nc_0$ respectively.

Example 2.4. Consider an n-sequence $x = (x_{i_1,i_2,...,i_n})$ defined by

$$x_{i_1,i_2,...,i_n} = \frac{1}{\prod\limits_{i_j=1}^{n} i_j}.$$

Let $\epsilon > 0$,

$$d(x_{i_1,i_2,...,i_n}, 0) = \left| \frac{1}{\prod\limits_{i_j=1}^{n} i_j} \right| < \epsilon,\ \forall\ i_1, i_2, ..., i_n > n_0 = \frac{1}{\epsilon}.$$

Then $P - \lim\limits_{i_1,i_2,...,i_n} x = 0$.

Definition 2.5. Let $E \subseteq \mathbb{N}^n$. Then E is said to have a n-density $\delta_n(E)$, if

$$\delta_n(E) = \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} \left(\sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \chi_E\big(i_1, i_2, ..., i_n\big) \right)$$

exists, where χ_E is a characteristic function of E. So,

$$\delta_n(E) = \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} |\{(i_1, i_2, ..., i_n) \in E : i_1 \leq m_1, i_2 \leq m_2, ..., i_n \leq m_n\}|.$$

Example 2.6. Let $E = \{(i_1, i_2, ..., i_n) : i_j \text{ is odd}, \forall j = 1, 2, ..., n\}$. Then

$$\begin{aligned} \delta_n(E) &= \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} \left(\frac{m_1}{2} \cdot \frac{m_2}{2} \cdot \ldots \cdot \frac{m_n}{2} \right) \\ &= \frac{1}{2^n}. \end{aligned}$$

Definition 2.7. An n-sequence $x = (x_{i_1,i_2,...,i_n})$ is said to be statistically convergent to a number x_0 in Pringsheim sense, if for a given $\epsilon > 0$,

$$\delta_n\Bigg(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : d(x_{i_1,i_2,...,i_n}, x_0) \geq \epsilon\ \} \Bigg) = 0.$$

We write $stat - \lim\limits_{i_1,i_2,...,i_n} x = x_0$. The space of such sequences is denoted by S_n.

EXAMPLE 2.8. Consider an n-sequence $x = (x_{i_1,i_2,...,i_n})$ defined by

$$x_{i_1,i_2,...,i_n} = \begin{cases} i_1 + i_2 + ... + i_n, & \text{if } \forall j = 1,2,...,n,\ i_j \text{ is a perfect square ;} \\ 2, & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$. Then,

$$\begin{aligned} &\delta_n\Bigg(\{(i_1,i_2,...,i_n) \in \mathbb{N}^n : d(x_{i_1,i_2,...,i_n}, 2) \geq \epsilon\ \}\Bigg) \\ &\leq \delta_n\Bigg(\{(i_1,i_2,...,i_n) \in \mathbb{N}^n :\ \forall j = 1,2,...,n,\ i_j \text{ is a perfect square}\}\Bigg) \\ &\leq \lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} \sqrt{m_1} \cdot \sqrt{m_2} \cdot ... \cdot \sqrt{m_n} \\ &= 0. \end{aligned}$$

Thus, $stat - \lim_{i_1,i_2,...,i_n} x = 2$.

This example also shows that even if an n-sequence is unbounded, it can still converge statistically.

We now define n-sequence of interval number together with their arithmetic properties and give its boundedness and convergence.

Definition 2.9. Define a transformation h from $\mathbb{N}^n$ to $\mathbb{IR}$ so that $h(i_1,i_2,...,i_n) = \bar{x}$, $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$. Then $\bar{x}$ will be n-sequence of interval numbers (or sequence of n-interval numbers) and $\bar{x}_{i_1,i_2,...,i_n}$ is called the $(i_1,i_2,...,i_n)^{th}$ term of the sequence $(\bar{x})$.

