Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 2 (2022), pp 97-108
DOI: 10.52547/ijmsi.17.2.97

A Note on Absolute Central Automorphisms of Finite p-Groups

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Abstract. Let G be a finite group. The automorphism σ of a group G is said to be an absolute central automorphism, if for all $x \in G$, $x^{-1}x^{\sigma} \in L(G)$, where $L(G)$ be the absolute centre of G. In this paper, we study some properties of absolute central automorphisms of a given finite p-group.

Keywords: Absolute centre, Absolute central automorphisms, Finite p-groups.

2000 Mathematics subject classification: 20D45, 20D25, 20D15.

1. Introduction

Let G be a finite group and N a characteristic subgroup of G. Suppose σ is an automorphism of G. If $(Ng)^{\sigma} = Ng$ for all g in G or equivalently σ induces the identity automorphism on G/N, we shall say σ centralizes G/N. We let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms of G centralizing G/N. Clearly $\sigma \in \mathrm{Aut}^N(G)$ if and only if $x^{-1}x^{\sigma} \in N$ for all $x \in G$. Now let M be a normal subgroup of G. Let us denote by $C_{\mathrm{Aut}^N(G)}(M)$ the group of all automorphisms of $\mathrm{Aut}^N(G)$ centralizing M. Various authors have studied the groups $\mathrm{Aut}^Z(G)$, the central automorphisms of G, where $Z = Z(G)$, $\mathrm{Aut}^{G'}(G)$, the IA-automorphisms of G, where G' stands for the commutator subgroup of G, and $\mathrm{Aut}^{\Phi}(G)$, where Φ denote the Frattini subgroup of G, the intersection of all maximal subgroups of G, see for example [14, 17, 19, 20]. For any

Received 30 September 2018; Accepted 15 August 2019

element $g \in G$ and $\sigma \in \mathrm{Aut}(G)$, the element $[g, \sigma] = g^{-1}g^{\sigma}$ is called the autocommutator of g and σ. Also inductively, for all $\sigma_1, \sigma_2, ..., \sigma_n \in \mathrm{Aut}(G)$, define $[g, \sigma_1, \sigma_2, ..., \sigma_n] = [[g, \sigma_1, \sigma_2, ..., \sigma_{n-1}], \sigma_n]$. Hegarty [7], generalized the concept of centre into absolute centre $L(G)$ of a group G as

$$L(G) = \{g \in G \mid [g, \sigma] = 1, \forall \sigma \in \mathrm{Aut}(G)\}.$$

One can easily check that the absolute centre is a characteristic subgroup contained in the centre of G. Also he introduced the concept of the absolute central automorphism. An automorphism σ of G is called an absolute central automorphism if σ centralizes $G/L(G)$. We denote the set of all absolute central automorphisms of G by $\mathrm{Aut}^{L}(G)$. Singh and Gumber [18], Kaboutari Farimani [9], also Shabani-Attar [17] have given some necessary and sufficient conditions for a finite non-abelian p-group such that all absolute central automorphisms are inner. In this paper, we will characterize the finite non-abelian p-groups G such that $\mathrm{Aut}^{L}(G) = \mathrm{Aut}^{G'}(G)$. Then, we determine the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\mathrm{Aut}^{L}(G) = \mathrm{Aut}^{\Phi}(G)$. Finally, we classify all finite p-groups G of order $p^n (3 \leq n \leq 5)$, such that $\mathrm{Aut}^{L}(G) = \mathrm{Inn}(G)$.

Throughout this paper all groups are assumed to be finite and p always denotes a prime number. Most of our notation is standard, and can be found in [5], for example. In particular, a p-group G is said to be extraspecial if $G' = Z(G) = \Phi(G)$ is of order p. Let $L_1(G) = L(G)$ and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \sigma_1, \sigma_2, ..., \sigma_n] = 1, \forall \sigma_1, \sigma_2, ..., \sigma_n \in \mathrm{Aut}(G)\}.$$

A group G is called autonilpotent of class at most n if $L_n(G) = G$, for some $n \in \mathbb{N}$. If σ is an automorphism of G and x is an element of G, we write x^{σ} for the image of x under σ and $o(x)$ for the order of x. For a finite group G, $\exp(G)$, $d(G)$ and $\mathrm{cl}(G)$, denote the exponent of G, minimal number of generators of G and the nilpotency class of G, respectively. Recall that a group G is called a central product of its subgroups $G_1, ..., G_n$ if $G = G_1 \cdots G_n$ and $[G_i, G_j] = 1$, for all $1 \leq i < j \leq n$. In this situation, we shall write $G = G_1 * \cdots * G_n$. For $s \geq 1$, we use the notation G^{*s} for the iterated central product defined by $G^{*s} = G * G^{*(s-1)}$ with $G^{*1} = G$, where G is a finite p-group. We also make the convention $G^{*0} = 1$. Finally, we use X^n for the direct product of n-copies of a group X, C_n for the cyclic group of order n where $n \geq 1$, as usual, D_8 for the dihedral group, Q_8 for the quaternion group, of order 8, respectively and $M_p(n, m)$ and $M_p(n, m, 1)$ for the minimal non-abelian p-groups of order p^{n+m} and p^{n+m+1} defined respectively by

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle,$$

where $n \geq 2$, $m \geq 1$ and

$$\langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle,$$

where $n \geq m \geq 1$ and if $p = 2$, then $m + n > 2$.

