Iranian Journal of Mathematical Sciences and Informatics
Vol. 15, No. 1 (2020), pp 85-97
DOI: 10.21859/IJMSI.15.1.85

Quotient G-systems and Green's relations

Sohrab Ostadhadi-Dehkordi[a,*], Kar Ping Shum[b]

[a]Department of Mathematics, University of Hormozgan, Bandar Abbas, Iran.
[b]Institute of Mathematics,Yunnan University, Kunming,650091, P.R. China.

E-mail: Ostadhadi-Dehkordi@hotmail.com
Ostadhadi@hormozgan.ac.ir
E-mail: kpshum@ynu.edu.cn

Abstract. In this paper, we first introduce the concepts of G-systems, quotient G-systems and isomorphism theorems on G-systems of n-ary semihypergroups. Also we consider the Green's equivalences on G-systems and further investigate some of their properties. A number of n-ary semihypergroups are constructed and presented as examples in this paper.

Keywords: n-Ary semihypergroup, G-Systems, Greens relations.

2010 Mathematics Subject Classification: 20N15 .

1. Introduction

The concept of Green relation introduced by Green [9] and have played a fundamental role in the development of semigroup theory. The Green's relations provide the necessary tools for using similar arguments on the monoid. The Green's relations are well known, and presented in deep detail in several places. The concept of an n-group was introduced by Dörnte in [7]. This concept is a natural generalization of a group. Since then there are numbers of papers concerning various n-ary algebras in the literature. It is noted that the algebraic hyperstructures are suitable generalizations of the classical algebraic structures. In a classical algebraic structure, the composition of two elements

*Corresponding Author

Received 10 May 2017; Accepted 08 September 2017

is an element, while in an algebraic hyperstructure, the composition of two elements is not necessarily an element but is a system. The notion of the hypergroup was introduced in 1934 by Marty [13] at the 8^{th} Congress of Scandinavian Mathematicians. He then published some notes on hypergroups,using them in different contexts such as algebraic functions,rational actions, non commutative groups. Since then, hundreds of research papers and several monographs have been published in this topic and several kind of hypergroups have been particularly studied, such as regular hypergroups, reversible regular hypergroups, canonical hypergroups, cogroups, cyclic hypergroups, associativity hypergroups, for example see ([2] and [3]). (the monograph by P. Corsini and V.Leoreanu)

A recent monograph on hyperstructures [5] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Moreover, Davvaz and Vougiouklis [6] have established a connection between the two do mains in the form of an extension of the concept of n-ary groups to the concept of n-ary hypergroups. They determined some connections between this hypergroupoid and Ivo Rosenberg s hypergroupoid associated with a binary relation. In [4] Cristea and Stefanescu, associated a hypergroupoid $(H, \otimes_\rho)$ with an n-ary relation ρ defined on a non-empty set H.The n-ary hyperoperations studied by many researchers, for example, see [2, 3, 4, 6, 10, 11, 12, 14, 15].

These Green's relations on a semigroup were first defined and studied by Green [9] dated back to 1951. These Green's relations on a semigroup played a fundamental role in the development of semigroup theory. In particular, Chinram and Siammai have considered and studied extensively the Green's relations on the Γ-semigroups and reductive Γ-semigroups [1]. Also, the Green's relations and congruences on n-ary semigroups were studied and are investigated by Sioson in 1967 [16].

In this paper, we define the left and right G-systems in the context of n-ary semihypergroups and introduce the concept of regular relation on the G-systems. Also, we consider the Green's equivalence relations on the G-systems and find some of their properties.

2. Basic definition

In this section, we present some definitions concerning the n-ary semihypergroups as a generalization of the n-ary semigroups and semigroups.

Let G be a non-empty set and f a mapping $f : G \times G \longrightarrow \wp^*(G)$, where $\wp^*(G)$ is the set of all non-empty subsets of G. Then, we call f is a *binary hyperoperation* on the set G. We denote by G^n the Cartesian product $G \times G.... \times G$, where G appears n times. The couple (G, f) is now called a *hypergroupoid*. For

any two non-empty subsets G_1 and G_2 of G, we define

$$G_1 \circ G_2 = \bigcup_{g_1 \in G_1, g_2 \in G_2} g_1 \circ g_2.$$

A hypergroupoid (G, f) is called a *semihypergroup* if for all g_1, g_2, g_3 of G, we have $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

In general, $f : G^n \longrightarrow \wp^*(G)$ is called an *n-ary hyperoperation* on G and we call (G, f) an *n-ary hypergroupoid.*(see [3]).

Let G_1, G_2,...,G_n be a non-empty subsets of G. Then, we define

$$f(G_1, G_2, ..., G_n) = \bigcup \{f(g_1, g_2, ..., g_n) : g_i \in G_i, i \in \{1, 2, ..., n\}\}.$$

The sequence $g_i, g_{i+1}, ..., g_{j+1}$ will be denoted by g_i^j. For $j < i$, g_i^j is a empty set.