Clearly $\bar{x}_{i_1,i_2,...,i_n}$ is a closed interval of real number. The first and last element of $\bar{x}_{i_1,i_2,...,i_n}$ is denoted by $\bar{x}_{f_{i_1,i_2,...,i_n}}$ and $\bar{x}_{l_{i_1,i_2,...,i_n}}$ and the set of all such sequence of n-number is denoted by w_n^i. w_n^i is a metric space with respect to the metric

$$d(\bar{x}_{i_1,i_2,...,i_n}, \bar{y}_{i_1,i_2,...,i_n}) = \max\{|x_{f_{i_1,i_2,...,i_n}} - y_{f_{i_1,i_2,...,i_n}}|, |x_{l_{i_1,i_2,...,i_n}} - y_{l_{i_1,i_2,...,i_n}}|\}. \tag{2.1}$$

Further, some algebraic properties of elements of w_n^i is given. Let $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ and $\bar{y} = (\bar{y}_{i_1,i_2,...,i_n})$ are two sequence in w_n^i and $\alpha \geq 0$. Then

$$\bar{x}_{i_1,i_2,...,i_n} + \bar{y}_{i_1,i_2,...,i_n} = [\bar{x}_{f_{i_1,i_2,...,i_n}} + \bar{y}_{f_{i_1,i_2,...,i_n}}, \bar{x}_{l_{i_1,i_2,...,i_n}} + \bar{y}_{l_{i_1,i_2,...,i_n}}]$$

and

$$\alpha\bar{x}_{i_1,i_2,...,i_n} = [\alpha\bar{x}_{f_{i_1,i_2,...,i_n}}, \alpha\bar{x}_{l_{i_1,i_2,...,i_n}}].$$

As $\mathbb{IR}$ is a quasivector space, w_n^i is also a quasivector space with the null element being $\bar{\theta} = (\bar{\theta}_{i_1,i_2,...,i_n}) = ([0,0])$ and unity being ([1,1]).

Our next example explains the structure of n-interval numbers and the algebraic properties of w_n^i.

EXAMPLE 2.10. Let $\bar{x}$ and $\bar{y}$ are two sequences such that

$\bar{x}_{i_1,i_2,...,i_n} = [i_1, i_1+1] + [i_2, i_2+1] + ... + [i_n, i_n+1]$, then

$\bar{x}_{i_1,i_2,...,i_n} = [i_1 + i_2, ... + i_n, i_1 + i_2 + ... + i_n + n]$ and let
$\bar{y}_{i_1,i_2,...,i_n} = [\min\{i_1, i_2, ..., i_n\}, \max\{i_1, i_2, ..., i_n\}]$, then

$(\bar{x} + \bar{y}) = ([i_1 + i_2, ... + i_n + \min\{i_1, i_2, ..., i_n\}, i_1 + i_2 + ... + i_n + n + \max\{i_1, i_2, ..., i_n\}])$.

Definition 2.11. An n-interval sequence $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ *such that* $i_1, i_2, ..., i_n \in \mathbb{N}$, is said to be bounded if there exists a positive number H such that

$$d(x_{i_1,i_2,...,i_n}, \bar{\theta}) \leq H, \ \forall \ i_1, i_2, ..., i_n \in \mathbb{N}.$$

The set of all such sequences is denoted by $\bar{l}^n_\infty$.

Example 2.12. Let

$$\bar{x}_{i_1,i_2,...,i_n} = \begin{cases} [4,5], \text{ if } \sum\limits_{i_j=1}^{n} i_j \text{ is even;} \\ [3,6], \text{ if } \sum\limits_{i_j=1}^{n} i_j \text{ is odd.} \end{cases}$$

Then $\forall \ i_1, i_2, ..., i_n \in \mathbb{N}$,

$$\begin{aligned} d(x_{i_1,i_2,...,i_n}, \bar{\theta}) &= \max\{|x_{f_{i_1,i_2,...,i_n}} - 0|, |x_{l_{i_1,i_2,...,i_n}} - 0|\} \\ &= \max\{4, 6\} \\ &= 6. \end{aligned}$$

Thus, $(x_{i_1,i_2,...,i_n}) \in \bar{l}^n_\infty$.

