Graph Convergence for $H(\cdot, \cdot)$-Co-Accretive Mapping with over-relaxed Proximal Point Method for Solving a Generalized Variational Inclusion Problem

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Abstract. In this paper, we use the concepts of graph convergence of $H(\cdot, \cdot)$-co-accretive mapping introduced by [R. Ahmad, M. Akram, M. Dilshad, Graph convergence for the $H(\cdot, \cdot)$-co-accretive mapping with an application, Bull. Malays. Math. Sci. Soc., doi: 10.1007/s40840-014-0103-z, 2014] and define an over-relaxed proximal point method to obtain the solution of a generalized variational inclusion problem in Banach spaces. Our results can be viewed as an extension of some previously known results in this direction.

Keywords: Graph convergence, Proximal point method, Accretive mapping, Variational inclusion, Convergence.


1. Introduction

Variational inequalities were extended and generalized in various ways using different concepts and obtained application oriented shapes. They are widely applied in mechanics, physics, optimization, economics, engineering sciences.
and general sciences etc. Variational inclusions are generalized forms of variational inequalities, which is mainly due to Hassouni and Moudafi [10]. The one of the most efficient and effective technique for solving variational inclusions is the resolvent operator technique, see e.g. [2, 3, 5, 6, 7, 9, 12, 13].

Over-relaxed factors are significant parameters affecting the convergence of a numerical scheme. They represent the fraction of the solution being carried forward from one iteration to the next for the various equations being solved during the simulation.

Verma [26] introduced a general framework for the over-relaxed A-proximal point algorithm based on A-maximal monotonicity and stated that it is application oriented. Pan et al. [23] solved a general nonlinear mixed set-valued variational inclusions by constructing an over-relaxed A-proximal point algorithm based on \((A, \eta)\)-accretive mappings. For related work, see [17, 27, 20].

Li and Huang [18] introduced the concepts of graph convergence for \(H(\cdot, \cdot)\)-accretive mappings and applied it to solve a variational inclusion problem. After that, Ahmad et al. [4] introduced the concept of graph convergence for \(H(\cdot, \cdot)\)-co-accretive mappings for solving a generalized variational inclusion problem. Very recently, Lan [19] designed the graph convergence analysis of over-relaxed \((A, \eta, m)\)-proximal point iterative methods for solving general nonlinear operator equations. A quite reasonable work is done in this direction to solve some classes of variational inclusion problems. For more details of the related work, we refer to [1, 8, 22, 14, 15, 21, 24, 25] and references therein.

In this communication, we design an over-relaxed proximal point algorithm for solving a generalized variational inclusion problem by using the concept of \(H(\cdot, \cdot)\)-co-accretive mapping due to Ahmad et al. [4]. We prove an existence result for generalized variational inclusion problem and show that the sequences generated by our algorithm converge to a solution of generalized variational inclusion problem.

2. Preliminaries

Let \(X\) be a real Banach space with its norm \(\| \cdot \|\), \(X^*\) be the topological dual of \(X\) and \(d\) be the metric induced by the norm \(\| \cdot \|\). Let \(\langle \cdot, \cdot \rangle\) be the dual pair between \(X\) and \(X^*\), \(CB(X)\) (respectively, \(2^X\)) be the family of all nonempty closed and bounded subsets (respectively, all non-empty subsets) of \(X\) and \(D\) be the Hausdorff metric on \(CB(X)\) defined by

\[
D(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\},
\]

where \(A, B \in CB(X)\), \(d(x, B) = \inf_{y \in B} d(x, y)\) and \(d(A, y) = \inf_{x \in A} d(x, y)\).
The generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \}, \forall x \in X,$$

where $q > 1$ is a constant. In particular, $J_2$ is the usual normalized duality mapping. It is well known that $J_q(x) = \|x\|^{q-1}J_2(x)$, for all $x(\neq 0) \in X$. If $X$ is a Hilbert space, then $J_2$ becomes the identity mapping on $X$.