Definition 2.1. [8] The n-ary hypergroupoid (G, f) is called an *n-ary semihypergroup* if for any $i, j \in \{1, 2, ..., n\}$ and g_1^{2n-1},

$$f(g_1^{i-1}, f(g_i^{n+i-1}), g_{n+i}^{2n-1}) = f(g_1^{i-1}, f(g_j^{n+j-1}), g_{n+j}^{2n-1}).$$

We call G an n-ary semihypergroup with identity if there is an element $e \in G$ such that

$$x \in f\left(e^{(i-1)}, x, e^{(n-i)}\right).$$

Let (G, f) be an n-ary semihypergroup and H be a non-empty subset of G. Then, H is an n-ary subhypergroup of G if it is closed under the n-ary hyperoperation f, i.e., for every $(h_1, h_2, ..., h_n) \in H^n$ implies that $f(h_1, h_2, ..., h_n) \subseteq H$.

The n-ary semihypergroup (G, f) with the equation $g \in f\left(g_1^{i-1}, x_i, g_{i+1}^n\right)$ has the solution $x_i \in G$ for any $g_1^{i-1}, x_i, g_{i+1}^n \in G$ and $1 \leq i \leq n$, is called an *n-ary hypergroup.*

An n-ary semihypergroup (G, f) is *commutative* if for all $g_1^n \in G$ and for any permutation σ of $\{1, 2, ..., n\}$, we have

$$f(g_1^n) = f(g_{\sigma(1)}, g_{\sigma(2)}, ..., g_{\sigma(n)}).$$

Let (G_1, f_1) and (G_2, f_2) be two n-ary semihypergroups. Then, a mapping $\varphi : G_1 \longrightarrow G_2$ is called a *homomorphism* if for all $x_1^n \in G_1$ we have

$$\varphi(f_1(x_1, x_2, ..., x_n)) = f_2(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n)).$$

when G_1 and G_2 are n-ary semihypergroups with a scaler identity, $\varphi(e_1) = e_2$.

The following examples are easy examples of n-ary hypergroup.

EXAMPLE 2.2. Let $(G,+)$ be a semihypergroup and f an n-ary hyperoperation on G defined by

$$f(g_1^n)=\sum_{i=1}^{n} g_i,\ \forall g_1^n \in G.$$

Then, we can verify that (G,f) is an n-ary semihypergroup.

EXAMPLE 2.3. Let G be a group and $< x,y >$ a subgroup generated by x,y. Define

$$f(g_1,g_2,...,g_n)=< g_1,g_2,...,g_n >,$$

on G. Then, one can verify that (G,f) is an n-ary hypergroup.

EXAMPLE 2.4. Let G be a semigroup and N a normal subsemigroup of G. For all $g_1^n \in G$, define $f(g_1,g_2,...,g_n)=g_1g_2...g_nH$. Then, obviously, (G,f) is an n-ary semihypergroup.

EXAMPLE 2.5. Let D be an integral domain and F be its field of fractions. Denote by U the group of the invertible elements of D. Define the following n-ary hyperoperation on F/U. For all $\overline{g_i}$, with $1 \leq i \leq n$,

$$f(\overline{g_1},\overline{g_2},...,\overline{g_n})=\{\overline{g} : \exists u_1^n \in U^n, g=u_1g_1+u_2g_2+...+u_ng_n\}.$$

Then, we can easily verify that $(F/U,f)$ is an n-ary semihypergroup.

We now construct the following non-trivial n-ary semihypergroups.

EXAMPLE 2.6. Let V be a vector space over an ordered field F and $x_1,x_2,...,x_n \in V$. Then, we define

$$f(x_1,x_2,...,x_n)=\{\lambda_1x_1+\lambda_2x_2+...+\lambda_nx_n : \lambda_i > o, \Sigma_{i=1}^{n}\lambda_i=1\}.$$

Hence, (V,f) is an n-ary semihypergroup.

EXAMPLE 2.7. Let G be a semigroup and $\{A_g\}_{g\in G}$ be a collection of non-empty distinct sets and $S=\bigcup_{g\in G} A_g$. For every $x_1,x_2,...,x_n \in S$, we define

$$f(x_1,x_2,...,x_n)=A_{g_1g_2,...,g_n},$$

where $x_i \in A_{g_i}$ for some $1 \leq i \leq n$. Then, S is an n-ary semihypergroup.

3. G-SYSTEMS AND REGULAR RELATIONS

In this section, we define the G-systems, the regular relations and prove the isomorphism theorems for G-systems.

Let (G,f) be an n-ary semihypergroup with identity and X be a non-empty set. We say that X is a *left G-system* if there is an action $h : G^{n-1} \times X \longrightarrow X$ with the properties:

$$h\left(g_1^{n-1},h\left(u_1^{n-1},x\right)\right)=h\left(f(g_1^{n-1},u_1),f(g_1^{n-1},u_2),...,f(g_1^{n-1},u_{n-1}),x\right)$$
$$h\left(e^{n-1},x\right)=x,$$

for every $x \in X$ and $g_1^{2n-1}, u_1^{n-1} \in G$.

