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Fuzzy Sumudu Transform for System of Fuzzy Differential Equations with Fuzzy Constant Coefficients

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Abstract. In this study, we employ fuzzy Sumudu transform to find the solution for system of linear fuzzy differential equations where the system possesses fuzzy constant coefficients instead of crisp. For this purpose, fuzzy Sumudu transform has been revisited and a brief comparison with fuzzy Laplace transform is provided alongside, particularly on the scale preserving property. For the sake of comparison, we introduce to the literature a time scaling theorem for fuzzy Laplace transform. Next, the system with fuzzy constant coefficients is interpreted under the strongly generalized differentiability. From here, new procedures for solving the systems are proposed. A numerical example is then carried out for solving a system adapted from fuzzy radioactive decay model. Conclusion is drawn in the last section and some potential research directions are given.

Keywords: Fuzzy Sumudu transform, Fuzzy differential equations, System of fuzzy differential equations, Fuzzy Laplace transform, Radioactive decay model.

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1. Introduction

For centuries, scientists have struggled to model real world phenomenon effectively and precisely. Among several modelling tools, scientists prefer to construct their models based on system of linear differential equations with initial value [19, 32, 26, 11, 14]. This is because this kind of modelling are easier to be solved and analysed. However, the drawback of this modelling is very obvious and it is far from ideal. This is resulting from the lack of our knowledge about the world. Uncertainties or imprecisions occur in almost every aspect in our life, especially when dealing with real life phenomenon such as microbes, air quality and population. Using system of ordinary differential equations, the uncertainties are not dealt accordingly, thus, this leads to an inexact model.

To handle this shortcoming, scientist make use of Zadeh fuzzy set theory where the author emphasized that a number can be classified into certain membership function rather than we represent it as a discrete or crisp number [35]. The new modelling tool is referred to as fuzzy differential equations (FDEs). Whenever structures of the model appear to be non probabilistic, FDEs are constructed as a replacement or improvement. Unlike ordinary differential equations, FDEs take accounts the uncertainties or imprecisions observed circulating around the problem. There are a vast literature that can be found discussing FDEs as well as fuzzy derivatives [13, 34, 30, 25, 12, 10].

When considering FDEs, there are several interpretations of derivatives involved. One of the most recent is the strongly generalized differentiability. Under this interpretation, not much solving method have been introduced. So, the effort to construct methods for finding the solutions of the considered FDEs, particularly with strongly generalized differentiability become an urgent matter. This includes both analytical and numerical methods. However, before we develop numerical methods for dealing with FDEs, it is important for us to construct analytical methods.

In early 2015, Ahmad and Abdul Rahman [5, 1] introduced into the literature fuzzy Sumudu transform (FST) for solving FDEs with fuzzy initial values. The work was done considering FDEs with strongly generalized differentiability. The authors successfully introduced some of FST fundamental properties and theories, and later demonstrated the proposed method on several numerical examples. As stated by the authors, the advantage of FST is significant as it reduces the complexity of the calculation when handling FDEs. Plus, the final unit of the solution can be seen without even completing the solving process. This is due to its scale preserving property. Then, FST is used to solve fuzzy partial differential equations [2] before later, Haydar used FST to solve nth-order FDEs with fuzzy initial values [20]. The studies are the followed by the application of FST on fuzzy fractional differential equations where the authors considered the β or fractional order to be between 0 and 1 [3]. Meanwhile, Volterra integral equations under fuzzy settings are solved in [4]. Recently, FST has been used to solve FDE with Z-numbers [21] and FDEs of order 2 by intuitionistic fuzzy number [31].

Scientists had continue utilizing FDEs to construct a more complex model which consists of several FDEs. The pioneer works for this effort is the development of system of fuzzy differential equations (SLFDEs) that is of the first order and the SLFDEs considered is linear. There has been several works regarding SLFDEs with fuzzy constant coefficients (FCCs) in the literature [15, 17, 6, 24, 27, 28]. These include the implementation of variational iteration method and homotopy analysis method. When dealing with FDEs interpreted under strongly generalized differentiability concept, there are two cases of differentiability to be considered [12]. The previous works done did not demonstrated both cases efficiently, for example, the work in [15]. Particularly, only the first case is demonstrated by the authors. Because of this, we intend to use FST for solving SLFDEs with FCCs, and both cases of differentiability interpreted under the mentioned concept will be fully demonstrated. Plus, as far as we have researched, this can be regard as the initial work in utilizing FST to deal with such system.

