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Additive Maps Preserving Idempotency of Products or Jordan Products of Operators

Ali Taghavi, Roja Hosseinzadeh*, Hamid Rohi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, P. O. Box 47416-1468, Babolsar, Mazandaran, Iran.

E-mail: taghavi@umz.ac.ir
ro.hosseinzadeh@umz.ac.ir
h.rohi@umz.ac.ir

Abstract. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional Hilbert spaces, while $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ denote the algebras of all linear bounded operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. We characterize the forms of additive mappings from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ that preserve the nonzero idempotency of either Jordan products of operators or usual products of operators in both directions.

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1. Introduction

The study of maps on operator algebras preserving certain properties or subsets is a topic which attracts much attention of many authors. See the references.

Some problems are concerned with preserving a certain property of usual product or other products of operators. For example see $[4, 6-10, 13, 15, 16]$.

Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings and $\phi : \mathcal{R} \to \mathcal{R}'$ be a map. Denote by $P_{\mathcal{R}}$ and $P_{\mathcal{R}'}$ the set of all idempotent elements of $\mathcal{R}$ and $\mathcal{R}'$, respectively. The triple Jordan product and the Jordan product of two elements A and B are defined as

*Corresponding Author

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ABA and $\frac{1}{2}(AB+BA)$, respectively. We say that ϕ preserves the idempotency of product of two elements, the idempotency of triple Jordan product of two elements and the idempotency of Jordan product of two elements, whenever we have

$$AB \in P_{\mathcal{R}} \Rightarrow \phi(A)\phi(B) \in P_{\mathcal{R}'},$$
$$ABA \in P_{\mathcal{R}} \Rightarrow \phi(A)\phi(B)\phi(A) \in P_{\mathcal{R}'}$$

and

$$\frac{1}{2}(AB+BA) \in P_{\mathcal{R}} \Rightarrow \frac{1}{2}(\phi(A)\phi(B)+\phi(B)\phi(A)) \in P_{\mathcal{R}'},$$

respectively. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional Hilbert spaces, while $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ denote the algebras of all linear bounded operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. In [8], authors characterized some forms of unital surjective maps on $B(X)$ preserving the nonzero idempotency of product of operators in both directions. Also in [15], authors characterized some forms of linear surjective maps on $B(X)$ preserving the nonzero idempotency of either products of operators or triple Jordan products of operators.

In this paper, we determine a form of additive mapping $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ such that the range of ϕ contains all minimal idempotents and I and also ϕ preserves the nonzero idempotency of Jordan products of operators in both directions. Moreover, we determine a form of surjective additive mapping $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ that preserves the nonzero idempotency of usual products of operators in both directions. Our main result are as follows.

Theorem 1.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional real or complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive map such that the range of ϕ contains all minimal idempotents and I. If ϕ preserves the nonzero idempotency of Jordan products of operators in both directions, then ϕ either annihilates minimal idempotents or there exists a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = \xi ATA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = \xi AT^tA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$, where $\xi = \pm 1$ (in the case that $\mathcal{H}$ and $\mathcal{K}$ are real, A is linear).*

Theorem 1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a surjective additive map. If ϕ preserves the nonzero idempotency of products of operators in both directions, then there exists a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = \xi ATA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = \xi AT^tA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$, where $\xi = \pm 1$.*

2. Proofs

In this section we prove our results. First we recall some notations. Let X and Y be Banach spaces. Recall that a standard operator algebra on X is a norm closed subalgebra of $B(X)$ which contains the identity and all finite rank operators. Denote the set of all idempotent operators of $\mathcal{B}(\mathcal{H})$ by $\mathcal{I}(\mathcal{H})$ and the Jordan product of A, B by $A \circ B = \frac{1}{2}(AB + BA)$. Also denote the dual space X by X^*.

For every nonzero $x \in X$ and $f \in X^*$, the symbol $x \otimes f$ stands for the rank one linear operator on X defined by

$$(x \otimes f)y = f(y)x. \quad (y \in X)$$

If $x, y \in \mathcal{H}$, then $x \otimes y$ stands for the rank one linear operator on $\mathcal{H}$ defined by

$$(x \otimes y)z = < z, y > x \quad (z \in \mathcal{H})$$

where $< z, y >$ denotes the inner product of z and y. We need some lemmas to prove our main result. Throughout this paper, $\mathcal{A} \subseteq B(X)$ and $\mathcal{B} \subseteq B(Y)$ are standard operator algebras.

