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Balanced Degree-Magic Labelings of Complete Bipartite Graphs under Binary Operations

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Abstract. A graph is called supermagic if there is a labeling of edges where the edges are labeled with consecutive distinct positive integers such that the sum of the labels of all edges incident with any vertex is constant. A graph G is called degree-magic if there is a labeling of the edges by integers $1, 2, ..., |E(G)|$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1 + |E(G)|)\deg(v)/2$. Degree-magic graphs extend supermagic regular graphs. In this paper we find the necessary and sufficient conditions for the existence of balanced degree-magic labelings of graphs obtained by taking the join, composition, Cartesian product, tensor product and strong product of complete bipartite graphs.

Keywords: Complete bipartite graphs, Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

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1. Introduction

We consider simple graphs without isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e\in E(G)} \eta(v,e)f(e) \quad \text{for every} \quad v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labeling* of G for an *index* λ if its index mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all} \quad v \in V(G).$$

A magic labeling f of a graph G is called a *supermagic labeling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever a supermagic (magic) labeling of G exists.

A bijective mapping f from $E(G)$ into $\{1, 2, ..., |E(G)|\}$ is called a *degree-magic labeling* (or only *d-magic labeling*) of a graph G if its index mapping f^* satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2}\deg(v) \quad \text{for all} \quad v \in V(G).$$

A d-magic labeling f of a graph G is called *balanced* if for all $v \in V(G)$, the following equation is satisfied

$$\begin{aligned}|\{e \in E(G) \quad &: \eta(v,e) = 1, f(e) \leq \lfloor |E(G)|/2 \rfloor\}| \\ &= |\{e \in E(G) : \eta(v,e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor\}|.\end{aligned}$$

We say that a graph G is *degree-magic* (*balanced degree-magic*) or only *d-magic* when a d-magic (balanced d-magic) labeling of G exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; see [6, 7, 10] for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [2] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degree-magic and balanced degree-magic complete bipartite graphs in [2]. They also characterized degree-magic complete tripartite graphs in [4]. Some of these concepts are investigated in [1, 3, 5]. We will hereinafter use the auxiliary results from these studies.

Theorem 1.1. [2] *Let G be a regular graph. Then G is supermagic if and only if it is d-magic.*

Theorem 1.2. [2] *Let G be a d-magic graph of even size. Then every vertex of G has an even degree and every component of G has an even size.*

Theorem 1.3. [2] *Let G be a balanced d-magic graph. Then G has an even number of edges and every vertex has an even degree.*

Theorem 1.4. [2] *Let G be a d-magic graph having a half-factor. Then $2G$ is a balanced d-magic graph.*

Theorem 1.5. [2] *Let H_1 and H_2 be edge-disjoint subgraphs of a graph G which form its decomposition. If H_1 is d-magic and H_2 is balanced d-magic, then G is a d-magic graph. Moreover, if H_1 and H_2 are both balanced d-magic, then G is a balanced d-magic graph.*

Proposition 1.6. [2] *For $p, q > 1$, the complete bipartite graph $K_{p,q}$ is d-magic if and only if $p \equiv q \pmod 2$ and $(p, q) \neq (2, 2)$.*

Theorem 1.7. [2] *The complete bipartite graph $K_{p,q}$ is balanced d-magic if and only if the following statements hold:*
(i) $p \equiv q \equiv 0 \pmod 2$;
(ii) if $p \equiv q \equiv 2 \pmod 4$, then $\min\{p, q\} \geq 6$.