2. Preliminary results

In this section we give some results which will be used in the rest of the paper.

Let G and H be any two groups. We denote by $\mathrm{Hom}(G, H)$ the set of all homomorphisms from G into H. Clearly, if H is an abelian group, then $\mathrm{Hom}(G, H)$ forms an abelian group under the following operation $(fg)(x) = f(x)g(x)$, for all $f, g \in \mathrm{Hom}(G, H)$ and $x \in G$.

The following lemma is a well-known.

Lemma 2.1. *Let A, B and C be finite abelian groups. Then*

(i) $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$*;*
(ii) $\mathrm{Hom}(A, B \times C) \cong \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$*;*
(iii) $\mathrm{Hom}(C_m, C_n) \cong C_e$*, where e is the greatest common divisor of m and n.*

We have the following theorem due to Müller [14].

Theorem 2.2. [14, Theorem] *If G is a finite p-group which is neither elementary abelian nor extraspecial, then $Aut^{\Phi}(G)/\mathrm{Inn}(G)$ is a non-trivial normal p-subgroup of the group of outer automorphisms of G.*

The following preliminary lemma is well-known result [19, Lemma 2.2].

Lemma 2.3. *Let G be a group and M, N be normal subgroups of G with $N \leq M$ and $C_N(M) \leq Z(G)$. Then $C_{\mathrm{Aut}^N(G)}(M) \cong \mathrm{Hom}(G/M, C_N(M))$.*

Corollary 2.4. *If G is a finite group, then*

$$C_{\mathrm{Aut}^L(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), L(G)),$$

where $L = L(G)$.

Moghaddam and Safa [12], proved that for a finite group G,

$$\mathrm{Aut}^L(G) \cong \mathrm{Hom}(G/L(G), L(G)).$$

The following theorem states a useful result for finite p-groups.

Theorem 2.5. *Let G be a finite p-group different from C_2. Then* $\mathrm{Aut}^L(G) \cong \mathrm{Hom}(G, L(G))$.

Proof. Let $\theta \in \mathrm{Aut}^L(G)$. We define the map $f_\theta : G \to L(G)$ by $f_\theta(g) = g^{-1}g^\theta$. It is easy to see that f_θ is a homomorphism, and $\theta \mapsto f_\theta$ is an injective map from $\mathrm{Aut}^L(G)$ to $\mathrm{Hom}(G, L(G))$. Conversely, assume that $f \in \mathrm{Hom}(G, L(G))$. Then we define $\theta = \theta_f : G \to G$ by $g^\theta = gf(g)$. Since by [11, Corollary 3.7], $g^{-1}g^\theta \in L(G) \leq \Phi(G)$, for every element $g \in G$, we may write G as the product of the image of θ and the Frattini subgroup of G and so the image of θ must be G itself. Hence θ is an automorphism of G. Now $\theta = \theta_f \in \mathrm{Aut}^L(G)$ and $f_{\theta_f} = f$. Finally, suppose that $\alpha, \beta \in \mathrm{Aut}^L(G)$. Then for any $x \in G$,

$$f_{\alpha\beta}(x) = x^{-1}x^{\alpha\beta} = x^{-1}(xx^{-1}x^\alpha)^\beta = x^{-1}x^\beta x^{-1}x^\alpha = x^{-1}x^\alpha x^{-1}x^\beta,$$

since $x^{-1}x^\alpha \in L(G)$. Thus $f_{\alpha\beta}(x) = f_\alpha(x)f_\beta(x)$ and so $\theta \mapsto f_\theta$ is a homomorphism, which completes the proof. □

We next give a necessary and sufficient condition on a finite p-group G for the group $\mathrm{Aut}^L(G)$ to be elementary abelian.

Corollary 2.6. *Let G be a finite p-group. Then $\mathrm{Aut}^L(G)$ is elementary abelian if and only if $\exp(G/G') = p$ or $\exp(L(G)) = p$.*

Proof. It is straightforward by Lemma 2.1 and Theorem 2.5. □

3. Main results

For a finite abelian p-group G, $|L(G)| = 1, 2$ by [11, Lemma 4.4] and so $|\mathrm{Aut}^L(G)| = 1$ or $\mathrm{Aut}^L(G) \cong C_2^d$, with $d = d(G)$. Thus we may assume that G is a non-abelian p-group. In this section, first we characterize the finite non-abelian p-groups G such that $\mathrm{Aut}^L(G) = \mathrm{Aut}^{G'}(G)$. Then, we determine the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\mathrm{Aut}^L(G) = \mathrm{Aut}^\Phi(G)$.