The modulus of smoothness of $X$ is the function $\rho_X : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

A Banach space $X$ is said to be uniformly smooth if,

$$\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0.$$

Also, $X$ is called $q$-uniformly smooth if, there exists a constant $C > 0$ such that

$$\rho_X(t) \leq Ct^q, \quad q > 1.$$

Note that $J_q$ is single-valued, if $X$ is uniformly smooth. In the study of characteristic inequalities in $q$-uniformly smooth Banach spaces, Xu [28] proved the following lemma.

**Lemma 2.1.** Let $q > 1$ be a real number and $X$ be a real smooth Banach space. Then $X$ is $q$-uniformly smooth if and only if, there exists a constant $C_q > 0$ such that for every $x, y \in X$

$$\|x + y\|^q \leq \|x\|^q + q(y, J_q(x)) + C_q\|y\|^q.$$

Throughout the paper unless otherwise specified, we take $X$ to be $q$-uniformly smooth Banach space. Now, we recall some definitions and results which will be used in subsequent section.

**Definition 2.2.** A mapping $A : X \rightarrow X$ is said to be

(i) accretive if,

$$\langle Ax - Ay, J_q(x - y) \rangle \geq 0, \forall x, y \in X;$$

(ii) strongly accretive if,

$$\langle Ax - Ay, J_q(x - y) \rangle > 0, \forall x, y \in X,$$

and the equality holds if and only if $x = y$;

(iii) $\delta$-strongly accretive if, there exists a constant $\delta > 0$ such that

$$\langle Ax - Ay, J_q(x - y) \rangle \geq \delta\|x - y\|^q, \forall x, y \in X;$$

(iv) $\beta$-relaxed accretive if, there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, J_q(x - y) \rangle \geq (-\beta)\|x - y\|^q, \forall x, y \in X;$$
(v) \( \mu \)-cocoercive if, there exists a constant \( \mu > 0 \) such that 
\[
\langle Ax - Ay, J_q(x - y) \rangle \geq \mu \|Ax - Ay\|^q, \forall x, y \in X;
\]

(vi) \( \gamma \)-relaxed cocoercive if, there exists a constant \( \gamma > 0 \) such that 
\[
\langle Ax - Ay, J_q(x - y) \rangle \geq (-\gamma)\|Ax - Ay\|^q, \forall x, y \in X;
\]

(vii) \( \sigma \)-Lipschitz continuous if, there exists a constant \( \sigma > 0 \) such that 
\[
\|Ax - Ay\| \leq \sigma \|x - y\|, \forall x, y \in X;
\]

(viii) \( \eta \)-expansive if, there exists a constant \( \eta > 0 \) such that 
\[
\|Ax - Ay\| \geq \eta \|x - y\|, \forall x, y \in X,
\]

if \( \eta = 1 \), then it is expansive.

**Definition 2.3.** A set-valued mapping \( T : X \rightarrow CB(X) \) is said to be \( \mathcal{D} \)-Lipschitz continuous if, there exists a constant \( \lambda_{\mathcal{D}} > 0 \) such that 
\[
\mathcal{D}(T(x), T(y)) \leq \lambda_{\mathcal{D}} \|x - y\|, \forall x, y \in X.
\]

**Definition 2.4.** Let \( H : X \times X \rightarrow X \) and \( A, B : X \rightarrow X \) be single-valued mappings. Then 

(i) \( H(A, \cdot) \) is said to be \( \mu_1 \)-cocoercive with respect to \( A \) if for a fixed \( u \in X \), there exists a constant \( \mu_1 > 0 \) such that 
\[
\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \mu_1 \|Ax - Ay\|^q, \forall x, y \in X;
\]

(ii) \( H(\cdot, B) \) is said to be \( \gamma_1 \)-relaxed cocoercive with respect to \( B \) if for a fixed \( u \in X \), there exists a constant \( \gamma_1 > 0 \) such that 
\[
\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq (-\gamma_1) \|Bx - By\|^q, \forall x, y \in X;
\]