Let M be an Orlicz function and $p = (p_{i_1,i_2,...,i_n})$ be an n-sequence of bounded positive real numbers such that $0 < p_{i_1,i_2,...,i_n} \leq \sup\limits_{i_1,i_2,...,i_n} p_{i_1,i_2,...,i_n} = D < \infty$ and $H = \max(1, 2^{D-1})$. Then the real number sequences $(x_{i_1,i_2,...,i_n})$ and $(y_{i_1,i_2,...,i_n})$ satisfy the following

$$|x_{i_1,i_2,...,i_n} + y_{i_1,i_2,...,i_n}|^{p_{i_1,i_2,...,i_n}} \leq H(|x_{i_1,i_2,...,i_n}|^{p_{i_1,i_2,...,i_n}} + |y_{i_1,i_2,...,i_n}|^{p_{i_1,i_2,...,i_n}}). \tag{2.2}$$

Definition 2.13. Consider an n-interval sequence $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ such that $i_1, i_2, ..., i_n \in \mathbb{N}$. If for a given $\epsilon > 0$, $\exists \ n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) < \epsilon, \ \forall \ i_1, i_2, ..., i_n > n_0,$$

then we say that n-interval sequence $\bar{x}$ is convergent in Pringsheim's sense to the interval number $\bar{x}_0$ and we write $P - \lim\limits_{i_1,i_2,...,i_n} \bar{x} = L$. The space of all convergent n-interval sequence in Pringsheim sense is denoted by $\bar{c}_n$. The space of all null n-interval sequence is denoted by ${}_n\bar{c}_0$.

For the sake of simplicity we will write lim instead of P-limit.

Example 2.14. Consider an n-sequence $\bar{x} = (x_{i_1,i_2,...,i_n})$ defined by

$$\bar{x}_{i_1,i_2,...,i_n} = \left[-\frac{2i_1}{i_1+1}, \frac{i_1}{i_1+1} \right].$$

Now,

$$\begin{aligned} d(\bar{x}_{i_1,i_2,...,i_n},[-2,1]) &= d\left(\left[-\frac{2i_1}{i_1+1},\frac{i_1}{i_1+1}\right],[-2,1]\right) \\ &= \max\left\{\left|-\frac{2i_1}{i_1+1}-2\right|,\left|\frac{i_1}{i_1+1}-1\right|\right\} \\ &= \max\left\{\left|\frac{-2}{i_1+1}\right|,\left|\frac{-1}{i_1+1}\right|\right\} \\ &= \frac{2}{i_1+1}. \end{aligned}$$

Thus for any given $\epsilon > 0$, take a natural number $n_0 \geq \frac{2}{\epsilon} - 1$ such that

$d(\bar{x}_{i_1,i_2,...,i_n},[-2,1]) < \epsilon,\ \forall\ i_1,i_2,...,i_n > \frac{2}{\epsilon} - 1.$

Hence,

$P - \lim_{i_1,i_2,...,i_n} \bar{x} = [-2,1].$

Now, we are in the position to introduce following sequence spaces defined by Orlicz function.

$$_n\bar{w}(M) = \left\{\bar{x} = (\bar{x}_{i_1,i_2,...,i_n}) \in w_n^i : P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^n m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right] = 0,\ \text{for some } \rho > 0\right\},$$

$$_n\bar{w}(M,p) = \left\{\bar{x} = (\bar{x}_{i_1,i_2,...,i_n}) \in w_n^i : P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^n m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} = 0,\ \text{for some } \rho > 0\right\},$$

$$_n\bar{w}_0(M,p) = \left\{\bar{x} = (\bar{x}_{i_1,i_2,...,i_n}) \in w_n^i : P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^n m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{\theta})}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} = 0,\ \text{for some } \rho > 0\right\},$$

and

$$_n\bar{w}_\infty(M,p) = \left\{\bar{x} = (\bar{x}_{i_1,i_2,...,i_n}) \in w_n^i : \sup_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^n m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{\theta})}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} < \infty,\ \text{for some } \rho > 0\right\}.$$

Remark 2.15. From the definition of these spaces we can write the inequality $_n\bar{w}(M,p) \subset {_n\bar{w}_0(M,p)} \subset {_n\bar{w}_\infty(M,p)}$. Also if we take $p = 1$, then $_n\bar{w}(M,p) = {_n\bar{w}(M)}$.

