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On the Properties of Balancing and Lucas-Balancing p-Numbers

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Abstract. The main goal of this paper is to develop a new generalization of balancing and Lucas-balancing sequences namely balancing and Lucas-balancing p-numbers and derive several identities related to them. Some combinatorial forms of these numbers are also presented.

Keywords: Balancing p-numbers, Lucas-balancing p-numbers, Incomplete balancing p-numbers, Incomplete Lucas-balancing p-numbers.

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1. Introduction

A number sequence closely associated to the famous Fibonacci sequence is the balancing sequence. Behera and Panda [1] in 1999 defined a natural number n as a balancing number if it is the solution of a simple Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$, calling r as the balancer corresponding to n. In general if B_n denotes the $n-th$ balancing number, then the balancing sequence is defined recursively as $B_n = 6B_{n-1} - B_{n-2}$, for $n \geq 2$ with seeds $B_0 = 0$ and $B_1 = 1$. The sequence companion to balancing sequence is the Lucas-balancing sequence whose recurrence relation is given by $C_n = 6C_{n-1} - C_{n-2}$, for $n \geq 2$ with seeds $C_0 = 1$ and $C_1 = 3$, where C_n denotes the n-th Lucas-balancing number. It is known that the ratio of two adjacent

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balancing numbers B_n and Lucas-balancing numbers C_n tends to a definite proportion $3+\sqrt{8}$ as $n \to \infty$. This number $\lambda_1 = 3+\sqrt{8}$ and its conjugate $\lambda_2 = 3-\sqrt{8}$ are indeed the roots of the characteristic equation $x^2-6x+1=0$. Binet's formulas are well-known in the theory of the balancing numbers, these formulas allow all balancing numbers B_n and Lucas-balancing numbers C_n to be represented by the roots of the characteristic equation as

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} \tag{1.1}$$

and

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}. \tag{1.2}$$

The theory of balancing numbers is broadly studied by many authors, the interested readers may see [1, 3, 4, 6, 11] for a detail review. The combinatorial forms for balancing and Lucas-balancing numbers were almost studied by Patel et al. [7]. They defined incomplete balancing and Lucas-balancing numbers as

$$B_n(k) = \sum_{j=0}^{k} (-1)^j \binom{n-1-j}{j} 6^{n-2j-1};\ 0 \leqslant k \leqslant \tilde{n},\ \tilde{n} = \lfloor \frac{n-1}{2} \rfloor$$

and

$$C_n(k) = 3\sum_{j=0}^{k} (-1)^j \frac{n}{n-j} \binom{n-j}{j} 6^{n-2j-1};\ 0 \leqslant k \leqslant \hat{n},\ \hat{n} = \lfloor \frac{n}{2} \rfloor.$$

Balancing and Lucas-balancing sequences are generalized in many ways. For details, one can see for example [2, 5, 6, 7, 9].

In this note, we generalize balancing and Lucas-balancing sequences by introducing balancing and Lucas-balancing p-numbers and deduce some of their properties. Further, we also present some of the combinatorial forms of these number sequences.

2. Balancing and Lucas-balancing p-numbers

In this section we introduce balancing and Lucas-balancing p-numbers and establish some of their properties.

Definition 2.1. For any given non-negative integer p, the balancing p-sequence is recursively defined as

$$B_p(n) = 6B_p(n-1) - B_p(n-p-1), \tag{2.1}$$

with seeds

$$B_p(n) = 6^{n-1};\ \text{for } n = 1, 2, \ldots, p+1 \text{ and } B_p(0) = 0. \tag{2.2}$$

For different values of p the recurrence relation (2.1) generates some interesting known sequences. For example, for the case $p = 0$, the recurrence relation (2.1) is reduced to the identity $B_0(n) = 5B_0(n-1)$, which generates the sequence of power of five, that is $B_0(n) = \{5^0, 5^1, 5^2, 5^3, \dots\}$ for $n = 1, 2, \dots$ with the given initials $B_0(0) = 0$ and $B_0(1) = 1$.

For the case $p = 1$, the basic recurrence relation (2.1) takes the form $B_1(n) = 6B_1(n-1) - B_1(n-2)$, with the initials $B_1(2) = 6^1 = 6$ and $B_1(1) = 6^0 = 1$ and which generates the classical balancing sequence $B_1(n) = B_n = \{1, 6, 35, 204, 1189, 6930, \dots\}$ for all $n \in \mathbb{N}$.

Definition 2.2. For any given non-negative integer p, Lucas-balancing p-numbers are defined by the following recurrence relation:

$$C_p(n) = 6C_p(n-1) - C_p(n-p-1), \tag{2.3}$$

with seeds

$$C_p(p+1) = 3\left(6^p - \frac{p+1}{6}\right) \text{ and } C_p(n) = 3 \cdot 6^{n-1},\ for\ n = 1, 2, \dots, p. \tag{2.4}$$

Notice that $C_p(0) = \frac{p+1}{2}$. Furthermore, for the initials $C_1(1) = 3$ and $C_1(2) = 17$, the recurrence relation (2.3) generates the classical Lucas-balancing numbers $C_n = C_1(n) = \{3, 17, 99, 577, \dots\}$.