Lemma 1.8. [4] *Let m, n and o be even positive integers. Then the complete tripartite graph $K_{m,n,o}$ is balanced d-magic.*

2. Balanced Degree-Magic Labelings in the Join of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the *join* of graphs G and H, denoted by $G+H$, consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H. For any positive integers p and q, we consider the join $K_{p,q} + K_{p,q}$ of complete bipartite graphs. Let $K_{p,q} + K_{p,q}$ be a d-magic graph. Since $\deg(v)$ is $p + 2q$ or $2p + q$ and $f^*(v) = (2pq + (p + q)^2 + 1)\deg(v)/2$ for any $v \in V(K_{p,q} + K_{p,q})$, we have

Proposition 2.1. *Let $K_{p,q} + K_{p,q}$ be a d-magic graph. Then p or q is even.*

Proposition 2.2. *Let $K_{p,q} + K_{p,q}$ be a balanced d-magic graph. Then both p and q are even.*

Proposition 2.3. *Let p and q be even positive integers. Then $K_{p+q,p+q}$ is a balanced d-magic graph.*

Proof. Applying Theorem 1.7, $K_{p+q,p+q}$ is a balanced d-magic graph. □

In the next result we show a sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs $K_{p,q} + K_{p,q}$.

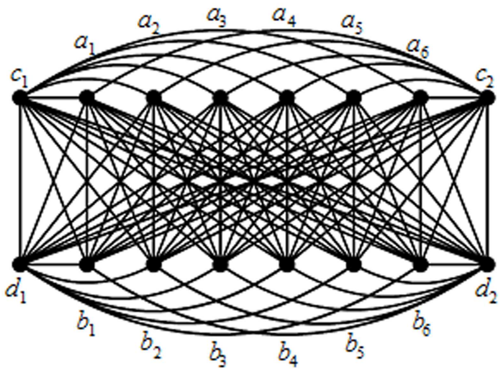

FIGURE 1. A balanced d-magic graph $K_{2,6} + K_{2,6}$ with 16 vertices and 88 edges.

Theorem 2.4. *Let p and q be even positive integers. Then $K_{p,q} + K_{p,q}$ is a balanced d-magic graph.*

Proof. Let p and q be even positive integers. We consider the following two cases:
Case I. If $(p, q) = (2, 2)$, the graph $K_{2,2} + K_{2,2}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{2,4}$. According to Theorem 1.5, $K_{2,2} + K_{2,2}$ is a balanced d-magic graph.
Case II. If $(p, q) \neq (2, 2)$, then $K_{p+q,p+q}$ is balanced d-magic by Proposition 2.3, and $2K_{p,q}$ is balanced d-magic by Theorem 1.4. Since $K_{p,q} + K_{p,q}$ is the graph such that $K_{p+q,p+q}$ and $2K_{p,q}$ form its decomposition, by Theorem 1.5, $K_{p,q} + K_{p,q}$ is a balanced d-magic graph. □

We know that $K_{2,6}$ is d-magic, but it is not balanced d-magic. Applying Theorem 2.4, we can construct a balanced d-magic graph $K_{2,6} + K_{2,6}$ (see Figure 1) with the labels on edges of $K_{2,6} + K_{2,6}$ in Table 2.

We will now generalize to find the necessary and sufficient conditions for the existence of balanced d-magic labelings of the join of complete bipartite graphs in a general form. For any positive integers p, q, r and s, we consider the join $K_{p,q} + K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} + K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is $p + r + s$, $q + r + s$, $p + q + r$ or $p + q + s$ and $f^*(v) = (pq + (p + q)(r + s) + rs + 1) \deg(v)/2$ for any $v \in V(K_{p,q} + K_{r,s})$, we have

Proposition 2.5. *Let $K_{p,q} + K_{r,s}$ be a d-magic graph. Then the following conditions hold:*
(i) only one of p, q, r and s is even or
(ii) only two of p, q, r and s are even or
(iii) all of p, q, r and s are even.

Vertices	a_1	a_2	a_3	a_4	a_5	a_6	c_1	c_2	d_1	d_2
b_1	15	70	75	26	23	62	18	67	1	88
b_2	74	16	17	63	66	24	71	25	11	78
b_3	69	19	14	68	61	27	76	22	3	86
b_4	36	57	56	37	44	49	29	48	85	4
b_5	31	54	59	42	39	46	34	51	84	5
b_6	58	32	33	47	50	40	55	41	83	6
d_1	20	73	72	21	28	65	13	64	-	-
d_2	53	35	30	52	45	43	60	38	-	-
c_1	77	87	79	9	8	7	-	-	-	-
c_2	12	2	10	80	81	82	-	-	-	-

Table 1. The labels on edges of balanced d-magic graph $K_{2,6} + K_{2,6}$.