EXAMPLE 2.16. Consider an Orlicz function $M(x) = x^2$ and n-sequence $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ defined by

$$\bar{x}_{i_1,i_2,...,i_n} = \left[-a - \frac{1}{\sum\limits_{i_j=1}^{n} {i_j}^2}, a + \frac{1}{\sum\limits_{i_j=1}^{n} i_j} \right] \text{ and } p_{i_1,i_2,...,i_n} = 2.$$

Now,

$$\begin{aligned} d(\bar{x}_{i_1,i_2,...,i_n}, [-a,a]) &= d\left(\left[-a - \frac{1}{\sum\limits_{i_j=1}^{n} {i_j}^2}, a + \frac{1}{\sum\limits_{i_j=1}^{n} i_j} \right], [-a,a] \right) \\ &= \max\left\{ \left| -\frac{1}{\sum\limits_{i_j=1}^{n} {i_j}^2} \right|, \left| \frac{1}{\sum\limits_{i_j=1}^{n} i_j} \right| \right\} \\ &= \frac{1}{\sum\limits_{i_j=1}^{n} i_j}. \end{aligned}$$

$$\begin{aligned} &P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}} \\ &= P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[\frac{1}{\rho^2 (i_1 + i_2 + ... + i_n)^2} \right]^2 \\ &= 0. \end{aligned}$$

Then $\bar{x} \in {}_n\bar{w}(M,p)$. From Remark 2.15, it is clear that it will also be in ${}_n\bar{w}_0(M,p)$ and ${}_n\bar{w}_\infty(M,p)$.

Theorem 2.17. *If* $0 < p_{i_1,i_2,...,i_n} < q_{i_1,i_2,...,i_n}$ *and* $\left(\frac{p_{i_1,i_2,...,i_n}}{q_{i_1,i_2,...,i_n}} \right)$ *is bounded, then*

${}_n\bar{w}(M,p) \subset {}_n\bar{w}(M,q)$.

Proof. If $x \in {}_n\bar{w}(M,p)$, then $\exists\ \rho > 0$ such that

$$P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}} = 0. \tag{2.3}$$

Since, Orlicz function is non-negative, therefore

$M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \leq 1,$

also, M is non-decreasing. Thus

$$\begin{aligned} &\sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{q_{i_1,i_2,...,i_n}} \\ &\qquad \leq \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}}. \end{aligned} \tag{2.4}$$

From Equations (2.3) and (2.4), we get that

$$P-\lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{q_{i_1,i_2,...,i_n}}=0.$$

Hence, $x\in {}_n\bar{w}(M,q)$. □

Corollary 2.18. ***(a)*** *If $0<p_{i_1,i_2,...,i_n}<1$, then ${}_n\bar{w}(M,p)\subset {}_n\bar{w}(M)$, and*
(b) *If $1<p_{i_1,i_2,...,i_n}<\infty$, then ${}_n\bar{w}(M)\subset \bar{w}(M,p)$.*

Proof. On taking $q=1$ in Theorem 2.17, we get the first part of the corollary and for second part, take $p=1$ in the same theorem. □

Theorem 2.19. *Let M_1 and M_2 be two Orlicz functions. Then ${}_n\bar{w}(M_1,p)\cap$ ${}_n\bar{w}(M_2,p)$
$\subset {}_n\bar{w}(M_1+M_2,p)$.*