Proposition 2.3. *For any particular positive integer p the sum of the balancing p-numbers $B_p(n)$ for all non-negative integers n is*

$$\sum_{i=0}^{n} B_p(i) = \frac{1}{4}\{B_p(n+1) - \sum_{i=0}^{p-1} B_p(n-i) - B_p(p+1) + (6^p - 1)\}.$$

Proof. We will prove this by using the principle of mathematical induction on n. Clearly the result is true for $n = 0, 1$ and 2. Let us assume the statement is true for $n = k$, and is

$$\sum_{i=0}^{k} B_p(i) = \frac{1}{4}\{B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1\}.$$

Now $\sum_{i=0}^{k+1} B_p(i)$ can be written as

$$\begin{aligned}\sum_{i=0}^{k+1} B_p(i) &= \sum_{i=0}^{k} B_p(i) + B_p(k+1)\\ &= \frac{1}{4}\{B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1\} + B_p(k+1)\\ &= \frac{1}{4}\{5B_p(k+1) - \sum_{i=0}^{p-1} B_p(k-i) - B_p(p+1) + 6^p - 1\}\\ &= \frac{1}{4}\{6B_p(k+1) - B_p(k-p+1) - B_p(k+1) - \sum_{i=0}^{p-2} B_p(k-i)\\ &\quad - B_p(p+1) + 6^p - 1\}\\ &= \frac{1}{4}\{6B_p(k+1) - B_p(k-p+1) - \sum_{i=0}^{p-1} B_p(k+1-i)\\ &\quad - B_p(p+1) + 6^p - 1\}\\ &= \frac{1}{4}\{B_p(k+2) - \sum_{i=0}^{p-1} B_p(k+1-i) - B_p(p+1) + 6^p - 1\},\end{aligned}$$

which proves the result. □

Proposition 2.4. *For any particular positive integer p the sum of the Lucas-balancing p-numbers $C_p(n)$ for all positive integer n is*

$$\sum_{i=1}^{n} C_p(i) = \frac{1}{4}\{C_p(n+1) - \sum_{i=0}^{p-1} C_p(n-i) - C_p(p+1) + 3(6^p - 1)\}.$$

Proof. The proof has similar approach to the above. □

As the limit of the ratio of two adjacent balancing and Lucas-balancing p-numbers $B_p(n)$ and $C_p(n)$ respectively tends to a definite proportion, we have

$$\lim_{n\to\infty} \frac{B_p(n)}{B_p(n-1)} = x.$$

Which imply by recurrence formula that

$$\begin{aligned}\frac{B_p(n)}{B_p(n-1)} &= \frac{6B_p(n-1) - B_p(n-p-1)}{B_p(n-1)}\\ &= 6 - \frac{1}{\frac{B_P(n-1)}{B_p(n-p-1)}}.\end{aligned}$$

It follows that

$$\frac{B_p(n)}{B_p(n-1)} = 6 - \frac{1}{\frac{B_p(n-1)B_p(n-2)\cdots B_p(n-p)}{B_p(n-2)B_p(n-3)\cdots B_P(n-p-1)}}.$$

Taking $\lim_{n\to\infty}$ on both sides, we get the result

$$x^{p+1} - 6x^p + 1 = 0. \tag{2.5}$$

The result (2.5) is the algebraic equation of $(p+1)$-th degree and has $(p+1)$ roots namely be $x_1, x_2, x_3, ..., x_{p+1}$. Now we examine the equation (2.5) for different values of p. By taking $p = 0$, (2.5) is the trivial equation $x = 5$, and for $p = 1$, (2.5) becomes $x^2 - 6x + 1 = 0$. After solving this equation, we get two defined roots λ_1 and λ_2, and has Binet's formulas (1.1) and (1.2).

Now we derive the Binet's formula for $B_p(n)$ and $C_p(n)$. Let $x_1, x_2, \ldots, x_p, x_{p+1}$ be roots of the polynomial equation $x^{p+1} - 6x^p + 1 = 0$, then the Binet's formulas for balancing and Lucas-balancing p-numbers with $p > 0$, are of the forms

$$B_p(n) = k_1 x_1^n + k_2 x_2^n + \cdots + k_{p+1} x_{p+1}^n \tag{2.6}$$

and

$$C_p(n) = a_1 x_1^n + a_2 x_2^n + \cdots + a_{p+1} x_{p+1}^n, \tag{2.7}$$

respectively, where $k_1, k_2, \ldots, k_{p+1}$ and $a_1, a_2, \ldots, a_{p+1}$ are coefficient constants.