Proposition 2.6. *Let $K_{p,q} + K_{r,s}$ be a balanced d-magic graph. Then p, q, r and s are even.*

Now we are able to show a sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs $K_{p,q} + K_{r,s}$.

Theorem 2.7. *Let p, q, r and s be even positive integers. Then $K_{p,q} + K_{r,s}$ is a balanced d-magic graph.*

Proof. Let p, q, r and s be even positive integers. We consider the following two cases:
Case I. If at least one of p, q, r and s is not congruent to 2 modulo 4. Suppose that p is not congruent to 2 modulo 4. Thus, $K_{p,q}$ is balanced d-magic by Theorem 1.7. Since r, s and $p + q$ are even, $K_{r,s,p+q}$ is balanced d-magic by Lemma 1.8. The graph $K_{p,q} + K_{r,s}$ is decomposable into two balanced d-magic subgraphs isomorphic to $K_{p,q}$ and $K_{r,s,p+q}$. According to Theorem 1.5, $K_{p,q} + K_{r,s}$ is a balanced d-magic graph.
Case II. If p, q, r and s are congruent to 2 modulo 4. Thus $q + r, q + s$ and $p + q$ are not congruent to 2 modulo 4. By Theorem 1.7, $K_{p,q+r}, K_{r,q+s}$ and $K_{s,p+q}$ are balanced d-magic. The graph $K_{p,q} + K_{r,s}$ is decomposable into three balanced d-magic subgraphs isomorphic to $K_{p,q+r}$, $K_{r,q+s}$ and $K_{s,p+q}$. According to Theorem 1.5, $K_{p,q} + K_{r,s}$ is a balanced d-magic graph. □

Corollary 2.8. *Let p, q, r and s be even positive integers. If $p = q = r = s$, then $K_{p,q} + K_{r,s}$ is a supermagic graph.*

Proof. Applying Theorems 1.1 and 2.7. □

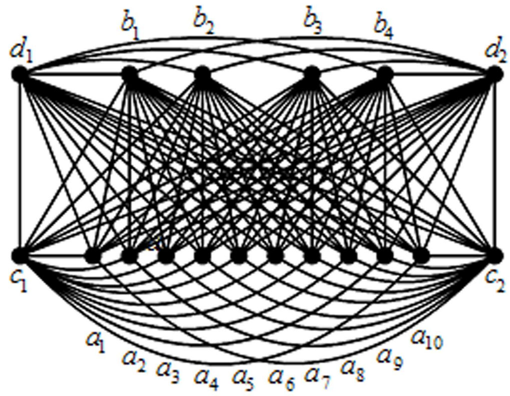

Figure 2. A balanced d-magic graph $K_{2,4} + K_{2,10}$ with 18 vertices and 100 edges.

Vertices	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	c_1	c_2	d_1	d_2
b_1	31	70	79	22	57	61	42	41	58	44	85	16	100	1
b_2	23	78	71	30	45	52	54	53	49	50	84	17	2	99
b_3	77	24	29	72	56	46	48	47	55	51	18	83	3	98
b_4	76	25	28	73	39	43	59	60	40	62	19	82	97	4
d_1	75	26	27	74	38	64	65	36	67	33	81	20	-	-
d_2	21	80	69	32	68	37	35	66	34	63	15	86	-	-
c_1	96	6	7	93	92	10	11	89	88	14	-	-	-	-
c_2	5	95	94	8	9	91	90	12	13	87	-	-	-	-

Table 2. The labels on edges of balanced d-magic graph $K_{2,4} + K_{2,10}$.

Since 4 is not congruent to 2 modulo 4, applying Theorem 2.7, a balanced d-magic graph $K_{2,4} + K_{2,10}$ is constructed (see Figure 2), and the labels on edges of $K_{2,4} + K_{2,10}$ are shown in Table 2.