The proof of the following lemma is similar to that of Lemma 2.2 in [15].

Lemma 2.1. [15] *Let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map such that preserves the nonzero idempotency of Jordan products of operators. If $N \in \mathcal{A}$ is a finite rank operator such that $N^2 = 0$, then $\phi(N)^4 = 0$.*

Lemma 2.2. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be an additive map. Then the following statements hold.*

(i) If ϕ preserves the nonzero idempotency of Jordan products of operators, then ϕ is injective.

(ii) If $I \in \mathrm{rng}\phi$ and ϕ preserves the nonzero idempotency of Jordan products of operators in both directions, then $\phi(I) = I$ or $\phi(I) = -I$.

Proof. (i) Assume $\phi(A) = 0$. We assert that A satisfies a quadratic polynomial equation. Otherwise, by the discussion in [11], there exists an $x \in X$ such that x, Ax and A^2x are linear independent. Then there is a linear functional f such that $f(x) = f(A^2x) = 0$ and $f(Ax) = 2$, because $\dim X \geq 3$. Setting $B = x \otimes f$, we have $A \circ B \in P_{\mathcal{A}} \setminus \{0\}$, implying that

$$\phi(A) \circ \phi(B) \in P_{\mathcal{B}} \setminus \{0\}.$$

This is a contradiction, because $\phi(A) \circ \phi(B) = 0$. So by the discussion in [11], A satisfies a quadratic polynomial equation.

Assume on the contrary that A is a nonzero operator. For any $B \in \mathcal{A}$ we have

$$\phi(A) \circ \phi(B) = 0.$$

However, there exists $B = x \otimes f$ such that $A \circ B = \frac{1}{2}Ax \otimes f + \frac{1}{2}x \otimes fA$ is a nonzero idempotent, a contradiction. We construct such B.

By the above assertion, A satisfies a quadratic polynomial equation. The spectrum of such A consists only of eigenvalues. If $A^2 \neq 0$, then A has a nonzero eigenvalue λ, because in this case there exist $r, s \in \mathbb{C}$ such that $rs \neq 0$ and λ satisfies a quadratic polynomial equation $\alpha^2 = r\alpha + s$. Since $rs \neq 0$,

$\alpha^2 = r\alpha + s$ has a nonzero root. Let x be its eigenvector. Choose a bounded functional f with $f(x) = \frac{1}{\lambda}$ to form $B = x \otimes f$ with the desired properties.

The remaining case is $A^2 = 0$. Since A is nonzero, we can find a vector x so that $Ax \neq 0$ and a functional f with $f(x) = 0$ and $f(Ax) = 1$ to form $B = x \otimes f$ with the desired properties. The proof is complete.

(ii) Since $I \in \text{rng}\phi$, there exists a nonzero operator $U \in \mathcal{A}$ such that $\phi(U) = I$. We show that $U = I$ or $-I$. We have $\phi(U) \circ \phi(U) \in \mathcal{P_B} \setminus \{0\}$. Hence we obtain

$$U^2 = U \circ U \in \mathcal{P_A} \setminus \{0\}. \tag{1}$$

This implies that for any $x \in X$, x, Ux and U^2x are linear dependent. Thus U satisfies a quadratic polynomial equation, by the discussion in [10]. This together with (1) yields that there exist $a, b \in \mathbb{C}$ such that we have

$$U^2 = aU + bI. \tag{2}$$

From (1) and (2), we obtain the answers $(0, 1)$, $(1, 0)$ and $(-1, 0)$ for (a, b) which imply that $U^2 = I$, $U^2 = U$and $U^2 = -U$.