By considering the balancing p-numbers given by the recurrence relation (2.1) and by using (2.2) and (2.6), we will get a set of following results.

$$\begin{aligned} B_p(0) &= k_1 + k_2 + \cdots + k_{p+1} = 0; \\ B_p(1) &= k_1 x_1 + k_2 x_2 + \cdots + k_{p+1} x_{p+1} = 1; \\ B_p(2) &= k_1 x_1^2 + k_2 x_2^2 + \cdots + k_{p+1} x_{p+1}^2 = 6; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_p(p) &= k_1 x_1^p + k_2 x_2^p + \cdots + k_{p+1} x_{p+1}^p = 6^{p-1}. \end{aligned} \tag{2.8}$$

Similarly by considering (2.3) and by using (2.4) and (2.7), we get

$$\begin{aligned} C_p(0) &= a_1 + a_2 + \cdots + a_{p+1} = \frac{p+1}{2}; \\ C_p(1) &= a_1 x_1 + a_2 x_2 + \cdots + a_{p+1} x_{p+1} = 3; \\ C_p(2) &= a_1 x_1^2 + a_2 x_2^2 + \cdots + a_{p+1} x_{p+1}^2 = 18; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C_p(p) &= a_1 x_1^p + a_2 x_2^p + \cdots + a_{p+1} x_{p+1}^p = 3 \cdot 6^{p-1}. \end{aligned} \tag{2.9}$$

Solving the above sets of equations, we get the approximate values of all constants $k_1, k_2, \ldots, k_{p+1}$ and $a_1, a_2, \ldots, a_{p+1}$.

For the case $p = 1$, the characteristic equation $x^{p+1} - 6x^p + 1 = 0$ is $x^2 - 6x + 1 = 0$, which implies the roots $x_1 = \lambda_1 = 3 + \sqrt{8}$ and $x_2 = \lambda_2 = \frac{1}{\lambda_1} = 3 - \sqrt{8}$. Hence for $p = 1$, equation (2.6) becomes the Binet's formula for balancing 1-number and is

$$B_1(n) = k_1 x_1^n + k_2 x_2^n = k_1(3 + \sqrt{8})^n + k_2(3 - \sqrt{8})^n \tag{2.10}$$

To find out the values of k_1 and k_2, use equation (2.8) and get $k_1 = \frac{1}{2\sqrt{8}}$ and $k_2 = \frac{-1}{2\sqrt{8}}$. Hence by manipulating k_1 and k_2 in (2.10), we get the desired Binet's formula (1.1).

In a similar way we find the Binet's formula for Lucas-balancing 1-numbers $C_1(n)$, equation (2.7) implies

$$C_1(n) = a_1 x_1^n + a_2 x_2^n = a_1(3+\sqrt{8})^n + a_2(3-\sqrt{8})^n. \tag{2.11}$$

To find out the values of a_1 and a_2, use equation (2.9) and get $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$. Hence by manipulating a_1 and a_2 in (2.11), we get the desired Binet's formula (1.2).

For $p = 2$, from the algebraic equation $x^{p+1} - 6x^p + 1 = 0$ we get $x^3 - 6x^2 + 1 = 0$, which gives $x_1 = -0.39543$, $x_2 = 0.42347$ and $x_3 = 5.9720$. Again for $p = 2$, the Binet's formula (2.6) and equation (2.8) become

$$B_2(n) = k_1 x_1^n + k_2 x_2^n + k_3 x_3^n \tag{2.12}$$

and

$$\begin{aligned} &k_1 + k_2 + k_3 = 0;\\ &k_1 x_1 + k_2 x_2 + k_3 x_3 = 1;\\ &k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2 = 6, \end{aligned}$$

respectively, and solving this system of equations, we get $k_1 = -0.0758435$, $k_2 = -0.0931908$ and $k_3 = 0.169034$.
Finally, (2.12) can be written as

$$\begin{aligned} B_2(n) \quad = \quad & (-0.0758435)(-0.39543)^n + (-0.0931908)(0.42347)^n \\ & +(0.169034)(5.9720)^n, \end{aligned}$$

which is the Binet's formula for balancing 2-numbers for any integers $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Similarly we can calculate the Binet's formula for the Lucas-balancing 2-numbers. Put $p = 2$ in the algebraic equation $x^{p+1} - 6x^p + 1 = 0$, we get the desired equation $x^3 - 6x^2 + 1 = 0$, which acquire same roots $x_1 = -0.39543$, $x_2 = 0.42347$ and $x_3 = 5.9720$. Again by using $p = 2$, the Binet's formula (2.7) and equation (2.9) become

$$C_2(n) = a_1 x_1^n + a_2 x_2^n + a_3 x_3^n \tag{2.13}$$

and

$$\begin{aligned} &a_1 + a_2 + a_3 = \frac{3}{2};\\ &a_1 x_1 + a_2 x_2 + a_3 x_3 = 3;\\ &a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 18, \end{aligned}$$

respectively and solving this system of equations, we get $a_1 = 0.499979$, $a_2 = 0.500028$ and $a_3 = 0.499993$.
Finally, (2.13) can be written as

$$C_2(n) = (0.499979)(-0.39543)^n + (0.500028)(0.42347)^n + (0.499993)(5.9720)^n,$$

which is the Binet's formula for the Lucas-balancing 2-numbers for any integers $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

In this way we can find out the Binet's formulas for all remaining balancing and Lucas-balancing p-numbers for occurrence of $p = 3, 4, \cdots$. In general the Binet's formulas for balancing and Lucas-balancing p-numbers are of the form given by (2.6) and (2.7) in which the coefficients $k_1, k_2, \cdots, k_{p+1}$ and $a_1, a_2, \cdots, a_{p+1}$ can be calculated by using the equations (2.8) and (2.9).