(iii) \( H(A, B) \) is said to be symmetric cocoercive with respect to \( A \) and \( B \) if, \( H(A, \cdot) \) is cocoercive with respect to \( A \) and \( H(\cdot, B) \) is relaxed cocoercive with respect to \( B \);

(iv) \( H(A, \cdot) \) is said to be \( \alpha_1 \)-strongly accretive with respect to \( A \) if for a fixed \( u \in X \), there exists a constant \( \alpha_1 > 0 \) such that 
\[
\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geq \alpha_1 \|x - y\|^q, \forall x, y \in X;
\]

(v) \( H(\cdot, B) \) is said to be \( \beta_1 \)-relaxed accretive with respect to \( B \) if for a fixed \( u \in X \), there exists a constant \( \beta_1 > 0 \) such that 
\[
\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geq (-\beta_1) \|x - y\|^q, \forall x, y \in X;
\]

(vi) \( H(A, B) \) is said to be symmetric accretive with respect to \( A \) and \( B \) if, \( H(A, \cdot) \) is strongly accretive with respect to \( A \) and \( H(\cdot, B) \) is relaxed accretive with respect to \( B \);
Definition 2.5. Let $M : X \times X \rightarrow 2^X$ be a set-valued mapping. Then

(i) $M(f, \cdot)$ is said to be $\alpha$-strongly accretive with respect to $f$ if, there exists a constant $\alpha > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq \alpha \| x - y \|^q, \forall x, y, w \in X,$$

and for all $u \in M(f(x), w), v \in M(f(y), w)$;

(ii) $M(\cdot, g)$ is said to be $\beta$-relaxed accretive with respect to $g$ if, there exists a constant $\beta > 0$ such that

$$\langle u - v, J_q(x - y) \rangle \geq (-\beta) \| x - y \|^q, \forall x, y, w \in X,$$

and for all $u \in M(w, g(x)), v \in M(w, g(y))$;

(iii) $M(f, g)$ is said to be symmetric accretive with respect to $f$ and $g$ if, $M(f, \cdot)$ is strongly accretive with respect to $f$ and $M(\cdot, g)$ is relaxed accretive with respect to $g$.

Definition 2.6. A sequence $\{x_i\}$ is said to converge linearly to $x^*$ if, there exists a constant $0 < c < 1$ such that

$$\| x_{i+1} - x^* \| \leq c \| x_i - x^* \|,$$

for all $i > m$, for some natural number $m > 0$.

Definition 2.7 ([4]). Let $A, B, f, g : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Let $M : X \times X \rightarrow 2^X$ be the set-valued mapping. The mapping $M$ is said to be $H(\cdot, \cdot)$-co-accretive with respect to $A, B, f, g$ if, $H(A, B)$ is symmetric cocoercive with respect to $A$ and $B$, $M(f, g)$ is symmetric accretive with respect to $f$ and $g$, and

$$[H(A, B) + \lambda M(f, g)](X) = X, \forall \lambda > 0.$$

Theorem 2.8 ([4]). Let $A, B, f, g : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Let $M : X \times X \rightarrow 2^X$ be an $H(\cdot, \cdot)$-co-accretive mapping with respect to $A, B, f, g$. Let $A$ be $\eta$-expansive and $B$ be $\sigma$-Lipschitz continuous and $\alpha > \beta$, $\mu > \gamma$ and $\eta \geq \sigma$. Then the mapping $[H(A, B) + \lambda M(f, g)]^{-1}$ is single-valued, for every $\lambda > 0$. 
Definition 2.9 ([4]). Let $A, B, f, g : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Let $M : X \times X \rightarrow 2^X$ be an $H(\cdot, \cdot)$-co-accretive mapping with respect to $A, B, f$ and $g$. Then the resolvent operator $R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)} : X \rightarrow X$ is defined by

$$R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) = [H(A, B) + \lambda M(f, g)]^{-1}(u), \ \forall u \in X, \lambda > 0.$$  