3. Balanced Degree-Magic Labelings in the Composition of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the *composition* of graphs G and H, denoted by $G \cdot H$, is a graph such that the vertex set of $G \cdot H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G \cdot H$ if and only if either u is adjacent with x in G or $u = x$ and v is adjacent with y in H. For any positive integers p, q, r and s, we consider the composition $K_{p,q} \cdot K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \cdot K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is $(r + s)p + r$, $(r + s)p + s$, $(r + s)q + r$ or $(r + s)q + s$ and

$f^*(v) = (pq(r+s)^2 + rs(p+q) + 1)\deg(v)/2$ for any $v \in V(K_{p,q} \cdot K_{r,s})$, we have

Proposition 3.1. *Let $K_{p,q} \cdot K_{r,s}$ be a d-magic graph. Then the following conditions hold:*
(i) only one of p, q, r and s is even or
(ii) at least both r and s are even.

Proposition 3.2. *Let $K_{p,q} \cdot K_{r,s}$ be a balanced d-magic graph. Then at least both r and s are even.*

In the next result we find a sufficient condition for the existence of balanced d-magic labelings of the composition of complete bipartite graphs $K_{p,q} \cdot K_{r,s}$.

Theorem 3.3. *Let p and q be positive integers, and let r and s be even positive integers. Then $K_{p,q} \cdot K_{r,s}$ is a balanced d-magic graph.*

Proof. Let p and q be positive integers, and let $k = \min\{p, q\}$ and $h = \max\{p, q\}$. It is clear that $K_{r+s,r+s}$, $K_{r,s} + K_{r,s}$ and $K_{r,s,r+s}$ are balanced d-magic by Proposition 2.3, Theorem 2.4 and Lemma1.8, respectively. The graph $K_{p,q} \cdot K_{r,s}$ is decomposable into k balanced d-magic subgraphs isomorphic to $K_{r,s}+K_{r,s}$, $h(k-1)$ balanced d-magic subgraphs isomorphic to $K_{r+s,r+s}$ and $h-k$ balanced d-magic subgraphs isomorphic to $K_{r,s,r+s}$. According to Theorem 1.5, $K_{p,q} \cdot K_{r,s}$ is a balanced d-magic graph. □

Notice that the graph composition $K_{p,q} \cdot K_{r,s}$ is naturally nonisomorphic to $K_{r,s} \cdot K_{p,q}$ except for the case $(p, q) = (r, s)$.

Corollary 3.4. *Let p and q be positive integers, and let r and s be even positive integers. If $p = q$ and $r = s$, then $K_{p,q} \cdot K_{r,s}$ is a supermagic graph.*

Proof. Applying Theorems 1.1 and 3.3. □

The following example is a balanced d-magic graph $K_{1,2} \cdot K_{2,2}$ (see Figure 3) with the labels on edges of $K_{1,2} \cdot K_{2,2}$ in Table 3.

4. Balanced Degree-Magic Labelings in the Cartesian Product of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the *Cartesian product* of graphs G and H, denoted by $G \times H$, is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G \times H$ if and only if either $u = x$ and v is adjacent with y in H or $v = y$ and u is adjacent with x in G. For any positive integers p, q, r and s, we consider the Cartesian product $K_{p,q} \times K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \times K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is $p+r$, $p+s$, $q+r$ or $q+s$ and $f^*(v) = (pq(r+s) + rs(p+q) + 1)\deg(v)/2$ for any $v \in V(K_{p,q} \times K_{r,s})$, we have

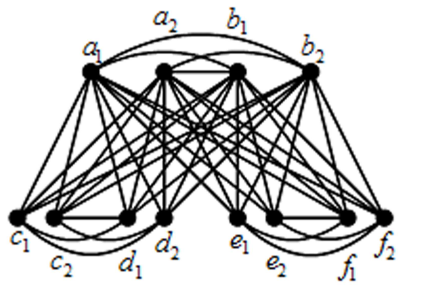

FIGURE 3. A balanced d-magic graph $K_{1,2} \cdot K_{2,2}$ with 12 vertices and 44 edges.