Before going to prove the following theorem it is better to discuss one more thing, that is, if $x_1, x_2, x_3, \cdots, x_{p+1}$ are roots of the characteristic equation $x^{p+1} - 6x^p + 1 = 0$, then these roots can be written in balancing and Lucas-balancing p-numbers as in form:

$$x_k^n = 6.x_k^{n-1} - x_k^{n-p-1} = x_k(6.x_k^{n-2} - x_k^{n-p-2}) = x_k \cdot x_k^{n-1}, \tag{2.14}$$

for all integer values n and $k = 1, 2, 3, \cdots, p+1$.

Theorem 2.5. *For any given positive integers $p(p > 0)$, balancing p-numbers can be written for $(n = 0, \pm 1, \pm 2, \pm 3, \cdots)$ in the form:*

$$B_p(n) = k_1 x_1^n + k_2 x_2^n + \cdots + k_{p+1} x_{p+1}^n, \tag{2.15}$$

where $k_1, k_2, \cdots, k_{p+1}$ are coefficient constants and $x_1, x_2, \cdots, x_{p+1}$ are roots of the polynomial equation $x^{p+1} - 6x^p + 1 = 0$.

Proof. We can easily find out the first p-terms for $n = 0, 1, 2, \cdots, p$ of the balancing p-numbers by using (2.6), (2.8) and algebraic equation $x^{p+1} - 6x^p + 1 = 0$. Now our seek is to prove $B_p(n) = k_1 x_1^n + k_2 x_2^n + \cdots + k_{p+1} x_{p+1}^n$ for remaining positive integers. For the case $n = p+1$, we have

$$\begin{aligned} B_p(p+1) =& k_1 x_1^{p+1} + k_2 x_2^{p+1} + \cdots + k_{p+1} x_{p+1}^{p+1} \\ =& 6[k_1 x_1^p + k_2 x_2^p + \cdots + k_{p+1} x_{p+1}^p] - [k_1 x_1^0 + k_2 x_2^0 + \cdots \\ & + k_{p+1} x_{p+1}^0]. \end{aligned}$$

Therefore according to (2.8), we have

$$B_p(p+1) = 6B_p(p) - B_p(0),$$

which is the basic recurrence relation (2.1) for $n = p+1$.

Similarly it is easy to prove that equation (2.15) is true for all remaining positive values from $n = p+2$.
Finally, we have to prove equation (2.15) is true for all negative values of n. For the case $n = -1$:

$$B_p(-1) = k_1 x_1^{-1} + k_2 x_2^{-1} + \cdots + k_{p+1} x_{p+1}^{-1}. \tag{2.16}$$

Let write (2.14) in the form:

$$x_k^{n-p-1} = 6 \cdot x_k^{n-1} - x_k^n. \tag{2.17}$$

By puting $n = p$ in (2.17), we get

$$x_k^{-1} = 6 \cdot x_k^{p-1} - x_k^p. \tag{2.18}$$

Apply (2.18) in (2.16), we get

$$B_p(-1) = 6[k_1 x_1^{p-1} + k_2 x_2^{p-1} + \cdots + k_{p+1} x_{p+1}^{p-1}] - [k_1 x_1^p + k_2 x_2^p + \cdots + k_{p+1} x_{p+1}^p]. \tag{2.19}$$

Using (2.8), expression (2.19) will become

$$B_p(-1) = 6B_p(p-1) - B_p(p) = 0,$$

which is the balancing p-number $B_p(-1) = 0$.

Similarly, for negative values of $n = -2, -3, -4, \cdots$, we will get all balancing p-numbers. Hence the equation (2.15) is true for all $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. This completes the proof. □

Using a similar approach to Theorem 2.5, we can also prove the following theorem for Lucas-balancing p-numbers

Theorem 2.6. *For any given positive integers $p(p > 0)$, Lucas-balancing p-numbers can be written for $(n = 0, \pm 1, \pm 2, \pm 3, \cdots)$ in the form:*

$$C_p(n) = a_1 x_1^n + a_2 x_2^n + \cdots + a_{p+1} x_{p+1}^n,$$

where $a_1, a_2, \cdots, a_{p+1}$ are coefficient constants and $x_1, x_2, \cdots, x_{p+1}$ are roots of the polynomial equation $x^{p+1} - 6x^p + 1 = 0$.