Theorem 2.10 ([4]). Let $A, B, f, g : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Let $M : X \times X \rightarrow 2^X$ be an $H(\cdot, \cdot)$-co-accretive mapping with respect to $A, B, f$ and $g$. Let $A$ be $\eta$-expansive and $B$ be $\sigma$-Lipschitz continuous and $\alpha > \beta$, $\mu > \gamma$ and $\eta \geq \sigma$. Then the resolvent operator $R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}$ is Lipschitz continuous with constant $\theta$, i.e.,

$$\|R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) - R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(v)\| \leq \theta\|u - v\|, \ \forall u, v \in X, \lambda > 0,$$

where $\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu\eta - \gamma\sigma)}$.

Definition 2.11. Let $M : X \times X \rightarrow 2^X$ be a set-valued mapping. The graph of $M$ is denoted by $G(M)$ and defined by

$$G(M) = \{(x, y, z) : z \in M(x, y), \ \forall x, y \in X\}.$$

Definition 2.12. Let $A, B, f, g : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Let $M_n, M : X \times X \rightarrow 2^X$ be $H(\cdot, \cdot)$-co-accretive mappings, for $n = 0, 1, 2, \cdots$. The sequence $M_n$ is said to be graph convergent to $M$, denoted by $M_n \xrightarrow{G} M$ if, for every $((f(x), g(x)), z) \in G(M)$, there exists a sequence $((f(x_n), g(x_n)), z_n) \in G(M_n)$ such that

$$f(x_n) \rightarrow f(x), \ g(x_n) \rightarrow g(x) \ \text{and} \ z_n \rightarrow z, \ \text{as} \ n \rightarrow \infty.$$

Theorem 2.13 ([4]). Let $M_n, M : X \times X \rightarrow 2^X$ be $H(\cdot, \cdot)$-co-accretive mappings with respect to $A, B, f$ and $g$. Let $H : X \times X \rightarrow X$ be a single-valued mapping such that $H(A, B)$ is $\xi_1$-Lipschitz continuous with respect to $A$ and $\xi_2$-Lipschitz continuous with respect to $B$. Suppose that $f$ is $\tau$-expansive mapping. Then, $M_n \xrightarrow{G} M$ if and only if

$$R_{\lambda, M_n(\cdot, \cdot)}^{H(\cdot, \cdot)}(u) \rightarrow R_{\lambda, M(\cdot, \cdot)}^{H(\cdot, \cdot)}(u), \ \forall u \in X, \lambda > 0.$$

3. Formulation of the problem and Algorithm framework

Let $T : X \rightarrow CB(X)$ and $M : X \times X \rightarrow 2^X$ be set-valued mappings and $f, g : X \rightarrow X$ be single-valued mappings. We consider the following problem of finding $x \in X$, $w \in T(x)$ such that

$$0 \in w + M(f(x), g(x)). \ \ (3.1)$$

Problem (3.1) is called generalized variational inclusion problem.

Special Cases:
Lemma 3.1. The elements $x \in X$, $w \in T(x)$ are the solutions of the generalized variational inclusion problem (3.1) if and only if, they satisfy the following equation:

$$x = R_{\lambda,M}^{H;\cdot} [H(Ax,Bx) - \lambda w],$$

where $\lambda > 0$ and $R_{\lambda,M}^{H;\cdot}(x) = [H(A,B) + \lambda M(f,g)]^{-1}(x)$, $\forall x \in X$.

Algorithm 3.2. Step 1. Choose an arbitrary initial point $x_0 \in X$ and $w_0 \in T(x_0)$.