Dually, we call a non-empty set X is a right G-system if there is an action $X \times G^{n-1} \longrightarrow X$,

$$h\left(h\left(x, u_1^{n-1}\right), g_1^{n-1}\right) = h\left(x, f\left(u_1^{n-1}, g_1\right), f\left(u_1^{n-1}, g_2\right), ..., f\left(u_1^{n-1}, g_{n-1}\right)\right).$$

$$h\left(x, e^{n-1}\right) = x.$$

We first state the G-systems and discuss the G-systems.

Example 3.1. Let G be an n-ary semihypergroup with identity and X be an n-ary sub-semihypergroup of G. We define

$$\begin{array}{rl} h : G^{n-1} \times X & \longrightarrow X. \\ (g_1^{n-1}, x) & \longmapsto e, \end{array}$$

where e is an identity and $g_1^{n-1} \in G$ and $x \in X$. Then, X is a G-system.

Example 3.2. Let $G = \bigcup_{n\geq 0} A_n$, $A_0 = \{0\}$, $A_n = [n, n+1)$ and $X = \mathbb{Z}^+$. Then, we define

$$\begin{array}{rl} f : G^n & \longrightarrow P^*(G) \\ (g_1^n) & \longmapsto A_t, \end{array}$$

where $t = max\{m_1, m_2, ..., m_n\}$ and $g_i \in A_{m_i}$. Then, (G, f) is an n-ary semihypergroup. Also,

$$\begin{array}{rl} h : G^{n-1} \times X & \longrightarrow X \\ (g_1^{n-1}, x) & \longmapsto max\{m_1, m_2, ..., m_{n-1}, x\}. \end{array}$$

Then, X is a G-system.

Let G and H be n-ary semihypergroups. Then, we say that X is a (G, H)-*system* if it is a left G-system by action $h_1 : G^{n-1} \times X \longrightarrow X$ and a right H-system by action $h_2 : X \times H^{n-1} \longrightarrow X$ and

$$h_2(h_1(g_1^{n-1}, x), t_1^{n-1}) = h_1(g_1^{n-1}, h_2(x, t_1^{n-1})),$$

where $g_1^{n-1} \in G^{n-1}, t_1^{n-1} \in H^{n-1}$ and $x \in X$.

Let X and Y be left G-systems and $h_1 : G^{n-1} \times X \longrightarrow X$ and $h_2 : G^{n-1} \times Y \longrightarrow Y$. Then, a map $\varphi : X \longrightarrow Y$ is a *morphism* when

$$\varphi\left(h_1\left(g_1^{n-1}, x\right)\right) = h_2\left(g_1^{n-1}, \varphi(x)\right).$$

Let $Mor(X, Y)$ be the set of all G-morphism from X into Y and X, Y are left G-systems such that $h_1 : G^{n-1} \times X \longrightarrow X$ and $h_2 : G^{n-1} \times Y \longrightarrow Y$. Then, we define

$$\begin{array}{rl} h : G^{n-1} \times Mor(X, Y) & \longrightarrow Mor(X, Y) \\ (g_1^{n-1}, \varphi) & \longmapsto \overline{\varphi}, \end{array}$$

where $\overline{\varphi} : X \longrightarrow Y$ and $\overline{\varphi}(x) = h_2\left(g_1^{n-1}, \varphi(x)\right)$. Hence

$$\begin{aligned}\overline{\varphi}(h_1\left(g_1^{n-1}, x\right)) &= h_2\left(g_1^{n-1}, \varphi(h_1\left(g_1^{n-1}, x\right))\right)\\ &= h_2\left(g_1^{n-1}, h_2\left(g_1^{n-1}, \varphi(x)\right)\right)\\ &= h_2\left(g_1^{n-1}, \overline{\varphi}(x)\right).\end{aligned}$$

This implies that $\overline{\varphi} \in Mor(X, Y)$. Moreover, we have the following equalities :

$$\begin{aligned}h\left(g_1^{n-1}, h\left(k_1^{n-1}, \varphi\right)\right)(x) &= h_2\left(k_1^{n-1}, h\left(g_1^{n-1}, \varphi\right)(x)\right)\\ &= h_2\left(k_1^{n-1}, h_2\left(g_1^{n-1}, \varphi(x)\right)\right)\\ &= h_2\left(f\left(k_1^{n-1}, g_1\right), ..., f\left(k_1^{n-1}, g_{n-1}\right), \varphi(x)\right).\end{aligned}$$

Then,

$$h\left(g_1^{n-1}, h\left(k_1^{n-1}, \varphi\right)\right) = h\left(f\left(k_1^{n-1}, g_1\right), ..., f\left(k_1^{n-1}, g_{n-1}\right), \varphi\right)$$

Hence $Mor(X, Y)$ is a left G-system.

It is clear that the cartesian product $X \times Y$ of a left G_1-system X and a right G_2-system Y become (G_1, G_2)-system by the following definitions:

$$\overline{h}_1(g_1^{n-1}, (x, y)) = (h_1(g_1^{n-1}, x), y),$$

$$\overline{h}_2((x, y), t_1^{n-1}) = (x, h_2(y, t_1^{n-1}),$$

where $x \in X$, $y \in Y$, $g_i \in G_1$ and $t_j \in G_2$, for $1 \leq i \leq n$ and $1 \leq j \leq n$.