In [9], Kaboutari Farimani proved the following two results giving some information of absolute central automorphisms of a finite p-group.

Lemma 3.1. *Let G be a finite non-abelian p-group. Then $C_{\mathrm{Aut}^L(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if $G/L(G)$ is abelian and $L(G)$ is cyclic.*

Theorem 3.2. *Let G be a finite non-abelian p-group. Then $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $G/L(G)$ is abelian, $L(G)$ is cyclic and $Z(G) = L(G)G^{p^n}$ where $\exp(L(G)) = p^n$.*

Note that the Theorem 3.2 yields the following corollary that is the Corollary 1 of Singh and Gumber [18].

Let G be a finite non-abelian p-group such that $G' \leq L(G)$. Let $G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 1$. Also let $G/L(G) = C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times \cdots \times C_{p^{\beta_s}}$, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_s \geq 1$ and $L(G) = C_{p^{\gamma_1}} \times$

$C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_t}}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_t \geq 1$. Since $G/Z(G)$ is a quotient group of $G/L(G)$ by [2, Section 25], $r \leq s$ and $\alpha_i \leq \beta_i$ for all $1 \leq i \leq r$.

By the above notation, we prove the following corollary:

Corollary 3.3. [18, Corollary 1] *Let G be a finite non-abelian p-group. Then $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $G' \leq L(G)$, $L(G)$ is cyclic and either $L(G) = Z(G)$ or $d(G/L(G)) = d(G/Z(G))$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest integer such that $\beta_k > \gamma_1$.*

Proof. First assume that $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. Hence by Theorem 3.2, $G' \leq L(G)$ and $L(G)$ is cyclic. If $\exp(G/L(G)) \leq \exp(L(G))$, then

$$G/Z(G) \cong \mathrm{Aut}^L(G) \cong \mathrm{Hom}(G/L(G), L(G)) \cong G/L(G),$$

because $L(G)$ is cyclic and by [12, Proposition 1]. Therefore $L(G) = Z(G)$. Next, let $\exp(G/L(G)) > \exp(L(G))$ and k is the largest integer such that $\beta_k > \gamma_1$. Since $L(G)$ and $G/L(G)$ are abelian,

$$d(G/Z(G)) = d(\mathrm{Hom}(G/L(G), L(G))) = d(G/L(G))d(L(G)) = d(G/L(G)).$$

Now we have $\mathrm{Hom}(G/L(G), L(G)) \cong C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times \cdots \times C_{p^{\gamma_1}} \times C_{p^{\beta_{k+1}}} \times \cdots \times C_{p^{\beta_s}}$ and $\mathrm{Hom}(G/L(G), L(G)) \cong G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$. Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \gamma_1$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, as required.

Conversely if $L(G) = Z(G)$, then $\exp(G/Z(G)) = \exp(G') | \exp(Z(G))$, since $G' \leq L(G)$ and by [13, Lemma 0.4]. Now

$$\mathrm{Hom}(G/L(G), L(G)) = \mathrm{Hom}(G/Z(G), Z(G)) \cong G/Z(G),$$

because $Z(G)$ is cyclic and so $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. Next assume that $L(G) < Z(G)$, $s = d(G/L(G)) = d(G/Z(G)) = r$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest integer such that $\beta_k > \gamma_1$. We claim that $Z(G) = L(G)G^{p^{\gamma_1}}$. Since $\exp(G/Z(G)) = \exp(L(G))$, we have $L(G) \leq L(G)G^{p^{\gamma_1}} \leq Z(G)$. It follows that $G/Z(G)$ is a quotient group of $G/L(G)G^{p^{\gamma_1}}$. Now let $G/L(G)G^{p^{\gamma_1}} = C_{p^{\gamma_1}} \times C_{p^{\delta_2}} \times \cdots \times C_{p^{\delta_r}}$, where $\delta_1 = \gamma_1 \geq \delta_2 \geq \cdots \geq \delta_r \geq 1$, since $d(G/L(G)) = d(G/L(G)G^{p^{\gamma_1}})$ and $\exp(G/L(G)G^{p^{\gamma_1}}) = p^{\gamma_1}$. Therefore $\gamma_1 = \alpha_i \leq \delta_i \leq \gamma_1$ for $1 \leq i \leq k$, whence we have $\delta_i = \gamma_1 = \alpha_i$ for $1 \leq i \leq k$. As $\beta_i = \alpha_i \leq \delta_i \leq \beta_i$ for $k+1 \leq i \leq r$, it follows that $\delta_i = \alpha_i = \beta_i$ for $k+1 \leq i \leq r$. Hence $G/Z(G) = G/L(G)G^{p^{\gamma_1}}$ and consequently $Z(G) = L(G)G^{p^{\gamma_1}}$. Therefore by Theorem 3.2, $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. This completes the proof. □

As an application of Theorem 3.2, we get another proof of the main result of [15].