The rest of this paper is arranged in the following order. In the next section, we revisit a number of necessary basic definitions and concepts of fuzzy numbers, as well as fuzzy derivatives. It is followed by Section 3, providing the definition and notation of FST that has been proposed earlier to the literature. In addition, we also discuss on the scale preserving property possessed by FST and a brief comparison was done with fuzzy Laplace transform which has been proposed in [7]. Next in Section 4, we provide some details on the SLFDEs where two cases of the strongly generalized differentiability are considered. In the subsequent section, we introduce step-by-step procedures in handling SLFDEs, specifically SLFDEs with FCCs. While in Section 6, we obtain solutions for a numerical example to exhibit the practicality of the proposed method. Succeeding Section 6 is conclusions drawn in Section 7 which also highlights some research directions.

2. Preliminaries

Several important definitions and properties for fuzzy numbers and fuzzy functions are recalled in this section. Note that the real number and fuzzy number are denoted by $\mathbb{R}$ and $\mathcal{F}(\mathbb{R})$, respectively, throughout this paper.

Definition 2.1. [35] A fuzzy number is a mapping $\widetilde{u} : \mathbb{R} \to [0, 1]$ that satisfies the following conditions.

(1) For every $\widetilde{u} \in \mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is upper semi continuous,

(2) for every $\widetilde{u} \in \mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is fuzzy convex, i.e., $\widetilde{u}(\gamma x + (1-\gamma)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for all $x, y \in \mathbb{R}$, and $\gamma \in [0,1]$,
(3) for every $\widetilde{u} \in \mathcal{F}(\mathbb{R})$, $\widetilde{u}$ is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $\widetilde{u}(x_0) = 1$,
(4) supp $\widetilde{u} = \{x \in \mathbb{R} | \widetilde{u}(x) > 0\}$ is the support of $\widetilde{u}$, and it has a compact closure cl(supp $\widetilde{u}$).

Definition 2.2. [22] Let $\widetilde{u} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in]0,1]$. The α-level set of $\widetilde{u}$ is the crisp set $\widetilde{u}^\alpha$ that contains all the elements with membership degree greater than or equal to α, i.e.

$$\widetilde{u}^\alpha = \{x \in \mathbb{R} | \widetilde{u}(x) \geq \alpha\},$$

where $\widetilde{u}^\alpha$ denotes α-level set of fuzzy number $\widetilde{u}$.

It can be stated that the α-level of any fuzzy number is a bounded and closed set, and it is denoted by $[\underline{u}^\alpha, \overline{u}^\alpha]$, for both lower and upper bound of $\widetilde{u}^\alpha$, respectively.

Definition 2.3. [16, 25] A parametric form of an arbitrary fuzzy number $\widetilde{u}$ is an ordered pair $[\underline{u}^\alpha, \overline{u}^\alpha]$ of functions $\underline{u}^\alpha$ and $\overline{u}^\alpha$, for any $\alpha \in [0,1]$, that fulfil the following conditions.

(1) $\underline{u}^\alpha$ is a bounded left continuous monotonic increasing function in $[0,1]$,
(2) $\overline{u}^\alpha$ is a bounded left continuous monotonic decreasing function in $[0,1]$,
(3) $\underline{u}^\alpha \leq \overline{u}^\alpha$.

A fuzzy number is classified into certain membership function. In the literature, there are many types of membership function defined by researchers. For example, triangular, trapezoidal, Gaussian and generalized bell membership function. Among all of them, the triangular membership function is the most widely utilised by researchers. It is represented by three crisp numbers (a_1, a_2, a_3) and its α-level is computed as below [23].

$$\widetilde{u}^\alpha = [a_1 + (a_2 - a_3)\alpha, a_3 - (a_3 - a_2)\alpha], \quad \alpha \in [0,1]. \tag{2.1}$$

Basic operations between fuzzy numbers can be seen in [16].

Theorem 2.4. [33] *Let the fuzzy function* $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ *represented by* $[\underline{f}^\alpha(x), \overline{f}^\alpha(x)]$. *For any* $\alpha \in [0,1]$, *assume that* $\underline{f}^\alpha(x)$ *and* $\overline{f}^\alpha(x)$ *are both Riemann-integrable on* $[a,b]$ *and assume that there are two positive* $\underline{M}^\alpha$ *and* $\overline{M}^\alpha$ *where* $\int_a^b |\underline{f}^\alpha(x)|dx \leq \underline{M}^\alpha$ *and* $\int_a^b |\overline{f}^\alpha(x)|dx \leq \overline{M}^\alpha$, *for every* $b \geq a$. *Then,* $\widetilde{f}(x)$ *is improper fuzzy Riemann-integrable on* $[a, \infty[$ *and the improper fuzzy Riemann-integrable is a fuzzy number. Furthermore, we have*

$$\int_a^\infty \widetilde{f}(x)dx = \left[\int_a^\infty \underline{f}^\alpha(x)dx, \int_a^\infty \overline{f}^\alpha(x)dx\right].$$