Step 2. Compute the sequence $\{x_n\}$ and $\{w_n\}$ by the following iterative procedure:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \quad n \geq 0,$$

where for some $P_n \in T(y_n)$, $y_n$ satisfies

$$\|y_n - R_{\lambda,M}^{H;\cdot} [H(Ax_n,Bx_n) - \lambda w_n]\| \leq \sigma_n (\|y_n - x_n\| + \lambda \|P_n - w_n\|);$$

and

$$\|w_n - w_{n-1}\| \leq D(T(x_n),T(x_{n-1})).$$

where $\{\alpha_n\} \subseteq [0, \infty)$ is a sequence of over-relaxed factors, $\{\sigma_n\}$ is a scalar sequence, $n \geq 0$, $\lambda > 0$, $\sum_{n=0}^{\infty} \sigma_n < \infty$, $\sigma_n \to 0$ and $\alpha = \lim_{n \to \infty} \sup \alpha_n < 1$.

Step 3. If $\{x_n\}$ and $\{y_n\}$ satisfy (3.5), (3.6) and $\{w_n\}$ satisfies (3.7) to an amount of accuracy, Stop. Otherwise, set $n = n + 1$ and repeat the Step 2.

Theorem 3.3. Let $X$ be a $q$-uniformly smooth Banach space. Let $A, B : X \to X$ and $H : X \times X \to X$ be mappings such that $H$ is symmetric cocoercive with respect to $A$ and $B$ with constants $\mu$ and $\gamma$, respectively; $r_1$-Lipschitz continuous with respect to $A$ and $r_2$-Lipschitz continuous with respect to $B$; $A$ is $\eta$-expansive and $B$ is $\sigma$-Lipschitz continuous. Let $T : X \to CB(X)$ be $D$-Lipschitz continuous with constant $\delta_T$ and the mappings $M_n, M : X \times X \to 2^X$
be \(H(\cdot,\cdot)\)-co-accrative mappings such that \(M_n \xrightarrow{G} M\). In addition, if for some \(\lambda > 0\), the following condition holds:

\[
\theta(r_1 + r_2) + \lambda \delta_T < 1,
\]

(3.8)

where \(\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu \eta^2 - \gamma \sigma^2)}\), \(\mu > \gamma, \eta \geq \sigma\) and \(\alpha > \beta\). Then, the generalized variational inclusion problem (3.1) admits a solution \((x^*, w^*)\), \(x^* \in X\), \(w^* \in T(x^*)\), and the sequences \(\{x_n\}\) and \(\{w_n\}\) defined in Algorithm 3.2 converge linearly to \(x^*\) and \(w^*\), respectively.

Proof. For any \(\lambda > 0\), we define a mapping \(G : X \to X\) by

\[
G(x) = R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}(H(Ax, Bx) - \lambda w_1), \quad \forall x \in X, w_1 \in T(x).
\]

Since the resolvent operator \(R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}\) is \(\theta\)-Lipschitz continuous, \(H\) is \(r_1\)-Lipschitz continuous with respect to \(A\) and \((r_2)\)-Lipschitz continuous with respect to \(B\), \(T\) is \(\delta_T\)-Lipschitz continuous, hence, for any \(x, y \in X\), \(w_1 \in T(x), w_2 \in T(y)\), we estimate

\[
\|G(x) - G(y)\| = \left\|R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}[H(Ax, Bx) - \lambda w_1] - R^{H(\cdot,\cdot)}_{\lambda, M(\cdot,\cdot)}[H(Ay, By) - \lambda w_2]\right\|
\leq \theta \|H(Ax, Bx) - H(Ay, By) - \lambda(w_1 - w_2)\|
\leq \theta \|H(Ax, Bx) - H(Ay, By)\| + \lambda \theta \|w_1 - w_2\|
\leq \theta (r_1 + r_2)\|x - y\| + \lambda \theta \delta_T \|T(x) - T(y)\|
\leq \theta (r_1 + r_2)\|x - y\| + \lambda \theta \delta_T \|x - y\|
= (\theta (r_1 + r_2) + \lambda \theta \delta_T) \|x - y\|,
\]

which implies that

\[
\|G(x) - G(y)\| \leq P(\theta_1)\|x - y\|,
\]