Theorem 3.4. [15, Theorem 3.2] *Let G be a non-abelian autonilpotent finite p-group of class 2. Then $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $L(G) = Z(G)$ and $L(G)$ is cyclic.*

Proof. Suppose that $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. Hence $L(G)$ is cyclic and $Z(G) = L(G)G^{p^n}$, where $\exp(L(G)) = p^n$. Now by [15, Proposition 2.13], $\exp(G/L(G))$ divides $\exp(L(G))$ and so $Z(G) = L(G)G^{p^n} = L(G)$. Conversely, assume that $L(G) = Z(G)$ and $L(G)$ is cyclic. Since G be a non-abelian autonilpotent p-group of class 2, $\mathrm{Aut}^L(G) = \mathrm{Aut}(G)$, by [15, Lemma 2.11]. Therefore $\mathrm{Inn}(G) \leq \mathrm{Aut}^L(G)$, $G' \leq L(G)$ and $G/L(G)$ is abelian. Obviously, $Z(G) = L(G) = L(G)G^{p^n}$, where $\exp(L(G)) = p^n$, and so $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$, by Theorem 3.2, as required. □

Corollary 3.5. *Let G be an extraspecial p-group.*

(i) *If $p > 2$, then $L(G)$ and $\mathrm{Aut}^L(G)$ is trivial.*
(ii) *If $p = 2$, then $L(G) \cong C_2$ and $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$.*

Proof. Let G be an extraspecial p-group. First assume that $p > 2$. By [10, Theorem 3], $L(G)$ is trivial and so $\mathrm{Aut}^L(G) = 1$.
To prove (ii), since $|G'| = 2$, and G' is a characteristic subgroup of G, we have $G' \leq L(G) \leq Z(G)$. Thus $G' = L(G) = Z(G) = \Phi(G)$ is cyclic of order 2. Now by Theorem 3.2, $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. □

Let G be a finite non-abelian p-group such that $G/L(G)$ is abelian. Then G is of class 2 and $\mathrm{Aut}^{G'}(G) \leq \mathrm{Aut}^L(G)$. Let $G/G' = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$, where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Also let $L(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}}$, where $b_1 \geq b_2 \geq \cdots \geq b_l \geq 1$ and $G' = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_n}}$, where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$. Since $G' \leq L(G)$, by [2, Section 25] we have $n \leq l$ and $e_j \leq b_j$ for all $1 \leq j \leq n$. By the above notation, we prove the following theorem:

Theorem 3.6. *Let G be a finite non-abelian p-group. Then $\mathrm{Aut}^L(G) = \mathrm{Aut}^{G'}(G)$ if and only if $G' = L(G)$ or $G' < L(G)$, $d(G') = d(L(G))$ and $a_1 = e_t$, where t is the largest integer between 1 and n such that $b_t > e_t$.*

Proof. Suppose that $\mathrm{Aut}^L(G) = \mathrm{Aut}^{G'}(G)$ and $G' \neq L(G)$. By Theorem 2.5 and Lemma 2.3, we have $|\mathrm{Hom}(G/G', L(G))| = |\mathrm{Hom}(G/G', G')|$. First, we claim that $d(G') = d(L(G))$. Suppose, for a contradiction, that $d(G') = n < l = d(L(G))$. Since $b_j \geq e_j$ for all j such that $1 \leq j \leq n$, by Lemma 2.1,

$$
\begin{aligned}
|\mathrm{Aut}^{G'}(G)| &= |\mathrm{Hom}(G/G', G')| = |\mathrm{Hom}(G/G', C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_n}})| \\
&\leq |\mathrm{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_n}})| < |\mathrm{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_n}})| \\
&\times |\mathrm{Hom}(G/G', C_{p^{b_{n+1}}} \times \cdots \times C_{p^{b_l}})| = |\mathrm{Hom}(G/G', C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}})| \\
&= |\mathrm{Hom}(G/G', L(G))| = |\mathrm{Aut}^L(G)|,
\end{aligned}
$$

which is a contradiction. So $n = l$, as required. Next, since $|\mathrm{Aut}^L(G)| = |\mathrm{Aut}^{G'}(G)|$, we have

$$\prod_{1\leq i\leq k, 1\leq j\leq l} p^{\min\{a_i,b_j\}} = \prod_{1\leq i\leq k, 1\leq j\leq l} p^{\min\{a_i,e_j\}}.$$