Let $U^2 = U$. We assert that $U = I$. Assume on the contrary that $U \neq I$. From $U \circ I \in \mathcal{P_A} \setminus \{0\}$, we obtain

$$\phi(I) = I \circ \phi(I) \in \mathcal{P_B} \setminus \{0\}. \tag{3}$$

On the other hand, there exists an idempotent operator T such that $U - T$ is not idempotent. In fact, $U + S - I$ isn't idempotent, where $S = I - T$. Thus

$$(U + S - I) \circ I = U + S - I \notin \mathcal{P_A} \setminus \{0\}$$

which implies that

$$\phi(U + S - I) \circ \phi(I) \notin \mathcal{P_B} \setminus \{0\}$$

This together with (3) yields that

$$\phi(I) \circ \phi(S) \notin \mathcal{P_B} \setminus \{0\}$$

which implies that

$$S = I \circ S \notin \mathcal{P_A} \setminus \{0\}.$$

This is a contradiction, because S is idempotent. So the proof of assertion is completed.

With a similar proof, the assumption $U^2 = -U$ yields that $U = -I$.

Now let $U^2 = I$. We assert that U is a multiple of I. Assume on the contrary that U is a non-scalar operator. Since I and U are linear independent, there is a nonzero vector $x \in X$ such that x and Ux are linear independent. Hence there exists $f \in X^*$ such that $f(x) = 0$ and $f(Ux) = 2$. Setting $B = x \otimes f$, we obtain

$$U \circ B \in \mathcal{P_A} \setminus \{0\}$$

which implies that

$$\phi(B) = \phi(U) \circ \phi(B) \in \mathcal{P}_{\mathcal{B}} \setminus \{0\}.$$

This is a contradiction, because B is a nilpotent such that $B^2 = 0$ and so by Lemma 2.1, $\phi(B)$ is a nilpotent operator. So the proof of assertion is completed. By the proved assertion, there exists a nonzero complex number λ such that $U = \lambda I$. Since $U^2 = I$, we obtain $\lambda^2 = 1$ and this completes the proof. □

Theorem 2.3. [5] *Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional real or complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive map preserving idempotents. Suppose that the range of ϕ contains all minimal idempotents. Then ϕ either annihilates minimal idempotents or there exists a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = AT^tA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ (in the case that $\mathcal{H}$ and $\mathcal{K}$ are real, A is linear).*

Proof of Theorem 1.1. Since by Lemma 2.2, $\phi(I) = I$ or $\phi(I) = -I$, from $P = I \circ P \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ we obtain that $\phi(P)$ or $-\phi(P)$ belongs to $\mathcal{I}(\mathcal{H}) \setminus \{0\}$. This together with $\phi(0) = 0$ implies that ϕ or $-\phi$ preserves the idempotent operators in both directions. Hence the forms of ϕ follows from Theorem 2.3.

Proposition 2.4. *Let $\dim \mathcal{H} \geq 3$. Let A be an arbitrary operator of $\mathcal{B}(\mathcal{H})$ and P be a rank one idempotent operator. Then $A \in \mathbb{C}^*P$ if and only if for every $T \in \mathcal{B}(\mathcal{H})$ such that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ we have $AT \notin \mathcal{I}(\mathcal{H}) \setminus \{0\}$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0, 1\}$.*

Proof. If $A \in \mathbb{C}^*P$, then there exists a $\lambda \in \mathbb{C}^*$ such that $A = \lambda P$. hence it is trivial that for every T such that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ then $\lambda PT \notin \mathcal{I}(\mathcal{H}) \setminus \{0\}$.

Conversely, Let $A \notin \mathbb{C}^*P$. Since P is rank one, by [2], there exists either an $x \in \mathcal{H}$ such that Px and Ax are linear independent or an $x \in \mathcal{H}$ and linear independent vectors $z_1, z_2 \in \mathcal{H}$ such that $P = x \otimes z_1$ and $A = x \otimes z_2$.

If Px and Ax are linear independent, then there exists $y \in \mathcal{H}$ such that $< Px, y >=< Ax, y >= 1$, because $\dim \mathcal{H} \geq 3$. Setting $T = x \otimes y$ follows that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ and also $AT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$. This is a contradiction.

If $P = x \otimes z_1$ and $A = x \otimes z_2$, then there exist $y, z_3 \in \mathcal{H}$ such that $< x, z_3 >=< y, z_1 >=< y, z_2 >= 1$. Setting $T = y \otimes z_3$ follows that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ and also $AT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$. This is a contradiction.