Let G be an n-ary semihypergroup and X be a left G-system. A relation ρ on the left G-system is called *left regular* if

$$\forall x_1, x_2 \in X,\ g_1^{n-1} \in G,\ x_1 \rho x_2 \Longrightarrow h\left(g_1^{n-1}, x_1\right) \rho\ h\left(g_1^{n-1}, x_2\right),$$

when X is a right G-system and

$$\forall x_1, x_2 \in X, g_1^{n-1} \in G,\ x_1 \rho x_2 \Longrightarrow h\left(x_1, g_1^{n-1}\right) \rho\ h\left(x_2, g_1^{n-1}\right),$$

then, X is a *right regular* relation. Now, we simply call a relation ρ is a *regular* relation if it is both a left and a right regular relation.

Let X be a left G-system and ρ be a regular relation on X. Then, $[X : \rho] = \{\rho(x) : x \in X\}$ is a left G-system by the following map:

$$\begin{aligned}\overline{h} : G^{n-1} \times [X : \rho] &\longrightarrow [X : \rho]\\ \left(g_1^{n-1}, \rho(x)\right) &\longmapsto \rho\left(h\left(g_1^{n-1}, x\right)\right).\end{aligned}$$

Let X_1 and X_2 be left G-systems and $\varphi : X_1 \longrightarrow X_2$ be a morphism. Then, we call a relation $ker(\varphi) = \{(x_1, x_2) \in X \times X : \varphi(x_1) = \varphi(x_2)\}$ the *kernel* of φ. This relation is obviously a left regular relation.

In the following theorem, we describe the morphism on a left G-system .

Theorem 3.3. *Let G be an n-ary semihypergroup, X be a left G-system and $\varphi : X_1 \longrightarrow X_2$ be a morphism. Then, $[X : (ker l\varphi)] \cong Im\varphi$.*

Proof. Suppose that $K = ker(\varphi)$ and $\psi : [X : K] \longrightarrow Im\varphi$ defined by $\psi(K(x)) = \varphi(x)$. Then the map ψ is well-defined and one to one. Indeed,we have the following equalities.:

$$K(x) = K(y) \Longleftrightarrow \varphi(x) = \varphi(y) \Longleftrightarrow \psi(K(x)) = \psi(K(y)).$$

On the other hand, we have

$$\begin{aligned} \psi\left(\overline{h}(g_1^{n-1}, K(x)\right) = \psi\left(K\left(h(g_1^{n-1}, x)\right)\right) &= \varphi(h(g_1^{n-1}, x)) \\ &= h\left(g_1^{n-1}, \varphi(x)\right) \\ &= \overline{h}\left(g_1^{n-1}, \psi(K(x)\right). \end{aligned}$$

This completes the proof. □

The following lemma is a crucial lemma of the morphisms on left G-systems.

Lemma 3.4. *Let ρ be a regular relation on a left G-system X and $\varphi : X \longrightarrow Y$ be a morphism such that $\rho \subseteq ker(\varphi)$. Then, there is a unique morphism $\overline{\varphi} : [X : \rho] \longrightarrow Y$ such that $Im(\overline{\varphi}) = Im(\varphi)$ and $\overline{\varphi} \circ \pi = \varphi$, where $\pi : X \longrightarrow [X : \rho]$ is a natural morphism.*

Proof. Suppose that $\overline{\varphi} : [X : \rho] \longrightarrow Y$ defined by $\overline{\varphi}(\rho(x)) = \varphi(x)$, where $x \in X$. Then, $\overline{\varphi}$ is well-defined, since, for all $x_1, x_2 \in X$

$$\rho(x_1) = \rho(x_2) \Longrightarrow (x_1, x_2) \in \rho \subseteq ker\varphi \Longrightarrow \varphi(x_1) = \varphi(x_2).$$

Also $\overline{\psi} \circ \pi = \varphi$. The uniqueness of $\overline{\varphi}$ is clear. □

Let ρ_1 and ρ_2 be regular relations on the left G-system X, where $\rho_1 \subseteq \rho_2$. By Theorem 3.4, there is a morphism $\overline{\varphi} : [X : \rho_1] \longrightarrow [X : \rho_2]$, such that $\overline{\varphi} \circ \pi_1 = \pi_2$, where $\pi_1 : X \longrightarrow [X : \rho_1]$ and $\pi_2 : G \longrightarrow [X : \rho_2]$ are nutural maps. The morphism $\overline{\varphi}$ is given by

$$\overline{\varphi}(\rho_1(x)) = \rho_2(x),\ x \in X.$$

Also, the regular relation $ker(\overline{\varphi})$ on $[X : \rho_1]$ given by

$$ker\overline{\varphi} = \{(\rho_1(x_1), \rho_1(x_2)) \in [X : \rho_1] \times [X : \rho_1] : (x_1, x_2) \in \rho_2\}.$$

For the regular relations on the left G-system X, we have the following theorem.