Proof. Let $(x_{i_1,i_2,...,i_n})\in {}_n\bar{w}(M_1,p)\cap {}_n\bar{w}(M_2,p)$. Then

$$P-\lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M_1\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho_1}\right)\right]^{p_{i_1,i_2,...,i_n}}=0,$$

$\rho_1>0$,
and

$$P-\lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M_2\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho_2}\right)\right]^{p_{i_1,i_2,...,i_n}}=0,$$

$\rho_2>0$.
Let $\rho=\max\{\rho_1,\rho_2\}$. Following from the inequality (1), we get

$$\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[(M_1+M_2)\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}}\leq H\Bigg(\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M_1\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho_1}\right)\right]^{p_{i_1,i_2,...,i_n}}+\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n}\left[M_2\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho_2}\right)\right]^{p_{i_1,i_2,...,i_n}}\Bigg).$$

Therefore, $x\in {}_n\bar{w}(M_1+M_2,p)$. □

Next, the definition of subsequence and K-step are given, which we require to define monotone spaces in n-sequences.

Definition 2.20. Let $x=(x_{i_1,i_2,...,i_n})$, $i_1,i_2,...,i_n\in\mathbb{N}$ be an n-sequence. Choose

$K=\{((m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n})\in\mathbb{N}^n:(i_1,i_2,...,i_n)\in\mathbb{N}^n\}$,

such that K is a strictly increasing subset of $\mathbb{N}^n$ and $\bar{\delta}^n(K)>0$, then the sequence $(x_{(m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n}})$ or $(x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n)\in K}$ is called a subsequence of x.

Remark 2.21. Since we need an increasing subset of $\mathbb{N}^n$ in many definitions such as subsequence and hence K-step, canonical preimage and monotone, etc. We take $\mathbb{N}^n$ as an ordered set.

Definition 2.22. Let $K=\{((m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n})\in\mathbb{N}^n:(i_1,i_2,...,i_n)\in\mathbb{N}^n\}$, be an increasing subset of $\mathbb{N}^n$ and E be a sequence space. A K-step of E is a sequence space

$\lambda_K^E = \{x_{(m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n}} \in w_n : (m_1)_{i_1}, (m_2)_{i_2}, ..., (m_n)_{i_n} \in E\}$.

Definition 2.23. A canonical preimage of a sequence $\{x_{(m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n}}\} \in \lambda_K^E$ is a sequence $y = (y_{i_1,i_2,...,i_n}) \in w_n$ defined by

$$y_{i_1,i_2,...,i_n} = \begin{cases} x_{i_1,i_2,...,i_n}, & (i_1, i_2, ..., i_n) \in K; \\ 0, & otherwise. \end{cases}$$

The set of preimages of all elements in step space λ_K^E is known as canonical preimage of λ_K^E, i.e, y is in canonical preimage of λ_K^E if and only if x is the canonical preimage of some $x \in \lambda_K^E$.

Definition 2.24. A sequence space is monotone if it contains the canonical preimages of its step space.

Definition 2.25. A sequence space E is said to be solid if for all n-sequence of scalars $(\alpha_{i_1,i_2,...,i_n})_{i_1,i_2,...,i_n \in \mathbb{N}}$ with $|\alpha_{i_1,i_2,...,i_n}| \leq 1$, and $(x_{i_1,i_2,...,i_n})_{i_1,i_2,...,i_n \in \mathbb{N}} \in E$,

$$(\alpha_{i_1,i_2,...,i_n} x_{i_1,i_2,...,i_n}) \in E.$$

Lemma 2.26. [5] *Every solid space is monotone.*

Theorem 2.27. *The n-sequence space ${}_n\bar{w}_\infty(M, p)$ is solid and hence monotone.*