Vertices	c_1	c_2	d_1	d_2	e_1	e_2	f_1	f_2	b_1	b_2
a_1	12	34	43	2	27	26	20	17	35	9
a_2	33	11	1	44	19	18	25	28	10	36
b_1	8	38	39	5	32	14	15	29	-	-
b_2	37	7	6	40	13	31	30	16	-	-
d_1	4	42	-	-	-	-	-	-	-	-
d_2	41	3	-	-	-	-	-	-	-	-
f_1	-	-	-	-	23	22	-	-	-	-
f_2	-	-	-	-	21	24	-	-	-	-

TABLE 3. The labels on edges of balanced d-magic graph $K_{1,2} \cdot K_{2,2}$.

Proposition 4.1. *Let $K_{p,q} \times K_{r,s}$ be a d-magic graph. Then the following conditions hold:*
(i) only one of p, q, r and s is even or
(ii) all of p, q, r and s are either odd or even.

Proposition 4.2. *Let $K_{p,q} \times K_{r,s}$ be a balanced d-magic graph. Then p, q, r and s are either odd or even.*

In the next result we are able to find a sufficient condition for the existence of balanced d-magic labelings of the Cartesian product of complete bipartite graphs $K_{p,q} \times K_{r,s}$.

Theorem 4.3. *Let p, q, r and s be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Then $K_{p,q} \times K_{r,s}$ is a balanced d-magic graph.*

Proof. Let p, q, r and s be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Since $K_{p,q}$ and $K_{r,s}$ are d-magic by Proposition 1.6, $2K_{p,q}$ and $2K_{r,s}$ are balanced d-magic by Theorem 1.4. The graph $K_{p,q} \times K_{r,s}$ is decomposable into $(r + s)/2$ balanced d-magic subgraphs isomorphic to $2K_{p,q}$ and $(p + q)/2$

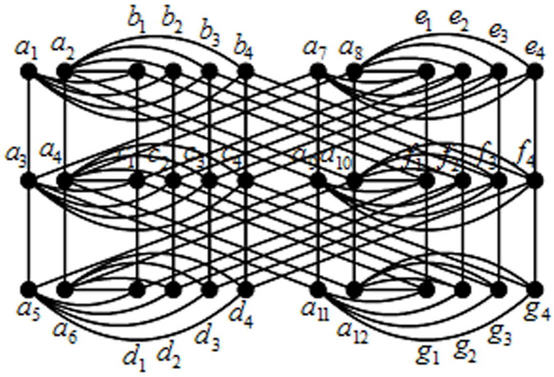

FIGURE 4. A balanced d-magic graph $K_{2,4} \times K_{2,4}$ with 36 vertices and 96 edges.

balanced d-magic subgraphs isomorphic to $2K_{r,s}$. According to Theorem 1.5, $K_{p,q} \times K_{r,s}$ is a balanced d-magic graph. □

Observe that the Cartesian product graph $K_{p,q} \times K_{r,s}$ is naturally isomorphic to $K_{r,s} \times K_{p,q}$.

Corollary 4.4. *Let p, q, r and s be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. If $p = q$ and $r = s$, then $K_{p,q} \times K_{r,s}$ is a supermagic graph.*

Proof. Applying Theorems 1.1 and 4.3. □

The following example is a balanced d-magic graph $K_{2,4} \times K_{2,4}$ (see Figure 4), and the labels on edges of $K_{2,4} \times K_{2,4}$ are shown in Table 4.

Vertices	b_1	b_2	b_3	b_4	e_1	e_2	e_3	e_4	a_3	a_4	a_9	a_{10}
a_1	96	2	3	93	-	-	-	-	72	-	25	-
a_2	1	95	94	4	-	-	-	-	-	64	-	33
a_7	-	-	-	-	8	90	91	5	27	-	70	-
a_8	-	-	-	-	89	7	6	92	-	35	-	62
c_1	48	-	-	-	51	-	-	-	88	9	-	-
c_2	-	32	-	-	-	67	-	-	10	87	-	-
c_3	-	-	40	-	-	-	59	-	11	86	-	-
c_4	-	-	-	56	-	-	-	43	85	12	-	-
f_1	49	-	-	-	46	-	-	-	-	-	16	81
f_2	-	65	-	-	-	30	-	-	-	-	82	15
f_3	-	-	57	-	-	-	38	-	-	-	83	14
f_4	-	-	-	41	-	-	-	54	-	-	13	84