These contradictions yields that $A \in \mathbb{C}^*P$ and this completes the proof. □

Lemma 2.5. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective additive map such that*

$$AB \in \mathcal{P}_{\mathcal{A}} \setminus \{0\} \Leftrightarrow \phi(A)\phi(B) \in \mathcal{P}_{\mathcal{B}} \setminus \{0\}$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then the following statements hold.

(i) $\phi(I) = I$ or $\phi(I) = -I$.

(ii) If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} \geq 3$ and $\mathcal{B} = \mathcal{B}(\mathcal{K})$, then $\phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P)$, for every rank one idempotent P.

Proof. (i) It is proved by using Lemma 2.1 and similar to the proof of Lemma 2.2 in [13].

(ii) By (i), $\phi(I) = I$ or $\phi(I) = -I$. Since $\phi(0) = 0$, we can conclude from (i) that ϕ or $-\phi$ preserves the idempotent operators in both directions. By Lemma 2.6 in [14], ϕ or $-\phi$ preserves the rank one idempotent operators in both directions. If $A \in \mathbb{C}^*P$, then by Proposition 2.4, for every $T \in \mathcal{B}(\mathcal{H})$ such that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ we have $AT \notin \mathcal{I}(\mathcal{H}) \setminus \{0\}$ which by surjectivity of ϕ imply that for every $T' \in \mathcal{B}(\mathcal{H})$ such that $\phi(P)T' \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ we have $\phi(A)T' \notin \mathcal{I}(\mathcal{H}) \setminus \{0\}$. Since $\phi(P)$ is a rank one idempotent, by Proposition 2.4 we can conclude that $\phi(A) \in \mathbb{C}^*\phi(P)$. This together with (i) and $\phi(0) = 0$ follows that $\phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P)$. This completes the proof. □

Proposition 2.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional real or complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive map. If ϕ preserves the idempotent operators, then ϕ preserves the square zero operators.*

Proof. Let $N \in \mathcal{B}(\mathcal{H})$ be a square zero operator. Then we have $\mathcal{H} = \ker N \oplus M$ for some closed subspace M of $\mathcal{H}$. Thus by this decomposition N has the following operator matrix

$$N = \begin{pmatrix} 0 & N_1 \\ 0 & 0 \end{pmatrix}.$$

If

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

then $A+nN \in \mathcal{I}(\mathcal{H})$ for every natural number n. It implies that $\phi(A)+n\phi(N) \in \mathcal{I}(\mathcal{K})$ for every natural number n. That is,

$$\phi(A) + n\phi(N) = \phi(A)^2 + n(\phi(A)\phi(N) + \phi(N)\phi(A)) + n^2\phi(N)^2$$

for all n. Setting $n = 1$ and $n = 2$ yield

$$\phi(N) = \phi(A)\phi(N) + \phi(N)\phi(A) + \phi(N)^2,$$

$$2\phi(N) = 2(\phi(A)\phi(N) + \phi(N)\phi(A)) + 4\phi(N)^2$$

which imply that $\phi(N)^2 = 0$ and this completes the proof. □

Theorem 2.7. [1] *Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a surjective additive map such that $\phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P)$ holds for every rank one operator P. Then ϕ preserves square zero in both directions if and only if there exists a nonzero scalar c and a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = cATA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = cAT^tA^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$.*

Proof of Theorem 1.2. By a similar proof to that of Lemma 2.3 in [15], we obtain that ϕ is injective. Since $\phi(0) = 0$, we can conclude from part (i) of Lemma 2.5 that ϕ or $-\phi$ preserves the idempotent operators in both directions. This together with Proposition 2.6 and the injectivity of ϕ implies that ϕ preserves the square zero operators in both directions. Moreover, by part (i) of Lemma 2.5, $\phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P)$, for every rank one idempotent P. Therefore the forms of ϕ follow from Theorem 2.7. The scalar c in Theorem 2.7 is the scalar that $\phi(I) = cI$ (by the proof of this theorem in [2]). This together with the part (i) of Lemma 2.5 completes the proof.

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