Proof. Let $(x_{i_1,i_2,...,i_n}) \in {}_n\bar{w}_\infty(M, p)$ and $(\alpha_{i_1,i_2,...,i_n})$ be a scalar sequence such that $|\alpha_{i_1,i_2,...,i_n}| \leq 1$ for all $i_1, i_2, ..., i_n \in \mathbb{N}$. Then
$M\left(\frac{d(\alpha_{i_1,i_2,...,i_n}\bar{x}_{i_1,i_2,...,i_n},\bar{\theta})}{\rho}\right) \leq M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{\theta})}{\rho}\right)$.
Hence,

$$\sup_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\alpha_{i_1,i_2,...,i_n}\bar{x}_{i_1,i_2,...,i_n},\bar{\theta})}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}}$$
$$\leq \sup_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{\theta})}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}}$$
$$< \infty.$$

Therefore, $(\alpha_{i_1,i_2,...,i_n} x_{i_1,i_2,...,i_n}) \in {}_n\bar{w}_\infty(M, p)$. Thus the space is solid and by using Lemma 2.26, it is also monotone. $\square$

Now, we give the definition of statistical convergence of n-sequence of interval number and give conditions for inclusion relations with ${}_n\bar{w}(M, p)$.

Definition 2.28. An n-sequence $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ is said to be statistically convergent to an interval number $\bar{x}_0$, if for every $\epsilon > 0$

$$P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod_{j=1}^{n} m_j} |\{i_1 \leq m_1, i_2 \leq m_2, ..., i_n \leq m_n : d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) \geq \epsilon\}| = 0,$$

we denote it as $\bar{s}_n - \lim\limits_{i_1,i_2,...,i_n} \bar{x}_{i_1,i_2,...,i_n} = \bar{x}_0$. The set of all such sequences is denoted by $\bar{s}_n$.

EXAMPLE 2.29. Let an n-interval sequence defined by $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ such that

$$\bar{x}_{i_1,i_2,...,i_n} = \begin{cases} {i_1}^2, \text{ if } i_j \text{ is prime}, \forall\ j = 1,2,...,n; \\ 0, \text{ otherwise}. \end{cases}$$

Like Example 2.8 for n-sequence, the n-interval unbounded sequence can also be statistically convergent.

Theorem 2.30. *Let M be an Orlicz function and $0 < h \leq \inf\limits_{i_1,i_2,...,i_n} p_{i_1,i_2,...,i_n} \leq \sup\limits_{i_1,i_2,...,i_n} p_{i_1,i_2,...,i_n} = H < \infty$. Then ${}_n\bar{w}(M,p) \subset \bar{s}_n$.*

Proof. Let $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n}) \in {}_n\bar{w}(M,p)$. Then there exists $\rho > 0$ such that

$$P - \lim_{m_1,m_2,...,m_n} \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} = 0. \tag{2.5}$$

For a given $\epsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}...\sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} \\
&= \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{\substack{i_1=1\ i_2=1 \quad i_n=1 \\ d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)<\epsilon}}^{m_1 \quad m_2 \quad m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} \\
&\quad + \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{\substack{i_1=1\ i_2=1 \quad i_n=1 \\ d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)\geq\epsilon}}^{m_1 \quad m_2 \quad m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} \\
&\geq \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{\substack{i_1=1\ i_2=1 \quad i_n=1 \\ d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)\geq\epsilon}}^{m_1 \quad m_2 \quad m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} \\
&\geq \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{\substack{i_1=1\ i_2=1 \quad i_n=1 \\ d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)\geq\epsilon}}^{m_1 \quad m_2 \quad m_n} \left[M\left(\frac{\epsilon}{\rho}\right)\right]^{p_{i_1,i_2,...,i_n}} \\
&\geq \frac{1}{\prod\limits_{j=1}^{n} m_j} \sum_{\substack{i_1=1\ i_2=1 \quad i_n=1 \\ d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0)\geq\epsilon}}^{m_1 \quad m_2 \quad m_n} \min\left\{M\left(\frac{\epsilon}{\rho}\right)^h, M\left(\frac{\epsilon}{\rho}\right)^H\right\} \\
&\geq \frac{1}{\prod\limits_{j=1}^{n} m_j} |\{(i_1,i_2,...,i_n) \in \mathbb{N}^n : d(\bar{x}_{i_1,i_2,...,i_n},\bar{x}_0) \geq \epsilon\}| \min\left\{M\left(\frac{\epsilon}{\rho}\right)^h, M\left(\frac{\epsilon}{\rho}\right)^H\right\}.
\end{aligned}$$