3. Incomplete balancing and Lucas-balancing p-numbers

In this section we introduce incomplete balancing and Lucas-balancing p-numbers and present some of their properties.

Definition 3.1. The incomplete balancing p-numbers denoted by $B_p^k(n)$ are defined by

$$B_p^k(n) = \sum_{j=0}^{k} (-1)^j \binom{n-1-pj}{j} 6^{n-(p+1)j-1}, \left(n = 1, 2, 3, \cdots; 0 \leqslant k \leqslant \lfloor \frac{n-1}{p+1} \rfloor\right). \tag{3.1}$$

In a similar manner incomplete Lucas-balancing p-numbers can also be defined as follows:

Definition 3.2. The incomplete Lucas-balancing p-numbers denoted by $C_p^k(n)$ are defined by

$$C_p^k(n) = 3\sum_{j=0}^{k}(-1)^j \frac{n}{n-pj}\binom{n-pj}{j}6^{n-(p+1)j-1}, \quad \left(n = 1, 2, 3, \cdots; 0 \leqslant k \leqslant \lfloor\frac{n}{p+1}\rfloor\right). \tag{3.2}$$

Notice that $B_1^{\lfloor\frac{n-1}{2}\rfloor}(n) = B_n$, $C_1^{\lfloor\frac{n}{2}\rfloor}(n) = C_n$ and $B_1^k(n) = B_n(k)$, $C_1^k(n) = C_n(k)$.

Some cases based on definitions 3.1 and 3.2 are

$$B_p^0 = 6^{n-1}; \text{ for all } n \geqslant 1,$$

$$B_p^1(n) = 6^{n-1} - 6^{n-p-2}(n-p-1); \text{ for all } n \geqslant p+2,$$

$$B_p^2(n) = 6^{n-1} - (n-1-p)6^{n-p-2} + 3(n-2p-1)(n-2p-2)6^{n-2p-4}; \text{ for all } n \geqslant 2p+1,$$

$$B_p^{\lfloor\frac{n-1}{p+1}\rfloor}(n) = B_p(n); \text{ for all } n \geqslant 1,$$

$$C_p^0(n) = 3.6^{n-1}; \text{ for all } n \geqslant 1,$$

$$C_p^1(n) = 3[6^{n-1} - n6^{n-p-2}]; \text{ for all } n \geqslant p+1,$$

$$C_p^2(n) = 3[6^{n-1} - n6^{n-p-2} + 3n(n-2p-1)6^{n-2p-4}]; \text{ for all } n \geqslant 2p+2$$

and

$$C_p^{\lfloor\frac{n}{p+1}\rfloor}(n) = C_p(n); \text{ for all } n \geqslant 1.$$

Proposition 3.3. *The recurrence relation of the incomplete balancing p-number is defined as:*

$$B_p^{k+1}(n) = 6B_p^{k+1}(n-1) - B_p^k(n-p-1); \ 0 \leqslant k \leqslant \frac{n-p-3}{p+1}. \tag{3.3}$$

Proof. By using Definition 3.1, the right hand side of (3.3) can be written as

$$
\begin{aligned}
&6\sum_{j=0}^{k+1}(-1)^j\binom{n-pj-2}{j}6^{n-(p+1)j-2}-\sum_{j=0}^{k}(-1)^j\binom{n-p-pj-2}{j}6^{n-p-(p+1)j-2}\\
=&\sum_{j=0}^{k+1}(-1)^j\binom{n-pj-2}{j}6^{n-(p+1)j-1}-\sum_{j=1}^{k+1}(-1)^{j-1}\binom{n-pj-2}{j-1}6^{n-(p+1)j-1}\\
=&\sum_{j=0}^{k+1}(-1)^j\binom{n-pj-2}{j}6^{n-(p+1)j-1}+\sum_{j=0}^{k+1}(-1)^j\binom{n-pj-2}{j-1}6^{n-(p+1)j-1}\\
&-\binom{n-2}{-1}6^{n-1}\\
=&\sum_{j=0}^{k+1}\left[\binom{n-pj-2}{j}+\binom{n-pj-2}{j-1}\right](-1)^j6^{n-(p+1)j-1}\\
=&\sum_{j=0}^{k+1}\binom{n-pj-1}{j}(-1)^j6^{n-(p+1)j-1}\\
=&B_p^{k+1}(n),
\end{aligned}
$$

and the result follows. $\square$

By virtue of Proposition 3.3 and equation (3.1), we get the following identity.

$$B_p^k(n)=6B_p^k(n-1)-B_p^k(n-p-1)+(-1)^k\binom{n-p(k+1)-2}{k}6^{n-(p+1)(k+1)-1}. \tag{3.4}$$

Proposition 3.4.