Theorem 3.5. *Let ρ_1 and ρ_2 be regular relations on a left G-system X such that $\rho_1 \subseteq \rho_2$. Then,*

$$[\rho_1 : \rho_2] = \{(\rho_2(a_1), \rho_2(a_2)) \in [X : \rho_2] \times [X : \rho_2] : (a_1, a_2) \in \rho_1\},$$

is regular on $[X : \rho_1]$ and $[[X : \rho_1] : [\rho_1 : \rho_2]] \cong [X : \rho_2]$.

Proof. The proof is straightforward and is hence omitted. □

Suppose that X is a (G, G)-system such that $h_1 : G^{n-1} \times X \longrightarrow X$, $h_2 : X \times G^{n-1} \longrightarrow X$ and ρ is an equivalence relation on X. We define

$$\rho^* = \left\{ \left(h_1 \left(g_1^{n-1}, h_2(x, t_1^{n-1}) \right), h_1 \left(g_1^{n-1}, h_2 \left(y, t_1^{n-1} \right) \right) \right) : g_1^{n-1}, t_1^{n-1} \in G, (x, y) \in \rho \right\}.$$

For the (G, G)-systems, we have the following propositions.

Proposition 3.6. *Let G be an n-ary semigroup and X be a (G, G)-system such that ρ is an equivalence relation on X. Then, ρ^* is the smallest regular relation containing ρ.*

Proof. It is clear that $\rho \subseteq \rho^*$. To show that ρ^* is a left regular, suppose that $(\overline{x}, \overline{y}) \in \rho^*$ and $k_1^{n-1} \in G$. Hence there are $g_1^{n-1}, t_1^{n-1} \in G$ and $x, y \in X$ such that

$$\overline{x} = h_1 \left(g_1^{n-1}, h_2(x, t_1^{n-1}) \right), \ \overline{y} = h_1 \left(g_1^{n-1}, h_2(y, t_1^{n-1}) \right).$$

Also, we deduce the following equalities:

$$\begin{aligned} h_1 \left(k_1^{n-1}, \overline{x} \right) &= h_1 \left(k_1^{n-1}, h_1 \left(g_1^{n-1}, h_2(x, t_1^{n-1}) \right) \right) \\ &= h_1 \left(f(k_1^{n-1}, g_1), ..., f(k_1^{n-1}, g_{n-1}), h_2(x, t_1^{n-1}) \right) \end{aligned}$$

and

$$\begin{aligned} h_1 \left(k_1^{n-1}, \overline{y} \right) &= h_1 \left(k_1^{n-1}, h_1 \left(g_1^{n-1}, h_2(y, t_1^{n-1}) \right) \right) \\ &= h_1 \left(f(k_1^{n-1}, g_1), ..., f(k_1^{n-1}, g_{n-1}), h_2(y, t_1^{n-1}) \right). \end{aligned}$$

This implies that $(h_1(k_1^{n-1}, \overline{x}), h_1(k_1^{n-1}, \overline{y})) \in \rho^*$.
By using the same arguments, we obtain the following equalities:

$$\begin{aligned} h_2 \left(\overline{x}, k_1^{n-1} \right) &= h_2 \left(h_1 \left(g_1^{n-1}, h_2 \left(x, t_1^{n-1} \right) \right), k_1^{n-1} \right) \\ &= h_2 \left(h_2 \left(h_1 \left(g_1^{n-1}, x \right), t_1^{n-1} \right), k_1^{k-1} \right) \\ &= h_2 \left(h_1 \left(g_1^{n-1}, x \right), f \left(t_1^{n-1}, k_1 \right), f \left(t_1^{n-1}, k_2 \right), ..., f \left(t_1^{n-1}, k_{n-1} \right) \right). \end{aligned}$$

and

$$\begin{aligned} h_2\left(\overline{y}, k_1^{n-1}\right) &= h_2\left(h_1\left(g_1^{n-1}, h_2\left(y, t_1^{n-1}\right)\right), k_1^{n-1}\right) \\ &= h_2\left(h_2\left(h_1\left(g_1^{n-1}, y\right), t_1^{n-1}\right), k_1^{k-1}\right) \\ &= h_2\left(h_1\left(g_1^{n-1}, y\right), f\left(t_1^{n-1}, k_1\right), f\left(t_1^{n-1}, k_2\right), ..., f\left(t_1^{n-1}, k_{n-1}\right)\right), \end{aligned}$$

which implies that $\left(h_2\left(\overline{x}, k_1^{n-1}\right), h_2\left(\overline{y}, k_1^{n-1}\right)\right) \in \rho^*$. Let σ be a left and right regular containing ρ. Then, for every $g_1^{n-1}, t_1^{n-1} \in G$ and $(x, y) \in \rho$,

$$\left(h_1\left(g_1^{n-1}, h_2(x, t_1^{n-1})\right), h_1\left(g_1^{n-1}, h_2(y, t_1^{n-1})\right)\right) \in \sigma.$$

Therefore, $\rho^* \subseteq \sigma$. □

For the equivalences on the G-systems,we have the following proposition.