Definition 2.5. [30] If $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R})$ and if there exists a fuzzy subset $\xi \in \mathcal{F}(\mathbb{R})$ such that $\xi + \widetilde{u} = \widetilde{v}$, then ξ is unique. In this case, ξ is called the Hukuhara difference, or simply H-difference, of u and v and is denoted by $\widetilde{v} -^H \widetilde{u}$.

The strongly generalized differentiability concept for a fuzzy function is described in the following definition.

Definition 2.6. [9, 10] Let $\widetilde{f} :]a,b[\to \mathcal{F}(\mathbb{R})$ be a fuzzy function and $x_0 \in]a,b[$. We say that $\widetilde{f}$ is strongly generalized differentiable on x_0, if there exists an element $\widetilde{f}'(x_0) \in \mathcal{F}(\mathbb{R})$, such that

(1) for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0+h) -^H \widetilde{f}(x_0), \widetilde{f}(x_0) -^H \widetilde{f}(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\widetilde{f}(x_0+h) -^H \widetilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\widetilde{f}(x_0) -^H \widetilde{f}(x_0-h)}{h} = \widetilde{f}'(x_0),$$

or

(2) for all $h > 0$ sufficiently small, $\exists \widetilde{f}(x_0) -^H \widetilde{f}(x_0+h), \widetilde{f}(x_0-h) -^H \widetilde{f}(x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\widetilde{f}(x_0) -^H \widetilde{f}(x_0+h)}{-h} = \lim_{h\to 0} \frac{\widetilde{f}(x_0-h) -^H \widetilde{f}(x_0)}{-h} = \widetilde{f}'(x_0).$$

In this paper, the first part of the Definition 2.6 is referred to as (i)-differentiable, while the second part as (ii)-differentiable.

Definition 2.7. [18] A fuzzy function $\widetilde{f} : [a,b] \to \mathcal{F}(\mathbb{R})$ is said to be continuous at $x_0 \in [a,b]$ if for each $\epsilon > 0$, there is $\delta > 0$ such that $D(\widetilde{f}(x), \widetilde{f}(x_0)) < \epsilon$, whenever $x \in [a,b]$ and $|x - x_0| < \delta$. We say that $\widetilde{f}$ is continuous on $[a,b]$ if $\widetilde{f}$ is continuous at each $x_0 \in [a,b]$.

Theorem 2.8. [12] *Let $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function and $\widetilde{f}(x) = [\underline{f}^{\alpha}(x), \overline{f}^{\alpha}(x)]$, for every $\alpha \in [0,1]$. Then*

(1) *if the fuzzy function $\widetilde{f}$ is (i)-differentiable, then $\underline{f}^{\alpha}(x)$ and $\overline{f}^{\alpha}(x)$ are both differentiable and*

$$f'(x) = [(\underline{f}')^{\alpha}(x), (\overline{f}')^{\alpha}(x)], \tag{2.2}$$

(2) *if the fuzzy function $\widetilde{f}$ is (ii)-differentiable, then $\underline{f}^{\alpha}(x)$ and $\overline{f}^{\alpha}(x)$ are both differentiable and*

$$f'(x) = [(\overline{f}')^{\alpha}(x), (\underline{f}')^{\alpha}(x)]. \tag{2.3}$$

The next section provides the previously constructed FST. The concept for SLFDEs under the strongly generalized differentiability concept is also provided in this section.

3. Fuzzy sumudu transform

In order to construct the solution for SLFDEs, we adopted the Definition of FST introduced in [1].

Definition 3.1. [1] Let $\widetilde{f} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy function. Suppose that $\widetilde{f}(ux) \odot e^{-x}$ is improper fuzzy Riemann-integrable on $[0, \infty[$, then $\int_0^\infty \widetilde{f}(ux) \odot e^{-x} dx$ is called fuzzy Sumudu transform and is denoted by

$$G(u) = \mathcal{S}[\widetilde{f}(x)](u) = \int_0^\infty \widetilde{f}(ux) \odot e^{-x} dx, \quad u \in [-\tau_1, \tau_2],$$

where the variable u is used to factor the variable x in the argument of the fuzzy function and $\tau_1, \tau_2 > 0$.