$$\sum_{j=0}^{h}\binom{h}{j}(-1)^{j+h}6^jB_p^{k+j}(n+p(j-1))=B_p^{k+h}(n+(p+1)h-p); \tag{3.5}$$

$$\left(0\leqslant k\leqslant\frac{n-p-h-1}{p+1}\right).$$

Proof. We shall prove this property by using principle of mathematical induction on h. The above sum (3.5) clearly holds for $h=0$ and $h=1$. Let us assume it holds for certain $h>1$. We will show that it holds for $h\to h+1$, now we have

$$\begin{aligned}
&\sum_{j=0}^{h+1}\binom{h+1}{j}(-1)^{j+h+1}6^j B_p^{k+j}(n+p(j-1))\\
&=\sum_{j=0}^{h+1}(-1)^{j+h+1}\left(\binom{h}{j}+\binom{h}{j-1}\right)B_p^{k+j}(n+p(j-1))6^j\\
&=\sum_{j=0}^{h+1}(-1)^{j+h+1}\binom{h}{j}B_p^{k+j}\Big(n+p(j-1)\Big)6^j+\sum_{j=0}^{h+1}(-1)^{j+h+1}\binom{h}{j-1}\\
&\quad\times B_p^{k+j}\Big(n+p(j-1)\Big)6^j\\
&=-B_p^{k+h}(n+(p+1)h-p)+\sum_{j=-1}^{h}(-1)^{j+h+2}\binom{h}{j}B_p^{k+j+1}\Big(n+pj\Big)6^{j+1}\\
&=-B_p^{k+h}(n+(p+1)h-p)+\sum_{j=0}^{h}(-1)^{j+h+2}\binom{h}{j}B_p^{k+j+1}\Big(n+pj\Big)6^j.6\\
&\quad+(-1)^{h+1}\binom{h}{-1}B_p^{k}(n-p)\\
&=-B_p^{k+h}(n+(p+1)h-p)+6\sum_{j=0}^{h}(-1)^{j+h}\binom{h}{j}B_p^{k+j+1}\Big(n+pj\Big)6^j\\
&=-B_p^{k+h}(n+(p+1)h-p)+6B_p^{k+h+1}(n+(p+1)h)\\
&=B_p^{k+h+1}(n+(p+1)h+1),
\end{aligned}$$

which follows the result. □

Proposition 3.5. *Let k be a non-negative integer. For $n \geqslant (p+1)k+p+2$, we have*

$$\sum_{j=0}^{h-1}6^{h-1-j}B_p^{k}(n-p+j)=6^hB_p^{k+1}(n)-B_p^{k+1}(n+h). \tag{3.6}$$

Proof. We shall prove this by using mathematical induction on h. The result is obvious for $h=1$ and $h=2$ by using (3.3).

Let us assume the given statement (3.6) is true for $h=t$ that is

$$\sum_{j=0}^{t-1}6^{t-1-j}B_p^{k}(n-p+j)=6^tB_p^{k+1}(n)-B_p^{k+1}(n+t).$$

Now it is enough to show that the sum (3.6) is true for $h=t+1$:

$$\sum_{j=0}^{t}6^{t-j}B_p^{k}(n-p+j)=6^{t+1}B_p^{k+1}(n)-B_p^{k+1}(n+t+1).$$

This implies

$$6\sum_{j=0}^{t-1}6^{t-1-j}B_p^k(n-p+j)+B_p^k(n-p+t)=6^{t+1}B_p^{k+1}(n)-B_p^{k+1}(n+t+1).$$

The above equality gives

$$6^{t+1}B_p^{k+1}(n)-6B_p^{k+1}(n+t)+B_p^k(n-p+t)=6^{t+1}B_p^{k+1}(n)-B_p^{k+1}(n+t+1).$$

Further simplification results

$$B_p^{k+1}(n+t+1)=6B_p^{k+1}(n+t)-B_p^k(n-p+t).$$

This completes the result in view of (3.3). □

Proposition 3.6.

$$2C_p^k(n)=6B_p^k(n)-(p+1)B_p^{k-1}(n-p);\ 0\leqslant k\leqslant\lfloor\frac{n-1}{p+1}\rfloor. \tag{3.7}$$

Proof. The right hand side of (3.7) can be written as

$$\begin{aligned}
&6\sum_{j=0}^{k}(-1)^j\binom{n-pj-1}{j}6^{n-(p+1)j-1}-(p+1)\sum_{j=0}^{k-1}(-1)^j\binom{n-p-pj-1}{j}\\
&\quad\times 6^{n-p-(p+1)j-1}\\
&=6\sum_{j=0}^{k}(-1)^j\binom{n-pj-1}{j}6^{n-(p+1)j-1}-(p+1)\sum_{j=1}^{k}(-1)^{j-1}\binom{n-pj-1}{j-1}\\
&\quad\times 6^{n-(p+1)j}\\
&=\sum_{j=0}^{k}(-1)^j\binom{n-pj-1}{j}6^{n-(p+1)j}+(p+1)\sum_{j=0}^{k}(-1)^j\binom{n-pj-1}{j-1}6^{n-(p+1)j}\\
&\quad-(p+1)\binom{n-1}{-1}6^n\\
&=\sum_{j=0}^{k}\left[\binom{n-pj-1}{j}+(p+1)\binom{n-pj-1}{j-1}\right](-1)^j6^{n-(p+1)j}\\
&=6\sum_{j=0}^{k}(-1)^j\frac{n}{n-pj}\binom{n-pj}{j}6^{n-(p+1)j-1}\\
&=2C_p^k(n),
\end{aligned}$$

and then the result follows. □

Proposition 3.7. *The recurrence relation of the incomplete Lucas-balancing p-numbers $C_p^k(n)$ is*

$$C_p^{k+1}(n)=6C_p^{k+1}(n-1)-C_p^k(n-p-1);\qquad \left(0\leqslant k\leqslant\frac{n-p-2}{p+1}\right). \tag{3.8}$$