By taking $P - \lim m_1,m_2,...,m_n \longrightarrow \infty$ and using Equation (2.5), we get the result. $\square$

Theorem 2.31. *Let M be an Orlicz function and $0 < h \leq \inf_{i_1,i_2,...,i_n} p_{i_1,i_2,...,i_n} \leq \sup_{i_1,i_2,...,i_n} p_{i_1,i_2,...,i_n} = H < \infty$. Then, for bounded n-interval sequences $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$, $\bar{s}_n \subset {}_n\bar{w}(M,p)$.*

Proof. Let $\bar{x} = (\bar{x}_{i_1,i_2,...,i_n})$ be a bounded and statistically convergent sequence. Since $\bar{x}$ is a bounded sequence, there exists a positive number H' such that $d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) \leq H'$.
For a given $\epsilon > 0$, we have

$$\frac{1}{\prod_{j=1}^{n} m_j} \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}}$$

$$= \frac{1}{\prod_{j=1}^{n} m_j} \sum_{\substack{i_1=1 \\ d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) < \epsilon}}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}}$$

$$+ \frac{1}{\prod_{j=1}^{n} m_j} \sum_{\substack{i_1=1 \\ d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) \geq \epsilon}}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0)}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}}$$

$$\leq \frac{1}{\prod_{j=1}^{n} m_j} \sum_{\substack{i_1=1 \\ d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) < \epsilon}}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \left[M\left(\frac{\epsilon}{\rho} \right) \right]^{p_{i_1,i_2,...,i_n}}$$

$$+ \frac{1}{\prod_{j=1}^{n} m_j} \sum_{\substack{i_1=1 \\ d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) \geq \epsilon}}^{m_1} \sum_{i_2=1}^{m_2} ... \sum_{i_n=1}^{m_n} \max \left\{ M\left(\frac{H'}{\rho} \right)^h, M\left(\frac{H'}{\rho} \right)^H \right\}$$

$$\leq \max \left\{ M\left(\frac{\epsilon}{\rho} \right)^h, M\left(\frac{\epsilon}{\rho} \right)^H \right\}$$

$$+ \frac{\max\left\{ M\left(\frac{H'}{\rho} \right)^h, M\left(\frac{H'}{\rho} \right)^H \right\}}{\prod_{j=1}^{n} m_j} |\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : d(\bar{x}_{i_1,i_2,...,i_n}, \bar{x}_0) \geq \epsilon\}|.$$

On taking $P - \lim m_1, m_2, ..., m_n \longrightarrow \infty$, we get the result. □

Corollary 2.32. $\bar{s}_n \cap \bar{l}^n_\infty = \bar{l}^n_\infty \cap {}_n\bar{w}(M,p)$.

Proof. The proof of corollary follows directly from Theorem 2.30 and 2.31. □

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References

1. N. Ahmad, S. K. Sharma, S. A. Mohiuddine, Generalized Entire Sequence Spaces Defined by Fractional Difference Operator and Sequence of Modulus Functions, *Journal of Applied And Engineering Mathematics*, **3**(2), (2017), 134-146.
2. Z. Birnbaum, Z. W. Orlicz, Über Die Verallgemeinerung Des Begriffes Der Zueinander Konjugierten Potenzen, *Studia Mathematica*, **3**(1), (1931), 1-67.