FST can also be parametrically written as follows.

$$\mathcal{S}[\widetilde{f}(x)](u) = [s[\underline{f}^\alpha(x)](u), s[\overline{f}^\alpha(x)](u)].$$

To explore theorems of FST, please see in [5, 1]. In the following subsection, we provide a brief discussion on the scale preserving property of FST and for the sake of comparison, we introduce the time scaling property of fuzzy Laplace Transform. This is the first time such property of fuzzy Laplace transform is introduced in the literature, and it is mainly to compare how scaling effects both FST and fuzzy Laplace transform.

3.1. Discussion on scale preserving property.

One of the most frequently highlighted property of FST is the scale preserving property. The theorems for scale preserving are as follows.

Theorem 3.2. [1](First preserving Theorem) *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function and a is an arbitrary constant, then*

$$\mathcal{S}[f(at)](u) = G(au).$$

Theorem 3.3. [1] (Second preserving Theorem) *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function, then*

$$\mathcal{S}\left[t \odot \frac{df(t)}{dt}\right](u) = u\frac{dG(u)}{du}.$$

The theorems listed are the first and second preserving Theorem, respectively. From the scale preserving Theorems, it can be said that FST may be used to solve problems without resorting to a new frequency domain. Unlike other integral transforms, the integral transforms need to be resorted to a new domain before they can be solved. For example, fuzzy Laplace transform, where its time scaling is given by the following theorem.

Theorem 3.4. *Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ be a continuous fuzzy-valued function and a is an arbitrary constant, then*

$$\mathcal{L}[f(at)](s) = \frac{1}{|a|} F(\frac{s}{a}),$$

where $\mathcal{L}$ is the fuzzy Laplace transform introduced in [7].

Proof. From [7], the parametric form of fuzzy Laplace transform is given as follows.

$$\begin{aligned}\mathcal{L}[f(at)](s) &= [l[f(at)](s), l[f(at)](s)], \\ &= \left[\int_0^\infty \underline{f}_\alpha(at)e^{-st}dt, \int_0^\infty \overline{f}_\alpha(at)e^{-st}dt\right].\end{aligned}$$

We let $m = at$, so, $\dfrac{dm}{a} = dt$. Then, we have

$$\begin{aligned}\mathcal{L}[f(at)](s) &= \left[\int_0^\infty \underline{f}_\alpha(m)e^{-s(m/a)}\frac{dm}{a}, \int_0^\infty \overline{f}_\alpha(m)e^{-s(m/a)}\frac{dm}{a}\right], \\ &= \frac{1}{a}\left[\int_0^\infty \underline{f}_\alpha(m)e^{-m(s/a)}dm, \int_0^\infty \overline{f}_\alpha(m)e^{-m(s/a)}dm\right].\end{aligned}$$

By the definition of fuzzy Laplace transform, finally,

$$\mathcal{L}[f(at)](s) = \frac{1}{a} F(\frac{s}{a}).$$

□

From Theorems 3.2, 3.3 and 3.4, we can see that, the scale is preserved when we apply FST to the original function but not when we apply fuzzy Laplace transform. This property of FST made it possible for researchers to treat the transformed function as a replica to the original function in many cases. In addition, this property also allows researchers to obtain the unit of the final solution even without transforming it back to the original domain. Such virtue is crucial for validation of the final results when complex systems are encountered.

4. System of first order linear fuzzy differential equations

In this part, we consider the following linear system that has been presented in several papers previously [8, 12, 22]. Based on the papers, the FDEs are in a system of ordinary differential equations. Additionally, in this paper, the system are extended to two cases, in the sense of differentiability as in Definition 2.6.

$$\widetilde{X}'(t) = \widetilde{A} \odot \widetilde{X}(t) \oplus \widetilde{f}(t), \tag{4.1}$$

subjects to the initial conditions

$$\widetilde{X}(0) = \widetilde{X}_0, \tag{4.2}$$

where the constant coefficients are A, $\widetilde{X}(t) = [X_1(t), X_2(t), \ldots, X_n(t)]^T$, $\widetilde{f}(t) = [f_1(t), f_2(t), \ldots, f_n(t)]^T$, $\widetilde{X}_0 = [X_{10}, X_{20}, \ldots, X_{n0}]$ and the elements of the vector $\widetilde{f}(t)$ are all fuzzy numbers. This means that the system consists of FDEs with FCCs.