Proposition 3.7. *Let ρ_1 and ρ_2 be equivalence relations on X. Then,*
1. $\rho_1 \subseteq \rho_2$ implies that $\rho_1^* \subseteq \rho_2^*$,
2. $(\rho_1 \cup \rho_2)^* = \rho_1^* \cup \rho_2^*$

Proof. The proof is straightforward and is hence omitted.

□

Proposition 3.8. *Let ρ be an equivalence on X and G be an n-ary semigroup. Then,*

$$\rho^\circ = \left\{(x, y) \in X \times X : \forall g_1^{n-1}, t_1^{n-1} \in G, \left(h_1\left(g_1^{n-1}, h_2(x, t_1^{n-1})\right), h_1\left(g_1^{n-1}, h_2(y, t_1^{n-1})\right)\right) \in \rho\right\},$$

is the largest regular relation on X contained in ρ.

Proof. Suppose that $(x, y) \in \rho^\circ$ and $k_1^{n-1} \in G$. Then for every $g_1^{n-1}, t_1^{n-1} \in G$, we have

$$\left(h_1\left(g_1^{n-1}, h_2(x, t_1^{n-1})\right), h_1\left(g_1^{n-1}, h_2(y, t_1^{n-1})\right)\right) \in \rho.$$

Hence

$$\begin{aligned} h_1\left(g_1^{n-1}, h_2(h_1\left(k_1^{n-1}, x\right), t_1^{n-1}\right) &= h_1\left(g_1^{n-1}, h_1(k_1^{n-1}, h_2\left(x, t_1^{n-1}\right)\right) \\ &= h_1\left(f(g_1^{n-1}, k_1), ..., f(g_1^{n-1}, k_{n-1}), h_2\left(x, t_1^{n-1}\right)\right). \end{aligned}$$

In a same way,

$$\begin{aligned} h_1\left(g_1^{n-1}, h_2(h_1\left(k_1^{n-1}, y\right), t_1^{n-1}\right) &= h_1\left(g_1^{n-1}, h_1(k_1^{n-1}, h_2\left(y, t_1^{n-1}\right)\right) \\ &= h_1\left(f(g_1^{n-1}, k_1), ..., f(g_1^{n-1}, k_{n-1}), h_2\left(y, t_1^{n-1}\right)\right). \end{aligned}$$

This implies that $(h_1(k_1^{n-1}, x), h_1(k_1^{n-1}, y)) \in \rho^\circ$. In a same way, we can see

$$(h_2(x, k_1^{n-1}), h_2(y, k_1^{n-1})) \in \rho^\circ.$$

Let R be a regular relation on X contained in ρ. Then,

$$\begin{aligned} (x, y) \in \rho &\Longrightarrow \forall g_1^{n-1}, t_1^{n-1} \in G, \left(h_2\left(x, t_1^{n-1}\right), h_2\left(y, t_1^{n-1}\right)\right) \in R \\ &\Longrightarrow \left(h_1\left(g_1^{n-1}, h_2\left(x, t_1^{n-1}\right)\right), h_1\left(t_1^{n-1}, h_2\left(y, t_1^{n-1}\right)\right)\right) \in R. \end{aligned}$$

This implies that $\rho^{\circ} \in R$. □

4. The Green's Relations

In this section, we introduce the Green's relations on a G-system and prove some properties.

Let G be an n-ary semihypergroup and X a (G,G)-system. Then, we define the relations L and R on the G-system G as follows:

$$(x,y) \in L \Longleftrightarrow \exists\ g_1^{n-1}, t_1^{n-1} \in G:\ h_1\left(g_1^{n-1}, x\right) = y,\ h_1(t_1^{n-1}, y) = x.$$

$$(x,y) \in R \Longleftrightarrow \exists\ k_1^{n-1}, s_1^{n-1} \in G:\ h_2(y, k_1^{n-1}) = x,\ h_2(x, s_1^{n-1}) = y.$$

The following Proposition is a basic result of the regular relations on a (G,G)-systems.

Proposition 4.1. *Let G be an n-ary semihypergroup and X be a (G,G)-system. Then, the relation R is a left regular relation and the relation L is a right regular relation.*

Proof. Suppose that $(x,y) \in L$ and $k_1^{n-1} \in G$. Then,

$$h_1\left(g_1^{n-1}, x\right) = y,\ h_1(t_1^{n-1}, y) = x,$$

where $g_1^{n-1}, t_1^{n-1} \in G$. We have the following equalities:

$$\begin{aligned} h_2\left(x, k_1^{n-1}\right) &= h_2\left(h_1(t_1^{n-1}, y), k_1^{n-1}\right) \\ &= h_1\left(t_1^{n-1}, h_2(y, k_1^{n-1})\right). \end{aligned}$$

In a same way, we can see $h_2\left(y, k_1^{n-1}\right) = h_1\left(g_1^{n-1}, h_2\left(y, k_1^{n-1}\right)\right)$.

These results lead that L is a right regular relation. By using similar arguments, we can show that R is a left regular relation. □

Corollary 4.2. *Let G be a commutative n-ary semihypergroup. Then, R and L are regular relations.*

For the n-ary semihypergroups, we have the following proposition.