Vertices	d_1	d_2	d_3	d_4	g_1	g_2	g_3	g_4	a_3	a_4	a_9	a_{10}
a_5	24	74	75	21	-	-	-	-	26	-	71	-
a_6	73	23	22	76	-	-	-	-	-	34	-	63
a_{11}	-	-	-	-	80	18	19	77	69	-	28	-
a_{12}	-	-	-	-	17	79	78	20	-	61	-	36
c_1	50	-	-	-	45	-	-	-	-	-	-	-
c_2	-	66	-	-	-	29	-	-	-	-	-	-
c_3	-	-	58	-	-	-	37	-	-	-	-	-
c_4	-	-	-	42	-	-	-	53	-	-	-	-
f_1	47	-	-	-	52	-	-	-	-	-	-	-
f_2	-	31	-	-	-	68	-	-	-	-	-	-
f_3	-	-	39	-	-	-	60	-	-	-	-	-
f_4	-	-	-	55	-	-	-	44	-	-	-	-

Table 4. The labels on edges of balanced d-magic graph $K_{2,4} \times K_{2,4}$.

5. Balanced Degree-Magic Labelings in the Tensor Product of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the *tensor product* of graphs G and H, denoted by $G \oplus H$, is a graph such that the vertex set of $G \oplus H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G \oplus H$ if and only if u is adjacent with x in G and v is adjacent with y in H. For any positive integers p, q, r and s, we consider the tensor product $K_{p,q} \oplus K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \oplus K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is pr, ps, qr or qs and $f^*(v) = (2pqrs + 1)\deg(v)/2$ for any $v \in V(K_{p,q} \oplus K_{r,s})$, we have

Proposition 5.1. *Let $K_{p,q} \oplus K_{r,s}$ be a balanced d-magic graph. Then p and q are even or r and s are even.*

Now we can prove a sufficient condition for the existence of balanced d-magic labelings of the tensor product of complete bipartite graphs $K_{p,q} \oplus K_{r,s}$.

Theorem 5.2. *Let p and q be positive integers with $(p, q) \neq (1, 1)$. Then $K_{p,q} \oplus K_{2,2}$ is a balanced d-magic graph.*

Proof. Let p and q be positive integers with $(p, q) \neq (1, 1)$. Let $k = \min\{p, q\}$ and $h = \max\{p, q\}$. Since $K_{2,2h}$ is d-magic by Proposition 1.6, $2K_{2,2h}$ is balanced d-magic by Theorem 1.4. The graph $K_{p,q} \oplus K_{2,2}$ is decomposable into k balanced d-magic subgraphs isomorphic to $2K_{2,2h}$. According to Theorem 1.5, $K_{p,q} \oplus K_{2,2}$ is a balanced d-magic graph. □

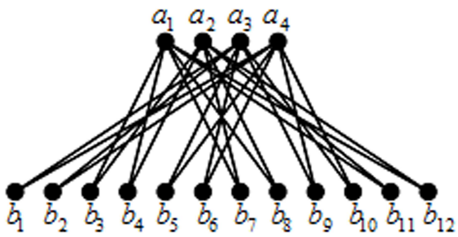

FIGURE 5. A balanced d-magic graph $K_{1,3} \oplus K_{2,2}$ with 16 vertices and 24 edges.

Vertices	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
a_1	-	-	1	11	-	-	3	21	-	-	20	19
a_2	-	-	24	14	-	-	22	4	-	-	5	6
a_3	13	23	-	-	15	9	-	-	8	7	-	-
a_4	12	2	-	-	10	16	-	-	17	18	-	-

TABLE 5. The labels on edges of balanced d-magic graph $K_{1,3} \oplus K_{2,2}$.