Since $b_j \geq e_j$ for all j such that $1 \leq j \leq l$, we have $\min\{a_i, b_j\} \geq \min\{a_i, e_j\}$, where $1 \leq i \leq k, 1 \leq j \leq l$. Thus $\min\{a_i, b_j\} = \min\{a_i, e_j\}$, for all $1 \leq i \leq k, 1 \leq j \leq l$. Next, since $G' < L(G)$, there exists some $1 \leq j \leq l$ such that $e_j < b_j$. Let t be the largest integer between 1 and n such that $e_t < b_t$. We show that $a_1 \leq e_t$. Suppose, on the contrary, that $a_1 > e_t$. Then by the above equality, we must have $\min\{a_1, b_t\} = \min\{a_1, e_t\} = e_t$, which is impossible. Hence $a_1 \leq e_t$. Let $\exp(G/Z(G)) = p^f$, where $f \in \mathbb{N}$. Since $\mathrm{cl}(G) = 2$, by [13, Lemma 0.4], $f = e_1$. But $a_1 \leq e_t \leq e_{t-1} \leq \cdots \leq e_1 = f \leq a_1$. Whence $a_1 = e_t$.

Conversely, if $G' = L(G)$, then $\mathrm{Aut}^{G'}(G) = \mathrm{Aut}^L(G)$. Assume that $G' < L(G)$, $d(G') = n = d(L(G)) = l$ and $a_1 = e_t$, where t is the largest integer between 1 and n such that $b_t > e_t$. Now by Lemma 2.3,

$$|\mathrm{Aut}^{G'}(G)| = |\mathrm{Hom}(G/G', G')| = \prod_{1\leq i\leq k, 1\leq j\leq l} p^{\min\{a_i,e_j\}},$$

and by Theorem 2.5,

$$|\mathrm{Aut}^L(G)| = |\mathrm{Hom}(G/G', L(G))| = \prod_{1\leq i\leq k, 1\leq j\leq l} p^{\min\{a_i,b_j\}}.$$

Since $a_1 = e_t$, we have $1 \leq a_k \leq \cdots \leq a_2 \leq a_1 = e_t \leq e_{t-1} \leq \cdots \leq e_2 \leq e_1$. Thus $b_j \geq e_j \geq a_i$ for all $1 \leq i \leq k$ and $1 \leq j \leq t$, which shows that $\min\{a_i, e_j\} = a_i = \min\{a_i, b_j\}$ for $1 \leq i \leq k$ and $1 \leq j \leq t$. Since $e_j = b_j$ for all $j \geq t+1$, we have $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \leq i \leq k$ and $t+1 \leq j \leq l$. Thus $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$. Therefore $|\mathrm{Aut}^{G'}(G)| = |\mathrm{Aut}^L(G)|$. Since $G' < L(G)$ we have $\mathrm{Aut}^{G'}(G) = \mathrm{Aut}^L(G)$, which completes the proof. $\square$

In [11], Meng and Guo proved that for a finite group G, if C_2 is not a direct factor of G, then $L(G) \leq \Phi(G)$. We end this section by characterizing the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\mathrm{Aut}^L(G) = \mathrm{Aut}^\Phi(G)$.

First, we give some basic results about the finite non-abelian p-groups G with cyclic Frattini subgroup.

Let $n > 1$. Following [1], we denote by $D^+_{2^{n+3}}$ and $Q^+_{2^{n+3}}$ the 2-groups of order 2^{n+3} defined by the following presentations.

$$D^+_{2^{n+3}} = \langle a, b, c \mid a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b, c] = 1\rangle,$$

$Q^+_{2^{n+3}} = \langle a, b, c \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b, c] = 1\rangle$. Note that if G is either $D^+_{2^{n+3}}$ or $Q^+_{2^{n+3}}$, then $\mathrm{cl}(G) = n + 1$.

In [1], Berger, Kovács and Newman proved the following result.

Theorem 3.7. [1, Theorem 2] *If G is a finite p-group with $Z(\Phi(G))$ cyclic, then*

$$G = E \times (G_0 * G_1 * \cdots * G_s),$$

where E is an elementary abelian, $G_1, ..., G_s$ are non-abelian of order p^3, of exponent p for p odd and dihedral for $p = 2$, while $G_0 > 1$ if $E > 1$, $|G_0| > 2$ if $s > 0$, and G_0 is one of the following types: cyclic, non-abelian with a cyclic maximal subgroup, $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D^+_{2^{n+3}}, Q^+_{2^{n+3}}, D^+_{2^{n+3}} * \mathbb{Z}_4$, all with $n > 1$. Conversely, every such group has cyclic Frattini subgroup.

Theorem 3.8. [20, Theorem 2.3] *Let G be a finite non-abelian p-group with cyclic Frattini subgroup $\Phi(G)$.*

(i) *If $p > 2$, or $p = 2$ and $cl(G) = 2$, then $\Phi(G) \leq Z(G)$.*
(ii) *If $cl(G) > 2$, then $G' = \Phi(G)$.*

Lemma 3.9. [20, Lemma 2.4] *Let G be a finite group with $\Phi(G) \leq Z(G)$. Then there is a bijection from $\mathrm{Hom}(G/G', \Phi(G))$ onto $\mathrm{Aut}^{\Phi}(G)$ associating to every homomorphism $f : G \to \Phi(G)$ the automorphism $x \mapsto xf(x)$ of G. In particular, if G is a p-group and $\exp(\Phi(G)) = p$, then $\mathrm{Aut}^{\Phi}(G) \cong \mathrm{Hom}(G/G', \Phi(G))$.*

In the following theorem, we will make use Theorem 3.7, which is the structural theorem for p-groups with cyclic Frattini subgroup.