(3.9)

where \(P(\theta_1) = \theta(r_1 + r_2) + \lambda \theta \delta_T\) and \(\theta = \frac{1}{\lambda(\alpha - \beta) + (\mu \eta^2 - \gamma \sigma^2)}\). It follows from condition (3.8) that \(0 \leq P(\theta_1) < 1\), and so \(G\) is a contraction mapping i.e., \(G\) has a unique fixed point in \(X\).

Next, we prove that \((x^*, w^*)\), \(x^* \in X\), \(w^* \in T(x^*)\) is a solution of the problem (3.1). It follows from Lemma 3.1 that

\[
x^* = (1 - \alpha_n)x^* + \alpha_n R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}[H(Ax^*, Bx^*) - \lambda w^*].
\]

(3.10)

Let

\[
z_{n+1} = (1 - \alpha_n)x_n + \alpha_n R^{H(\cdot,\cdot)}_{\lambda, M_n(\cdot,\cdot)}[H(Ax_n, Bx_n) - \lambda w_n].
\]

(3.11)
Using the Lipschitz continuity of the resolvent operator $R^{H(\cdot)}_{\lambda,M(\cdot)}$, we evaluate
\[
\|z_{n+1} - x^*\| = \|(1 - \alpha_n)(x_n - x^*) + \alpha_n \{ R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax_n, Bx_n) - \lambda w_n] - R^{H(\cdot)}_{\lambda,M(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*]\}\|
\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax_n, Bx_n) - \lambda w_n] - R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*]\| + \alpha_n \|R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax_n, Bx_n) - \lambda w_n] - R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*]\|
\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|H(Ax_n, Bx_n) - H(Ax^*, Bx^*) - \lambda(w_n - w^*)\| + \alpha_n \|R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax_n, Bx_n) - \lambda w_n] - R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*]\|
\]
(3.12)

By using Lemma 3.1 and as $H(A,B)$ is $\mu$-cocoercive with respect to $A$, $\gamma$-relaxed cocoercive with respect to $B$, $r_1$-Lipschitz continuous with respect to $A$ and $r_2$-Lipschitz continuous with respect to $B$, we have
\[
\|H(Ax_n, Bx_n) - H(Ax^*, Bx^*) - \lambda(w_n - w^*)\|^q
\leq \lambda^q\|w_n - w^*\|^q + C_q \|H(Ax_n, Bx_n) - H(Ax^*, Bx^*)\|^q - 2\lambda \mu \|Ax_n - Ax^*\|^q - \gamma \|Bx_n - Bx^*\|^q
\leq \lambda^q\|w_n - w^*\|^q + C_q (r_1 + r_2)^q \|x_n - x^*\|^q - 2\lambda \mu \|Ax_n - Ax^*\|^q - \gamma \|Bx_n - Bx^*\|^q
\leq [\lambda^q\delta^q_x + C_q (r_1 + r_2)^q - 2\lambda \mu \|Ax_n - Ax^*\|^q - \gamma \|Bx_n - Bx^*\|^q] \|x_n - x^*\|^q
\]
which implies that
\[
\|H(Ax_n, Bx_n) - H(Ax^*, Bx^*) - \lambda(w_n - w^*)\|
\leq \sqrt{\lambda \mu \delta^q_x + C_q (r_1 + r_2)^q - 2\lambda \mu \|Ax_n - Ax^*\|^q - \gamma \|Bx_n - Bx^*\|^q} \|x_n - x^*\|^q
\]
(3.13)

By Theorem 2.13, we have
\[
R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*] \rightarrow R^{H(\cdot)}_{\lambda,M(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*].
\]
Let
\[
b_n = R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*] - R^{H(\cdot)}_{\lambda,M_n(\cdot)}[H(Ax^*, Bx^*) - \lambda w^*], \quad (3.14)
\]
then, $b_n \rightarrow 0$ as $n \rightarrow \infty$.