Let $[X_m(t)]^\alpha = [(\underline{X}_m)^\alpha(t), (\overline{X}_m)^\alpha(t)]$, $m = 1, 2, \ldots, n$, if $X_m(t)$ is (i)-differentiable, then $[X'_m(t)]^\alpha = [(\underline{X}'_m)^\alpha(t), (\overline{X}'_m)^\alpha(t)]$. This is resulting from Theorem 2.8. Thus for $m = 1, 2, \ldots, n$, we have

$$\begin{aligned}
(\underline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n} \underline{a_{mj}^\alpha (X_j)^\alpha(t)} + (\underline{f}_m)^\alpha(t), \\
&\qquad (\underline{X}_m)^\alpha(0) = (\underline{X}_{m0})^\alpha, \\
(\overline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n} \overline{a_{mj}^\alpha (X_j)^\alpha(t)} + (\overline{f}_m)^\alpha(t), \\
&\qquad (\overline{X}_m)^\alpha(0) = (\overline{X}_{m0})^\alpha,
\end{aligned} \tag{4.3}$$

and if $X_m(t)$ is (ii)-differentiable, we have $[X'_m(t)]^\alpha = [(\overline{X}'_m)^\alpha(t), (\underline{X}'_m)^\alpha(t)]$. So,

$$\begin{aligned}
(\overline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n} \underline{a_{mj}^\alpha (X_j)^\alpha(t)} + (\underline{f}_m)^\alpha(t), \\
&\qquad (\overline{X}_m)^\alpha(0) = (\overline{X}_{m0})^\alpha, \\
(\underline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n} \overline{a_{mj}^\alpha (X_j)^\alpha(t)} + (\overline{f}_m)^\alpha(t), \\
&\qquad (\underline{X}_m)^\alpha(0) = (\underline{X}_{m0})^\alpha,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\underline{a_{mj}^\alpha (X_j)^\alpha(t)} &= \min\{AX | A \in [(\underline{a}_{mj})^\alpha, (\overline{a}_{mj})^\alpha], X \in [(\underline{X}_j)^\alpha(t), (\overline{X}_j)^\alpha(t)]\}, \\
\overline{a_{mj}^\alpha (X_j)^\alpha(t)} &= \max\{AX | A \in [(\underline{a}_{mj})^\alpha, (\overline{a}_{mj})^\alpha], X \in [(\underline{X}_j)^\alpha(t), (\overline{X}_j)^\alpha(t)]\}.
\end{aligned} \tag{4.5}$$

Consequently, if $X_m(t)$ is (i)-differentiable, Eq. (4.3) is interpreted as the following.

1. If a_{mj} is non negative,

$$
\begin{aligned}
(\underline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\underline{a}_{mj})^\alpha(\underline{X}_j)^\alpha(t) + (\underline{f}_m)^\alpha(t), \\
&\qquad\qquad (\underline{X}_m)^\alpha(0) = (\underline{X}_{m0})^\alpha, \\
(\overline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\overline{a}_{mj})^\alpha(\overline{X}_j)^\alpha(t) + (\overline{f}_m)^\alpha(t), \\
&\qquad\qquad (\overline{X}_m)^\alpha(0) = (\overline{X}_{m0})^\alpha,
\end{aligned} \tag{4.6}
$$

2. if a_{mj} is non positive,

$$
\begin{aligned}
(\underline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\underline{a}_{mj})^\alpha(\overline{X}_j)^\alpha(t) + (\underline{f}_m)^\alpha(t), \\
&\qquad\qquad (\underline{X}_m)^\alpha(0) = (\underline{X}_{m0})^\alpha, \\
(\overline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\overline{a}_{mj})^\alpha(\underline{X}_j)^\alpha(t) + (\overline{f}_m)^\alpha(t), \\
&\qquad\qquad (\overline{X}_m)^\alpha(0) = (\overline{X}_{m0})^\alpha.
\end{aligned} \tag{4.7}
$$

And if $X_m(t)$ is (ii)-differentiable, Eq. (4.4) is interpreted as follows.