Theorem 3.10. *Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Then $\mathrm{Aut}^L(G) = \mathrm{Aut}^{\Phi}(G)$ if and only if G is one of the following types: $C_2^m \times D_8^{*(s+1)}$ or $C_2^m \times (D_8^{*s} * Q_8)$, where $s, m \geq 0$.*

Proof. Let $\mathrm{Aut}^L(G) = \mathrm{Aut}^{\Phi}(G)$. Hence $\mathrm{Aut}^{\Phi}(G)$ is abelian, G is of class 2 and by Theorem 3.8, $\Phi(G) \leq Z(G)$. It follows that $\exp(G') = \exp(G/Z(G)) = p$ and so $|G'| = p$. Assume that $|\Phi(G) : G'| = p^a$. Then $\Phi(G) \cong C_{p^{a+1}}$ and we observe that $\exp(G/G') \leq p^{a+1} = |\Phi(G)|$. Together with Lemma 3.9, we have $|\mathrm{Aut}^{\Phi}(G)| = |\mathrm{Hom}(G, \Phi(G))| = |G|/p$. Next, we note that $G' \cap L(G) \neq 1$; otherwise, $G' \cap L(G) = 1$ and $G' \times L(G)$ would be a subgroup of $\Phi(G)$. Hence either $G' = 1$ or $L(G) = 1$, a contradiction. Whence $G' \leq L(G)$. Now we are able to show that $G' = L(G) \cong C_p$. To do this, first assume that $L(G) \neq \Phi(G)$. By similar argument that was applied for Theorem 3.6, we have $\exp(G/G') \leq \exp(L(G))$, which implies that $\exp(G/L(G)) \leq \exp(G/G') \leq \exp(L(G)) = |L(G)|$. If $L(G) = \Phi(G)$, then $\exp(G/L(G)) = \exp(G/\Phi(G)) \leq \exp(L(G)) = |L(G)|$. Thus $|\mathrm{Aut}^L(G)| = |G/L(G)| = |\mathrm{Aut}^{\Phi}(G)| = |G/G'|$, by [12, Proposition 1] and so $G' = L(G) \cong C_p$. Now, we will make use of the notation of Theorem 3.7.

Since $\mathrm{cl}(G) = 2$, by Theorem 3.7 and [5, Theorems 5.4.3 and 5.4.4], G_0 is one of the groups $M_p(n, 1)$, where $n \geq 3$, if $p = 2$; D_8 or Q_8.
We claim that $G' = G'_0$ and $\Phi(G) = \Phi(G_0)$. To see this, since $G'_0 \bigcap G'_i \neq 1$ for $1 \leq i \leq s$ and $|G'_i| = p$, we have $G'_i \leq G'_0$ and so $G' = G'_0$. Also $\Phi(G) = G'G^p = G'_0 E^p G_0^p G_1^p \cdots G_s^p = G'_0 G_0^p = \Phi(G_0)$. To continue the proof, we may consider two cases:

Case I. $E = 1$.
Let $G = G_0 * T$, where T be one of the groups $M_p(1,1,1)^{*s}$, while $p > 2$ or D_8^{*s}, where all $s \geq 0$. Note that if $s = 0$, then $G = G_0$ and $Z(G) = Z(G_0) = \Phi(G_0) = \Phi(G)$; otherwise, since $1 \neq G_0 \bigcap T = Z(T) \leq Z(G_0)$, then $Z(G) = Z(G_0)$, because $|Z(T)| = p$, which implies that $\Phi(G) = \Phi(G_0) = Z(G_0) = Z(G)$. We claim that G is an extraspecial p-group. To see this, since $G' = L(G) \cong C_p$, by Theorem 3.2, $\mathrm{Aut}^{\Phi}(G) = \mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. This shows that G is an extraspecial p-group, by Theorem 2.2. If $p > 2$, then by Corollary 3.5, $L(G) = 1$, which is impossible. Whence $p = 2$. If $G_0 \cong M_2(n, 1), n \geq 3$, then by [5, Theorem 5.4.3], $Z(G) = \Phi(G)$ is of order 2^{n-1}. This yields that $n = 2$, since $|Z(G)| = 2$, a contradiction. Therefore G_0 is isomorphic either to D_8 or Q_8, and G be one of the groups: $D_8^{*(s+1)}$ or $Q_8 * D_8^{*s}$, for some $s \geq 0$.

Case II. $E \neq 1$.
In this case $G_0 > 1$ and $G = E \times (G_0 * T)$, where T be one of the groups lying in Case I.