Proposition 4.3. *Let G be an n-ary semihypergroup and X be a (G,G)-system. Then, $L \circ R = R \circ L$.*

Proof. Suppose that G be an ary-semihypergroup and $(x,y) \in L \circ R$. Then, there exists $z \in X$ such that $(x,z) \in L$ and $(z,y) \in R$. Hence there exist $g_1^{n-1}, t_1^{n-1}, k_1^{n-1}, s_1^{n-1} \in G$ such that

$$\begin{aligned} h_1\left(g_1^{n-1}, x\right) &= z,\ h_1\left(t_1^{n-1}, z\right) = x, \\ h_2\left(z, k_1^{n-1}\right) &= y,\ h_2(y, s_1^{n-1}) = z. \end{aligned}$$

Let $w = h_2\left(h_1(t_1^{n-1}, z), k_1^{n-1}\right)$. Then,

$$\begin{aligned} h_2\left(w, s_1^{n-1}\right) &= h_2\left(h_2\left(h_1(t_1^{n-1}, z), k_1^{n-1}\right), s_1^{n-1}\right) \\ &= h_2\left(h_1\left(t_1^{n-1}, h_2\left(z, k_1^{n-1}\right)\right), s_1^{n-1}\right) \\ &= h_2\left(h_1\left(t_1^{n-1}, y\right), s_1^{n-1}\right) \\ &= h_1\left(t_1^{n-1}, h_2\left(y, s_1^{n-1}\right)\right) \\ &= h_1\left(t_1^{n-1}, z\right) = z. \end{aligned}$$

and

$$h_2\left(x, k_1^{n-1}\right) = h_2\left(h_1\left(t_1^{n-1}, z\right), k_1^{n-1}\right) = w.$$

Hence xRw. Moreover, we have

$$h_1\left(t_1^{n-1}, y\right) = h_1\left(t_1^{n-1}, h_2\left(z, k_1^{n-1}\right)\right) = h_2\left(h_1\left(t_1^{n-1}, z\right), k_1^{n-1}\right) = w.$$

and

$$\begin{aligned} h_1\left(g_1^{n-1}, w\right) &= h_1\left(g_1^{n-1}, h_2\left(h_1\left(t_1^{n-1}, z\right), k_1^{n-1}\right)\right) \\ &= h_1\left(g_1^{n-1}, h_2\left(x, k_1^{n-1}\right)\right) \\ &= h_2\left(h_1\left(g_1^{n-1}, x\right), k_1^{n-1}\right) \\ &= h_2\left(z, k_1^{n-1}\right) = y. \end{aligned}$$

Hence wLy. We deduce that $(x, y) \in R \circ L$. This implies that $L \circ R \subseteq R \circ L$. The reverse inclusion follows in a similar way. □

Let G be an n-ary semihypergroup and X be a (G, G)-system and $D = L \circ R$. We define a relation ζ on X as follows:

$$(x, y) \in \zeta \Longleftrightarrow \exists\ g_1^{n-1}, t_1^{n-1}, k_1^{n-1}, s_1^{n-1} \in G :$$

$$h_1\left(g_1^{n-1}, h_2\left(x, t_1^{n-1}\right)\right) = y,\ h_1\left(k_1^{n-1}, h_2\left(y, s_1^{n-1}\right)\right) = x.$$

It is easy to see that $L \subseteq \zeta$ and $R \subseteq \zeta$. Hence $D = L \circ R \subseteq \zeta$. It is easy to see that D is the smallest equivalence containing L and R.

Finally, we state the main theorem of this paper.

Theorem 4.4. *Let G be an n-ary semihypergroup, X be a (G, G)-system and $a, b \in X$ such that $(a, b) \in R$ and $g_1^{n-1}, t_1^{n-1} \in G$ such that $h_2\left(a, g_1^{n-1}\right) = b$ and $h_2\left(b, t_1^{n-1}\right) = a$. Then, there exist $\rho_{g_1^{n-1}} : X \longrightarrow X$ and $\rho_{t_1^{n-1}} : X \longrightarrow X$ such that $\rho_{g_1^{n-1}} \mid_{l(a)}$ and $\rho_{t_1^{n-1}} \mid_{l(b)}$ are mutually inverse R-classes preserving from $L(a)$ onto $L(b)$ and $L(b)$ onto $L(a)$.*

Proof. Suppose that $(a, b) \in R$. Then, by the definition of R, there exists $g_1^{n-1}, t_1^{n-1} \in G$ such that $h_2\left(a, g_1^{n-1}\right) = b$ and $h_2\left(b, t_1^{n-1}\right) = a$. We define

$$\begin{aligned} \rho_{g_1^{n-1}} :\ & X \longrightarrow X \\ & x \longmapsto h_2\left(x, g_1^{n-1}\right). \end{aligned}$$