1. If a_{mj} is non negative,

$$
\begin{aligned}
(\overline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\underline{a}_{mj})^\alpha(\underline{X}_j)^\alpha(t) + (\underline{f}_m)^\alpha(t), \\
&\qquad\qquad (\underline{X}_m)^\alpha(0) = (\underline{X}_{m0})^\alpha, \\
(\underline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\overline{a}_{mj})^\alpha(\overline{X}_j)^\alpha(t) + (\overline{f}_m)^\alpha(t), \\
&\qquad\qquad (\overline{X}_m)^\alpha(0) = (\overline{X}_{m0})^\alpha,
\end{aligned} \tag{4.8}
$$

2. if a_{mj} is non positive,

$$
\begin{aligned}
(\overline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\underline{a}_{mj})^\alpha(\overline{X}_j)^\alpha(t) + (\underline{f}_m)^\alpha(t), \\
&\qquad\qquad (\underline{X}_m)^\alpha(0) = (\underline{X}_{m0})^\alpha, \\
(\underline{X}'_m)^\alpha(t) &= \sum_{j=1}^{n}(\overline{a}_{mj})^\alpha(\underline{X}_j)^\alpha(t) + (\overline{f}_m)^\alpha(t), \\
&\qquad\qquad (\overline{X}_m)^\alpha(0) = (\overline{X}_{m0})^\alpha.
\end{aligned} \tag{4.9}
$$

5. Fuzzy sumudu transform for system of linear first order fuzzy differential equations

In this part, procedures for solving SLFDEs with FCCs using FST is constructed. First, consider the following SLFDEs [22].

$$\widetilde{X}'(t) = \widetilde{A} \odot \widetilde{X}(t) \oplus \widetilde{f}(t), \tag{5.1}$$

subjects to the initial conditions

$$\widetilde{X}(0) = \widetilde{X}_0. \tag{5.2}$$

Eq. (5.1) can also be rewritten as

$$\begin{aligned}
\widetilde{X}_1' &= \sum_{j=1}^{n} \widetilde{a}_{1j}\widetilde{X}_j(t) + \widetilde{f}_1(t),\\
\widetilde{X}_2' &= \sum_{j=1}^{n} \widetilde{a}_{2j}\widetilde{X}_j(t) + \widetilde{f}_2(t),\\
&\vdots\\
\widetilde{X}_n' &= \sum_{j=1}^{n} \widetilde{a}_{nj}\widetilde{X}_j(t) + \widetilde{f}_n(t),
\end{aligned} \tag{5.3}$$

or simply,

$$\widetilde{X}_m' = \sum_{j=1}^{n} \widetilde{a}_{mj}\widetilde{X}_j(t) + \widetilde{f}_m(t), \tag{5.4}$$

for $m = 1, 2, \ldots, n$, subjects to initial conditions $\widetilde{X}_0 = [X_{10}, X_{20}, \ldots, X_{n0}]$. By using FST on both sides of Eq. (5.4), we have

$$\mathcal{S}[\widetilde{X}_m'](u) = \mathcal{S}\left[\sum_{j=1}^{n} \widetilde{a}_{mj}\widetilde{X}_j(t) + \widetilde{f}_m(t)\right](u). \tag{5.5}$$

From Theorem 2.8, we know that the conditions of the fuzzy derivative happened by two cases, which are (i)-differentiable or (ii)-differentiable. So, for $m = 1, 2, \ldots, n$, if $\widetilde{X}_m$ is (i)-differentiable, we have $\widetilde{X}_m' = [\underline{X}_m', \overline{X}_m']$. If $\widetilde{X}_m$ is (ii)-differentiable, we have $\widetilde{X}_m' = [\overline{X}_m', \underline{X}_m']$.

By Theorem 3.1 in [5], if $\widetilde{X}_m$ is (i)-differentiable, we have

$$\mathcal{S}[\widetilde{X}_m'](u) = \frac{G(u) -^{H} \widetilde{X}_m(0)}{u},$$

and this means that,

$$
\begin{aligned}
s\left[\sum_{j=1}^{n}\underline{a_{mj}^{\alpha}(X_j)^{\alpha}(t)}+(\underline{f}_m)^{\alpha}(t)\right](u)&=\frac{s[(\underline{X}_m)^{\alpha}(t)](u)-(\underline{X}_m)^{\alpha}(0)}{u},\\
s\left[\sum_{j=1}^{n}\overline{a_{mj}^{\alpha}(X_j)^{\alpha}(t)}+(\overline{f}_m)^{\alpha}(t)\right](u)&=\frac{s[(\overline{X}_m)^{\alpha}(t)](u)-(\overline{X}_m)^{\alpha}(0)}{u},
\end{aligned}
\tag{5.6}
$$

where $m=1,2,\dots,n$.