We claim that $\mathrm{Aut}^{\Phi(G_0*T)}(G_0 * T) = \mathrm{Aut}^{L(G_0*T)}(G_0 * T)$. Choose a non-trivial element σ of $\mathrm{Aut}^{\Phi(G_0*T)}(G_0 * T)$. Then the map $\overline{\sigma}$ defined by $(ef)^{\overline{\sigma}} = ef^{\sigma}$, for all $e \in E, f \in G_0*T$ denotes an automorphism of $\mathrm{Aut}^{\Phi}(G) = \mathrm{Aut}^L(G)$. Since $G' \cap L(G_0*T) \neq 1$, then $L(G) \leq L(G_0*T)$ and so σ is in $\mathrm{Aut}^{L(G_0*T)}(G_0* T)$. This shows that $\mathrm{Aut}^{\Phi(G_0*T)}(G_0 * T) = \mathrm{Aut}^{L(G_0*T)}(G_0 * T)$, as required. Next, by a similar argument as mentioned for the previous case, G_0 be one of the groups: D_8 or Q_8. Therefore G has one of the following types: $C_2^m \times D_8^{*(s+1)}$ or $C_2^m \times (D_8^{*s} * Q_8)$, where $s \geq 0, m > 0$.

Conversely, assume that G be of the groups in Theorem 3.10. Hence $G' = L(G) \cong C_2$. Now the proof is complete, since $|\mathrm{Aut}^L(G)| = |\mathrm{Aut}^{\Phi}(G)| = |G|/2$. $\square$

4. Classify all finite p-groups G of order $p^n (3 \leq n \leq 5)$, such that $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$

Let G be a non-abelian group of order p^3. Then by Corollary 3.5, $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $p = 2$. In the following corollaries, we use Theorems 4.7 and 5.1 of [11] and classify all finite p-groups G of order $p^n (4 \leq n \leq 5)$, such that $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. First we recall the following concept, which was introduced by Hall in [6].

Definition 4.1. Two finite groups G and H are said to be isoclinic if there exist isomorphisms $\phi : G/Z(G) \to H/Z(H)$ and $\theta : G' \to H'$ such that, if $(x_1Z(G))^\phi = y_1Z(H)$ and $(x_2Z(G))^\phi = y_2Z(H)$, then $[x_1, x_2]^\theta = [y_1, y_2]$. Notice that isoclinism is an equivalence relation among finite groups and the equivalence classes are called isoclinism families.

Corollary 4.2. *Let G be a non-abelian group of order p^4. Then $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $p = 2$ and G is one of the following types: $M_2(3, 1)$ or $M_2(2, 1, 1)$.*

Proof. Assume that $|G| = p^4$ and $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. We claim that $|Z(G)| = p^2$. Suppose for a contradiction, that $|Z(G)| = p$. We observe that $G' \leq Z(G) \cong C_p$, by Theorem 3.2 and so G is an extraspecial p-group, a contradiction since the order of G is not of the form p^{2n+1}, for some natural number n. Therefore $G/Z(G) \cong C_p^2$, and hence $|G'| = p$. We consider two cases:

Case I. p an odd prime. It is straightforward to see that the map $\sigma : G \to G$ by $x^\sigma = x^{1+p}$, is an automorphism of G. Hence for any element x of $L(G)$, $x = x^\sigma = x^{1+p}$, and so $x^p = 1$. Thus $\exp(L(G)) = p$ and so $G' = L(G) \cong C_p$, by Theorem 3.2. If $G/L(G) \cong C_{p^3}$, then by [3, Theorem 2.2], G is cyclic, a contradiction. Next, we assume that $G/L(G) \cong C_{p^2} \times C_p$. Then G is an abelian group by [11, Theorem 5.1], which is impossible. Finally, if $G/L(G) \cong C_p^3$, then $L(G) = \Phi(G)$ and so $\mathrm{Aut}^\Phi(G) = \mathrm{Inn}(G)$. Therefore by Theorem 2.2, G is an extraspecial p-group, a contradiction.

Case II. $p = 2$. Since $|G'| = 2$, and G' be a characteristic subgroup of G, we have $G' \leq L(G) \leq Z(G)$. Thus $|L(G)| = 2$ or 4. If $|L(G)| = 4$, then $L(G) = Z(G)$ and $G/L(G) \cong C_2^2$. Hence by [11, Theorems 5.1 and 4.7], $G \cong M_2(2, 2)$, and $L(G) \cong C_2^2$, which is a contradiction by Theorem 3.2. Next we assume that $|L(G)| = 2$. So $G' = L(G)$ and $|G/L(G)| = 8$. By a similar argument, G is isomorphic to one of the following groups: $M_2(3, 1)$ or $M_2(2, 1, 1)$. The converse follows at once from Theorem 3.2. $\square$