By using of (3.13) and (3.14), (3.12) becomes
\[
\|z_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|b_n\|,
\]
(3.15)
where $L_1 = \sqrt{\lambda \mu \delta^q_x + C_q (r_1 + r_2)^q - 2\lambda \mu \|Ax_n - Ax^*\|^q - \gamma \|Bx_n - Bx^*\|^q}$. 

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Since \(x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n\), \(x_{n+1} - x_n = \alpha_n(y_n - x_n)\), it follows that

\[
\|x_{n+1} - z_{n+1}\| = \|(1 - \alpha_n)x_n + \alpha_n y_n - [(1 - \alpha_n)x_n + \\
\alpha_n R_{\lambda, M_n} H(Ax_n, Bx_n) - \lambda w_n]\| \\
= \|\alpha_n \left[ y_n - R_{\lambda, M_n} H(Ax_n, Bx_n) - \lambda w_n \right]\| \\
= \|\alpha_n \sigma_n \left[ \|y_n - x_n\| + \lambda \|P_n - w_n\| \right]\| \\
\leq \alpha_n \sigma_n \|\alpha_n y_n - x_n\| + \alpha_n \sigma_n \lambda D(T(y_n), T(x_n)) \\
\leq \alpha_n \sigma_n \|y_n - x_n\| + \alpha_n \sigma_n \lambda T(x_n) ||y_n - x_n|| \\
= \alpha_n \sigma_n (1 + \lambda T(x_n)) ||y_n - x_n|| \\
= \sigma_n (1 + \lambda T(x_n)) ||x_{n+1} - x_n||. \tag{3.16}
\]

Using the above discussed arguments, we obtain that

\[
\|x_{n+1} - x^*\| \leq \|x_{n+1} - z_{n+1}\| + \|z_{n+1} - x^*\| \\
\leq \sigma_n (1 + \lambda T(x_n)) ||x_{n+1} - x_n|| + \{(1 - \alpha_n) + \alpha_n \theta L_1\} ||x_n - x^*|| + \alpha_n \|b_n\| \\
= \sigma_n (1 + \lambda T(x_n)) ||x_{n+1} - x^* + x^* - x_n|| + \{(1 - \alpha_n) + \alpha_n \theta L_1\} ||x_n - x^*|| + \alpha_n \|b_n\| \\
\leq \sigma_n (1 + \lambda T(x_n)) ||x_{n+1} - x^*|| + \sigma_n (1 + \lambda T(x_n)) ||x_{n} - x^*|| + \\
\{(1 - \alpha_n) + \alpha_n \theta L_1\} ||x_n - x^*|| + \alpha_n \|b_n\|
\]

which implies that

\[
\|x_{n+1} - x^*\| \leq \frac{\sigma_n (1 + \lambda T(x_n)) + \{(1 - \alpha_n) + \alpha_n \theta L_1\} ||x_n - x^*|| + \alpha_n \|b_n\|}{1 - \sigma_n (1 + \lambda T(x_n))} \\
= \frac{\alpha_n}{1 - \sigma_n (1 + \lambda T(x_n))} \|b_n\|. \tag{3.17}
\]

From (3.14) and (3.17), it follows that \(x_n\) converges to \(x^*\) linearly. Also from Algorithm 3.2 and \(D\)-Lipschitz continuity \(T\), we have

\[
\|w_n - w_{n-1}\| \leq \mathcal{D}(T(x_n), T(x_{n-1})) \\
\leq \delta_T ||x_n - x_{n-1}||. \tag{3.18}
\]

Since \(x_n\) converges to \(x^*\) linearly, it follows from (3.18) that \(w_n\) converges to \(w\) linearly. This completes the proof.

\[\square\]

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References