Also by Theorem 3.1 in [5], if $\widetilde{X}_m$ is (ii)-differentiable, we have

$$
\mathcal{S}[\widetilde{X}_m](u)=\frac{-\widetilde{X}_m(0)\,-^{H}\,(-G(u))}{u},
$$

and this means that,

$$
\begin{aligned}
s\left[\sum_{j=1}^{n}\underline{a_{mj}^{\alpha}(X_j)^{\alpha}(t)}+(\underline{f}_m)^{\alpha}(t)\right](u)&=\frac{s[(\overline{X}_m)^{\alpha}(t)](u)-(\overline{X}_m)^{\alpha}(0)}{u},\\
s\left[\sum_{j=1}^{n}\overline{a_{mj}^{\alpha}(X_j)^{\alpha}(t)}+(\overline{f}_m)^{\alpha}(t)\right](u)&=\frac{s[(\underline{X}_m)^{\alpha}(t)](u)-(\underline{X}_m)^{\alpha}(0)}{u},
\end{aligned}
\tag{5.7}
$$

where $m=1,2,\dots,n$.

To solve Eq. (5.6) or (5.7), we assume that

$$
\begin{aligned}
s[(\underline{X}_m)^{\alpha}(t)](u)&=L_m^{\alpha}(u),\\
s[(\overline{X}_m)^{\alpha}(t)](u)&=U_m^{\alpha}(u),
\end{aligned}
\tag{5.8}
$$

where $L_1^{\alpha}(u),U_1^{\alpha}(u),L_2^{\alpha}(u),U_2^{\alpha}(u),\dots,L_n^{\alpha}(u)$ and $U_n^{\alpha}(u)$ are the solutions of Eq. (5.6) or (5.7). Then, by the inverse of FST, the final solutions are as the following.

$$
\begin{aligned}
(\underline{X}_m)^{\alpha}(t)&=s^{-1}[L_m^{\alpha}(u)],\\
(\overline{X}_m)^{\alpha}(t)&=s^{-1}[U_m^{\alpha}(u)].
\end{aligned}
\tag{5.9}
$$

6. Numerical example

The developed procedures is then applied on a numerical example. This is meant to demonstrate the applicability of FST in solving SLFDEs with FCCs.

Example 6.1. The following model is adapted from fuzzy radioactive decay model from [17]. In this model, r, N_1, N_2, $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ are considered to be triangular fuzzy numbers.

$$
\begin{aligned}
&N_1'(t) = -\widetilde{\lambda}_1 N_1(t) + r, \\
&N_2'(t) = \widetilde{\lambda}_1 N_1(t) - \widetilde{\lambda}_2 N_2(t), \\
&N_1(0) = N_1, \qquad N_2(0) = N_2.
\end{aligned} \tag{6.1}
$$

In [17], the authors stated that the values of r, N_1, N_2, $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ may be acquired from experts. The lowest, the highest and the most exact values for r, N_1, N_2 and $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ can be considered, and from this, five triangular fuzzy numbers are constructed. Their corresponding parametric representations can be calculated using Eq. (2.1). By using FST on both sides of Eq. (6.1),

$$
\begin{aligned}
&\mathbb{S}[N_1'(t)](u) = \mathbb{S}[-\lambda_1 N_1(t) + r](u), \\
&\mathbb{S}[N_2'(t)](u) = \mathbb{S}[\lambda_1 N_1(t) - \lambda_2 N_2(t)](u).
\end{aligned} \tag{6.2}
$$

This is equivalent to

$$
\begin{aligned}
&\mathbb{S}[N_1'(t)](u) = -\lambda_1 \mathbb{S}[N_1(t)](u) + r, \\
&\mathbb{S}[N_2'(t)](u) = \lambda_1 \mathbb{S}[N_1(t)](u) - \lambda_2 \mathbb{S}[N_2(t)](u).
\end{aligned} \tag{6.3}
$$

Then, the procedures for solving Eq. (6.3) are divided into two cases.

Case 1: First, consider both $N_1(t)$ and $N_2(t)$ to be (i)-differentiable. Then,

$$
\begin{aligned}
\frac{s[\underline{N}_1(t)](u) - \underline{N}_1(0)}{u} &= -\overline{\lambda}_1 s[\overline{N}_1(t)](u) + \underline{r}, \\
\frac{s[\overline{N}_1(t)](u) - \overline{N}_1(0)}{u} &= -\underline{\lambda}_1 s[\underline{N}_1(t)](u) + \overline{r}, \\
\frac{s[\underline{N}_2(t)](u) - \underline{N}_2(0)}{u} &= \underline{\lambda}_1 s[\underline{N}_1(t)](u) - \overline{\lambda}_2 s[\overline{N}_2(t)](u), \\
\frac{s[\overline{N}_2(t)](u) - \overline{N}_2(0)}{u} &= \overline{\lambda}_1 s[\overline{N}_1(t)](u) - \underline{\lambda}_2 s[\underline{N}_2(t)](u).
\end{aligned} \tag{6.4}
$$