Hence, $\rho_{g_1^{n-1}}(a) = h_2\left(a, g_1^{n-1}\right) = b$. Since L is regular, this implies that for every $x \in R(a)$,

$$h_2\left(x, g_1^{n-1}\right) \ L \ h_2\left(a, g_1^{n-1}\right) = b.$$

Hence $\rho_{g_1^{n-1}}(L(a)) \subseteq L(b)$. In a similar way, we can define

$$\begin{aligned} \rho_{t_1^{n-1}}: \quad & X \longrightarrow X \\ & x \longmapsto h_2\left(x, t_1^{n-1}\right) \end{aligned}$$

and $\rho_{t_1^{n-1}}(b) = h_2\left(b, t_1^{n-1}\right) = a$. Thus, for every $x \in L(a)$,

$$h_2\left(x, t_1^{n-1}\right) \ L \ h_2\left(b, t_1^{n-1}\right) = b.$$

This implies that $\rho_{t_1^{n-1}}(L(b)) \subseteq L(a)$ and $\rho_{t_1^{n-1}} \circ \rho_{g_1^{n-1}} : L(a) \longrightarrow L(a)$. Let $x \in L(a)$. Then, there exists $u_1^{n-1} \in G$ such that $x = h_1\left(u_1^{n-1}, a\right)$. Hence

$$\begin{aligned} \rho_{t_1^{n-1}} \circ \rho_{g_1^{n-1}}(x) = \rho_{t_1^{n-1}}\left(h_2\left(x, g_1^{n-1}\right)\right) &= h_2\left(h_2\left(x, g_1^{n-1}\right), t_1^{n-1}\right) \\ &= h_2\left(h_2\left(h_1\left(u_1^{n-1}, a\right), g_1^{n-1}\right), t_1^{n-1}\right) \\ &= h_2\left(h_1\left(u_1^{n-1}, h_2\left(a, g_1^{n-1}\right)\right), t_1^{n-1}\right) \\ &= h_2\left(h_1\left(u_1^{n-1}, b\right), t_1^{n-1}\right) \\ &= h_1\left(u_1^{n-1}, h_2\left(b, t_1^{n-1}\right)\right) \\ &= h_1\left(u_1^{n-1}, a\right) = x. \end{aligned}$$

Thus, $\rho_{t_1^{n-1}} \circ \rho_{g_1^{n-1}}$ is the identity map and we can show in a closely similar way that $\rho_{g_1^{n-1}} \circ \rho_{t_1^{n-1}}$ is the identity map on $L(b)$. This completes the proof. □

5. Conclusion

The Green's relations provide the necessary tools for using similar arguments on the monoid. When working in language theory using automata, several tools comes naturally into play. A typical example is the use of the decomposition of the graph of the automaton into strongly connected components, and the use of the connected components for driving an induction in a proof. Since the Green relation used in automata theory we introduced this concept on n-ary semihypergroup. In future works, we consider and used of G-systems and Green relation for solving automata related questions.

Acknowledgments

The authors would like to express their gratitude to the referees for their careful reading and helpful comments.

References

1. R. Chinram and P. Siammai, On Green s relations for Γ-semigroups and reductive Γ-semigroups, International Journal of Algebra, **2**(4), (2008), 187-195.
2. P. Corsini, *Prolegomena of hypergroup theory*, Second Edition, Aviani Editore (1993).
3. P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, (2003).
4. I. Cristea and M. Stefanescu, *Hypergroups and n-ary relations*, European Journal of Combinatorics, **31**(3), (2010), 780-789.
5. B. Davvaz and V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press USA (2007).
6. B. Davvaz and T. Vougiouklis, *n-ary hypergroups*, Iranian Journal of Science and. Technology, Transaction A, **30** (A2), (2006), 165-174.
7. W. Dörnte, *Unterschungen uber einen verallgemeinerten gruppenbegriff*, Mathematische Zeitschrift, 29, (1929), 1-19.
8. M. Farshi and B. Davvaz, *Relations and homomorphisms of n-hypergroups*, European Journal of Combinatorics, 44 (2015), 218-230.
9. J. A. Green, *On the structure of semigroups*. Annals of Mathematics, 54, (1951), 163-172.
10. K. Hila , B. Davvaz and K. Naka, *On hyperideal structure of ternary semihypergroup*, Iranian Journal of Mathematical Sciences and Informatics, **9**(1), (2014), 81-98.
11. V. Leoreanu-Fotea and B. Davvaz, *n-hypergroups and binary relations*, European Journal of Combinatorics, **29** (5), (2008), 1207-1218.
12. V. Leoreanu-Fotea, Ivo Rosenberg, B. Davvaz and T. Vougiouklis, *A new class of n-ary hyperoperations*, European Journal of Combinatorics, 44 (2015), 265-273.
13. F. Marty, *Sur une generalization de la notion de group*, 8^{th} Congres mathematics Scandinaves (1934), 45-49.
14. S. Mirvakili and B. Davvaz, *Relations on Krasner* (m, n)*-hyperrings*, European Journal of Combinatorics, **31**(3), (2010), 79-802
15. S. Ostadhadi-Dehkordi *m-Ary Hypervector Space: Convergent Sequence and Bundle Subsets*, Iranian Journal of Mathematical Sciences and Informatics, **11**(2), (2016), 23-41.
16. F. M. Sioson, *A note on congruences*, Proceedings of the Japan Academy, 43 (1967), 103-107.