Proof. Applying (3.3) and (3.7), we have

$$\begin{aligned}
&2C_p^{k+1}(n)\\
=&6\Big(6B_p^{k+1}(n-1)-(p+1)B_p^k(n-p-1)\Big)-\Big(6B_p^k(n-p-1)-(p+1)\\
&\times B_p^{k-1}(n-2p-1)\Big)\\
=&6\Big(2C_p^{k+1}(n-1)\Big)-2C_p^k(n-p-1).
\end{aligned}$$

Hence, $C_p^{k+1}(n)=6C_p^{k+1}(n-1)-C_p^k(n-p-1)$. □

Here we observe that by applying (3.2), the above relation (3.8) can be transformed into the non-homogeneous relation

$$\begin{aligned}
C_p^k(n)=&6C_p^k(n-1)-C_p^k(n-p-1)+3(-1)^k\frac{n-p-1}{n-(k+1)p-1}\\
&\times\binom{n-(k+1)p-1}{k}6^{n-(k+1)p-k-2}.
\end{aligned}\tag{3.9}$$

Proposition 3.8. *For $0\leqslant k\leqslant\frac{n-p-h}{p+1}$, we have*

$$\sum_{j=0}^{h}\binom{h}{j}(-1)^{j+h}6^jC_p^{k+j}\Big(n+p(j-1)\Big)=C_p^{k+h}\Big(n+(p+1)h-p\Big).$$

Proof. The proof is similar to Proposition 3.4. □

Proposition 3.9. *Let k be a non-negative integer. For $n\geqslant(p+1)(k+1)$, the identity*

$$\sum_{j=0}^{h-1}6^{h-1-j}C_p^k(n-p+j)=6^hC_p^{k+1}(n)-C_p^{k+1}(n+h)$$

holds.

Proof. The proof is analogous to Proposition 3.5. □

The following result which has already proved in [8] is useful while finding the generating functions of $B_p^k(n)$ and $C_p^k(n)$.

Lemma 3.10. *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the non-homogeneous recurrence relation*

$$s_n=6s_{n-1}-s_{n-p-1}+r_n,\quad n>p,$$

where r_n is a given complex sequence. Then the generating function $S_p^k(t)$ of the sequence $\{s_n\}_{n=0}^{\infty}$ is

$$S_p^k(t)=\frac{s_0-r_0+\sum_{i=1}^{p}t^i(s_i-6s_{i-1}-r_i)+G(t)}{1-6t+t^{p+1}},$$

where G(t) denotes the generating function of $\{r_n\}$.

Theorem 3.11. *The generating function of the incomplete balancing p-numbers* $B_p^k(n)$ $(k = 0, 1, 2, 3, \dots)$ *is given by*

$$\begin{aligned} R_p^k(t) &= \sum_{j=0}^{\infty} B_p^j(k) t^j \\ &= t^{k(p+1)+1} \Big[\Big\{ B_p(k(p+1)+1) + \sum_{i=1}^{p} t^i (B_p(k(p+1)+i+1) - \\ &\quad 6B_p(k(p+1)+i)) \Big\} (1-6t)^{k+1} + (-1)^k t^{p+1} \Big] \cdot \\ &\quad \Big[(1-6t+t^{p+1})(1-6t)^{k+1} \Big]^{-1}. \end{aligned}$$

Proof. We prove this theorem by using Lemma 3.10. Let k be a fixed positive integer, from (3.1) and (3.4), we have

$$B_p^k(n) = 0; \quad if\ 0 \leqslant n < k(p+1)+1$$

and

$$B_p^k(k(p+1)+1) = B_p(k(p+1)+1),$$

$$B_p^k(k(p+1)+2) = B_p(k(p+1)+2),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$B_p^k(k(p+1)+p+1) = B_p(k(p+1)+p+1),$$

and that

$$\begin{aligned} B_p^k(n) &= 6B_p^k(n-1) - B_p^k(n-p-1) \\ &\quad + (-1)^k \binom{n-p(k+1)-2}{n-k(p+1)-p-2} 6^{n-k(p+1)-p-2}, \end{aligned}$$