Corollary 4.3. *Let G be a non-abelian group of order p^5. Then $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ if and only if $p = 2$ and G is one of the following types: $M_2(3, 2)$, $M_2(4, 1)$, $M_2(2, 2, 1)$, D_8^{*2} or $D_8 * Q_8$.*

Proof. Let G be a finite group such that $|G| = p^5$ and $\mathrm{Aut}^L(G) = \mathrm{Inn}(G)$. We consider two cases:

Case I. $p > 2$. These groups lying in the isoclinism families (5), (4) or (2) of [8, 4.5] and we show that $\mathrm{Aut}^L(G) \neq \mathrm{Inn}(G)$.
First, let G denote one of the groups in the isoclinism family (5). Hence $|Z(G)| = p$ and $G' = Z(G) = \Phi(G) \cong C_p$, by Theorem 3.2. So G is an extraspecial p-group and by Corollary 3.5, $|L(G)| = 1$, a contradiction.
Next, let G be one of the groups in the isoclinism family (4). Then $G' \cong C_p^2$, which is a contradiction, since G' is cyclic.

Finally, let G denote one of the groups in the isoclinism family (2). Then $G/Z(G) \cong C_p^2$ and so $d(G/L(G)) > 1$. We observe that $G' = L(G) \cong C_p$ and $Z(G) = \Phi(G)$, by using Theorems 2.2, 3.2, [3, Theorem 2.2] and [11, Theorem 5.1]. So $d(G) = 2$ and by [16], G is a minimal non-abelian p-group. If $G/L(G) \cong C_{p^3} \times C_p$, then G is an abelian group, by [11, Theorem 5.1], a contradiction. If $G/L(G) \cong C_{p^2}^2$, then by [16], $G \cong M_p(3,2)$ or $G \cong M_p(2,2,1)$. Thus $L(G) = 1$, by [11, Theorem 4.7], a contradiction. Finally, assume that $G/L(G) \cong C_{p^2} \times C_p^2$ or $G/L(G) \cong C_p^4$. In this cases, $\mathrm{Aut}^L(G) \neq \mathrm{Inn}(G)$, by Theorem 2.5.

Case II. $p = 2$. We can see that $|L(G)| = 2, 4$, by [3, Theorem 2.2] and [11, Theorem 5.1]. First, we assume that $|L(G)| = 4$. Since G is a non-cyclic group, by [3, Theorem 2.2], $d(G/L(G)) > 1$. It follows that $G/L(G)$ is one of the groups C_2^3 or $C_4 \times C_2$. Now in the first case, $L(G) = \Phi(G)$ and so G is an extraspecial 2-group by Theorem 2.2. Hence $G' = L(G) \cong C_2$, a contradiction. Therefore $G/L(G) \cong C_4 \times C_2$ and by [11, Theorems 5.1 and 4.7], G is one of the groups: $M_2(2,3)$ or $M_2(3,1,1)$, and $L(G) \cong C_2^2$, a contradiction by Theorem 3.2. Now we may suppose that $|L(G)| = 2$. So $G' = L(G) \cong C_2$. We discuss the following cases.

If $G/L(G) \cong C_2^4$, then $L(G) = \Phi(G)$ and so $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$. Therefore by Theorem 2.2, G is an extraspecial 2-group. Thus G is one of the groups D_8^{*2} or $D_8 * Q_8$, by [21]. Next, suppose that $G/L(G) \cong C_4 \times C_2^2$. Hence $G/L(G) = \langle \bar{a}, \bar{b}, \bar{c} \rangle$, where $\bar{a} = aL(G), \bar{b} = bL(G)$, $\bar{c} = cL(G)$ and $o(\bar{a}) = 4$, $o(\bar{b}) = o(\bar{c}) = 2$. Therefore $G = \langle a, b, c, L(G) \rangle = \langle a, b, c \rangle$, by [11, Corollary 3.7]. Since $\langle a^2 \rangle \times G' \leq Z(G)$, we have either $Z(G) \cong C_4 \times C_2$ or C_2^2. If $Z(G) \cong C_4 \times C_2$, then $\mathrm{Aut}^L(G) \neq \mathrm{Inn}(G)$, by Theorem 2.5. Therefore $Z(G) \cong C_2^2$. Now by using GAP [4], we find that there are no such groups. Next, if $G/L(G) \cong C_8 \times C_2$, then $G \cong M_2(4,1)$, by [11, Theorem 5.1]. Finally, suppose that $G/L(G) \cong C_4^2$. Then $d(G) = 2$, by [11, Corollary 3.7] and $G' = L(G) \cong C_2$. Hence by [16], G is a minimal non-abelian 2-group. Thus G is isomorphic to the group $M_2(3,2)$ or $M_2(2,2,1)$. The converse follows at once from Theorem 3.2. $\square$

Acknowledgments

The author is grateful to the referees for their valuable suggestions. The paper was revised according to these comments. This research was in part supported by a grant from Payame Noor University.

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