3. P. S. Dwyer, Computation with Approximate Numbers, *Linear Computations*, **1951**, (1951), 11-34.
4. O. H. H. Edely, M. Mursaleen, A. Khan, Approximation for Periodic Functions Via Weighted Statistical Convergence, *Applied Mathematics and Computation*, **219**(15), (2013), 8231-8236.
5. A. Esi, Some Double Sequence Spaces of Interval Numbers Defined by Orlicz Function, *Journal of the Egyptian Mathematical Society*, **22**(3), (2014), 424-427.
6. H. Fast, Sur La Convergence Statistique, *Colloquium Mathematicae*, **2**(3), (1951), 241-244.
7. A. Khan, V. Sharma, Statistical Approximation by (p, q)-Analogue of Bernstein-Stancu Operators, *ArXiv Preprint ArXiv:1604.05339*, **16**(4), (2016).
8. V. A. Khan, Q. M. Lohani, Statistically Pre-Cauchy Sequences and Orlicz Functions, *Southeast Asian Bulletin of Mathematics*, **31**(6), (2007), 1107-1112.
9. S. A. Mohiuddine, A. Alotaibi, M. Mursaleen, Statistical Convergence of Double Sequences in Locally Solid Riesz Spaces, *Abstract and Applied Analysis*, **2012**, (2012), Art. ID 719729, 9 pp.
10. J. Lindenstrauss, L. Tzafriri, On Orlicz Sequence Spaces, *Israel Journal of Mathematics*, **10**(3), (1971), 379-390.
11. S. A. Mohiuddine, B. Hazarika, Some Classes of Ideal Convergent Sequences and Generalized Difference Matrix Operator, *Filomat*, **31**(6), (2017), 1827-1834.
12. R. E. Moore, *Automatic Error Analysis in Digital Computation*, Lockheed Missiles and Space Co, 1959.
13. R. E. Moore, C. T. Yang, *Interval Analysis*, Lockheed Missiles and Space Co, 1959.
14. F. Móricz, Statistical Convergence of Multiple Sequences, *Archiv Der Mathematik*, **81**(1), (2003), 82-89.
15. M. Mursaleen, M. A. Khan, Qamaruddin, Difference Sequence Spaces Defined by Orlicz Function. *Demonstratio Mathematica*, **32**(1), (1999), 145-150.
16. S. A. Mohiuddine, K. Raj, M. Mursaleen, A. Alotaibi, Linear Isomorphic Spaces of Fractional-Order Difference Operators, *Alexandria Engineering Journal*, **60**(1), (2021), 1155-1164.
17. S. D. Parashar, B. Choudhary, Sequence Spaces Defined by Orlicz Functions, *Indian Journal of Pure and Applied Mathematics*, **25**, (1994), 419-428.
18. W. L. C. Sargent, Some Sequence Spaces Related to the l_p Spaces, *Journal of the London Mathematical Society*, **1**(2), (1960), 161-171.
19. M. Sengonul, A. Eryılmaz, On the Sequence Spaces of Interval Numbers, *Thai Journal of Mathematics*, **8**(3), (2012), 503-510.
20. I. J. Schoenberg, The Integrability of Certain Functions and Related Summability Methods, *The American Mathematical Monthly*, **66**(5), (1959), 361-375.
21. H. Steinhaus, Sur la Convergence Ordinaire Et La Convergence Asymptotique, *Colloquium Mathematicae*, **2**(1), (1951), 73-74.
22. B. C. Tripathy, R. Goswami, Vector Valued Multiple Sequence Spaces Defined by Orlicz Function, *Boletim Da Sociedade Paranaense De Matematica*, **33**(1), (2015), 67-79.
23. T. Yaying, B. Hazarika, S. A. Mohiuddine, On Difference Sequence Spaces of Fractional-Order Involving Padovan Numbers, *Asian-European Journal of Mathematics*, **14**(6), (2012), 215-225.
24. T. Yaying, B. Hazarika, S. A. Mohiuddine, M. Mursaleen, K. J. Ansari, Sequence Spaces Derived by the Triple Band Generalized Fibonacci Difference Operator, *Advances in Difference Equations*, **2020**(1), (2020), Art. ID 639.