Theorem 5.3. *Let p and q be positive integers, and let r and s be even positive integers with $(r, s) \neq (2, 2)$. Then $K_{p,q} \oplus K_{r,s}$ is a balanced d-magic graph.*

Proof. Let p and q be positive integers, and let r and s be even positive integers with $(r, s) \neq (2, 2)$. Since $K_{r,s}$ is d-magic by Proposition 1.6, $2K_{r,s}$ is balanced d-magic by Theorem 1.4. The graph $K_{p,q} \oplus K_{r,s}$ is decomposable into pq balanced d-magic subgraphs isomorphic to $2K_{r,s}$. According to Theorem 1.5, $K_{p,q} \oplus K_{r,s}$ is a balanced d-magic graph. □

It is clear that the tensor product graph $K_{p,q} \oplus K_{r,s}$ is isomorphic to $K_{r,s} \oplus K_{p,q}$.

Corollary 5.4. *Let p, q be positive integers with $(p, q) \neq (1, 1)$, and let r, s be even positive integers. If $p = q$ and $r = s$, then $K_{p,q} \oplus K_{r,s}$ is a supermagic graph.*

Proof. Applying Theorems 1.1, 5.2 and 5.3. □

Below is an example of balanced d-magic graph $K_{1,3} \oplus K_{2,2}$ (see Figure 5), and the labels on edges of $K_{1,3} \oplus K_{2,2}$ are shown in Table 5.

6. Balanced Degree-Magic Labelings in the Strong Product of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the *strong product* of graphs G and H, denoted by $G \otimes H$, is a graph such that the vertex set of $G \otimes H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G \otimes H$ if and only if $u = x$ and v is adjacent with y in H, or $v = y$ and u is adjacent with x in G, or u is adjacent with x in G and v is adjacent with y in H. For any positive integers p, q, r and s, we consider the strong product $K_{p,q} \otimes K_{r,s}$ of complete bipartite graphs. Let $K_{p,q} \otimes K_{r,s}$ be a d-magic graph. Since $\deg(v)$ is $p + r + pr$, $p + s + ps$, $q + r + qr$ or $q + s + qs$ and $f^*(v) = (pq(r+s) + rs(p+q) + 2pqrs + 1)\deg(v)/2$ for any $v \in V(K_{p,q} \otimes K_{r,s})$, we have

Proposition 6.1. *Let $K_{p,q} \otimes K_{r,s}$ be a d-magic graph. Then the following conditions hold:*
(i) only one of p, q, r and s is even or
(ii) all of p, q, r and s are even.

Proposition 6.2. *Let $K_{p,q} \otimes K_{r,s}$ be a balanced d-magic graph. Then p, q, r and s are even.*

We conclude this paper with an identification of the sufficient condition for the existence of balanced d-magic labelings of the strong product of complete bipartite graphs $K_{p,q} \otimes K_{r,s}$.

Theorem 6.3. *Let p, q, r and s be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Then $K_{p,q} \otimes K_{r,s}$ is a balanced d-magic graph.*

Proof. Let p, q, r and s be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. Thus, $K_{p,q} \times K_{r,s}$ is balanced d-magic by Theorem 4.3, and $K_{p,q} \oplus K_{r,s}$ is balanced d-magic by Theorem 5.3. Since $K_{p,q} \otimes K_{r,s}$ is the graph such that $K_{p,q} \times K_{r,s}$ and $K_{p,q} \oplus K_{r,s}$ form its decomposition, by Theorem 1.5, $K_{p,q} \otimes K_{r,s}$ is a balanced d-magic graph. □

It is clear that the strong product graph $K_{p,q} \otimes K_{r,s}$ is isomorphic to $K_{r,s} \otimes K_{p,q}$.

Corollary 6.4. *Let p, q, r and s be even positive integers with $(p, q) \neq (2, 2)$ and $(r, s) \neq (2, 2)$. If $p = q$ and $r = s$, then $K_{p,q} \otimes K_{r,s}$ is a supermagic graph.*

Proof. Applying Theorems 1.1 and 6.3. □

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