if $n \geqslant k(p+1)+p+2$.
We let

$$s_0 = B_p^k(k(p+1)+1),\ s_1 = B_p^k(k(p+1)+2), ..., s_p = B_p^k(k(p+1)+p+1)$$

and $s_n = B_p^k(n+k(p+1)+1)$. Suppose that $r_0 = r_1 = r_2 = \dots = r_p = 0$ and

$$r_n = \binom{n-(p+1)+k}{n-(p+1)} (-1)^k 6^{n-(p+1)}.$$

Thus we can easily derive that the generating function of the sequence r_n is (see p.355 of [10])

$$G(t) = \frac{(-1)^k t^{p+1}}{(1-6t)^{k+1}}.$$

Then in view of Lemma 3.10, the generating function

$$\begin{aligned} S_p^k(t)(1-6t+t^{p+1}) - \frac{(-1)^k t^{p+1}}{(1-6t)^{k+1}} \\ =& B_p^k(k(p+1)+1) + \sum_{i=1}^{p} t^i (B_p(k(p+1)+i+1) \\ & - 6B_p(k(p+1)+i)), \end{aligned}$$

which implies

$$\begin{aligned} S_p^k(t) &= \Big[\big\{B_p^k(k(p+1)+1) + \sum_{i=1}^{p} t^i (B_p(k(p+1)+i+1) - 6B_p(k(p+1)+ \\ & \quad i))\big\}(1-6t)^{k+1} + (-1)^k t^{p+1}\Big]\Big[(1-6t+t^{p+1})(1-6t)^{k+1}\Big]^{-1}. \end{aligned}$$

Finally, we conclude that

$$R_p^k(t) = t^{k(p+1)+1} S_p^k(t).$$

This completes the proof. □

Theorem 3.12. *The generating function of the incomplete Lucas-balancing p-numbers* $C_p^k(n)$ $(k = 0, 1, 2, 3, ...)$ *is given by*

$$\begin{aligned} W_p^k(t) &= \sum_{j=0}^{\infty} C_p^j(k) t^j \\ &= t^{k(p+1)}\Big[\Big\{C_p(k(p+1)) + \sum_{i=1}^{p} t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+ \\ & \quad i-1))\Big\}(1-6t)^{k+1} + (-1)^k 3t^{p+1}(p(1-t)+1)\Big]\Big[(1-6t+t^{p+1}) \cdot \\ & \quad (1-6t)^{k+1}\Big]^{-1}. \end{aligned}$$

Proof. We prove this theorem by using Lemma 3.10. Let k be a fixed positive integer, from (3.2) and (3.9), we have

$$C_p^k(n) = 0; \quad if\ 0 \leqslant n < k(p+1)$$

and

$$C_p^k(k(p+1)) = C_p(k(p+1)),$$

$$C_p^k(k(p+1)+1) = C_p(k(p+1)+1),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$C_p^k(k(p+1)+p) = C_p(k(p+1)+p),$$

and that

$$\begin{aligned} C_p^k(n) =& 6C_p^k(n-1) - C_p^k(n-p-1) + 3(-1)^k \frac{n-p-1}{n-(k+1)p-1} \cdot \\ & \binom{n-p(k+1)-1}{n-k(p+1)-p-1} 6^{n-k(p+1)-p-2}; \end{aligned}$$

if $n \geqslant k(p+1)+p+1$. We let
$s_0 = C_p^k(k(p+1))$, $s_1 = C_p^k(k(p+1)+1)$, $\cdots$, $s_p = C_p^k(k(p+1)+p)$ and $s_n = C_p^k(n+k(p+1))$.
Suppose that $r_0 = r_1 = r_2 = \cdots = r_p = 0$ and

$$r_n = \frac{n+k(p+1)-p-1}{n+k-p-1} \binom{n-(p+1)+k}{n-(p+1)} 3(-1)^k 6^{n-(p+2)}.$$

Then the generating function of the sequence r_n is (p.355, [10])

$$G(t) = \frac{(-1)^k 3t^{p+1}(p(1-t)+1)}{(1-6t)^{k+1}}.$$

By virtue of Lemma 3.10, the generating function

$$\begin{aligned} S_p^k(t)(1 &-& 6t + t^{p+1}) - \frac{(-1)^k 3t^{p+1}(p(1-t)+1)}{(1-6t)^{k+1}} \\ &=& C_p^k(k(p+1)) + \sum_{i=1}^{p} t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+i-1)). \end{aligned}$$

Further simplification gives

$$\begin{aligned} S_p^k(t) &=& \Big[\Big\{ C_p^k(k(p+1)) + \sum_{i=1}^{p} t^i (C_p(k(p+1)+i) - 6C_p(k(p+1)+i-1)) \Big\} \cdot \\ && (1-6t)^{k+1} + (-1)^k 3t^{p+1}(p(1-t)+1) \Big] \Big[(1-6t+t^{p+1}) \cdot \\ && (1-6t)^{k+1} \Big]^{-1}. \end{aligned}$$

Finally, we conclude that

$$W_p^k(t) = t^{k(p+1)} S_p^k(t),$$

and hence the proof. □

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