Rearranging Eq. (6.4), it is obtained that

$$
\begin{aligned}
&s[\underline{N}_1(t)] + u\overline{\lambda}_1 s[\overline{N}_1(t)](u) = \underline{N}_1(0) + \underline{r}u, \\
&s[\overline{N}_1(t)] + u\underline{\lambda}_1 s[\underline{N}_1(t)](u) = \overline{N}_1(0) + \overline{r}u, \\
&s[\underline{N}_2(t)](u) - u\underline{\lambda}_1 s[\underline{N}_1(t)](u) + u\overline{\lambda}_2 s[\overline{N}_2(t)](u) = \underline{N}_2(0), \\
&s[\overline{N}_2(t)](u) - u\overline{\lambda}_1 s[\overline{N}_1(t)](u) + u\underline{\lambda}_2 s[\underline{N}_2(t)](u) = \overline{N}_2(0).
\end{aligned} \tag{6.5}
$$

Solving Eq. (6.5),

$$
\begin{aligned}
s[\underline{N}_1(t)](u) =& (-\overline{\lambda}_1\overline{N}_1(0)+\underline{r})\frac{u}{1-\underline{\lambda}_1\overline{\lambda}_1u^2}-\overline{\lambda}_1\overline{r}\frac{u^2}{1-\underline{\lambda}_1\overline{\lambda}_1u^2}+\underline{N}_1(0)\frac{1}{1-\underline{\lambda}_1\overline{\lambda}_1u^2},\\
s[\overline{N}_1(t)](u) =& (-\underline{\lambda}_1\underline{N}_1(0)+\overline{r})\frac{u}{1-\underline{\lambda}_1\overline{\lambda}_1u^2}-\underline{\lambda}_1\underline{r}\frac{u^2}{1-\underline{\lambda}_1\overline{\lambda}_1u^2}+\overline{N}_1(0)\frac{1}{1-\underline{\lambda}_1\overline{\lambda}_1u^2},\\
s[\underline{N}_2(t)](u) =& \frac{1}{(\underline{\lambda}_1\overline{\lambda}_1u^2-1)(\underline{\lambda}_2\overline{\lambda}_2u^2-1)}\left(\underline{r}\underline{\lambda}_1\overline{\lambda}_1\overline{\lambda}_2u^4+\underline{N}_2(0)+(\underline{\lambda}_1\overline{\lambda}_1\overline{\lambda}_2\overline{N}_2(0)\right.\\
&+\underline{\lambda}_1\overline{\lambda}_1\overline{\lambda}_2\underline{N}_1(0)-\overline{r}\underline{\lambda}_1\overline{\lambda}_1-\overline{r}\overline{\lambda}_2\overline{\lambda}_1)u^3-(\underline{\lambda}_1\overline{\lambda}_1\underline{N}_2(0)-\underline{r}\underline{\lambda}_1\\
&\left.+\overline{N}_1(0)\underline{\lambda}_1\overline{\lambda}_1+\overline{N}_1(0)\overline{\lambda}_2\overline{\lambda}_1)u^2-(\overline{\lambda}_2\overline{N}_2(0)-\underline{N}_1(0)\underline{\lambda}_1)u\right),\\
s[\overline{N}_2(t)](u) =& \frac{1}{(\overline{\lambda}_1\underline{\lambda}_1u^2-1)(\overline{\lambda}_2\underline{\lambda}_2u^2-1)}\left(\overline{r}\overline{\lambda}_1\underline{\lambda}_1\underline{\lambda}_2u^4+\overline{N}_2(0)+(\overline{\lambda}_1\underline{\lambda}_1\underline{\lambda}_2\underline{N}_2(0))\right.\\
&+\overline{\lambda}_1\underline{\lambda}_1\underline{\lambda}_2\overline{N}_1(0-\underline{r}\overline{\lambda}_1\underline{\lambda}_1-\underline{r}\underline{\lambda}_2\underline{\lambda}_1)u^3-(\overline{\lambda}_1\underline{\lambda}_1\overline{N}_2(0)-\overline{r}\overline{\lambda}_1\\
&\left.+\underline{N}_1(0)\overline{\lambda}_1\underline{\lambda}_1+\underline{N}_1(0)\underline{\lambda}_2\underline{\lambda}_1)u^2-(\underline{\lambda}_2\underline{N}_2(0)-\overline{N}_1(0)\overline{\lambda}_1)u\right).
\end{aligned}
$$

By using the inverse of FST, finally, the solutions are as follows.

$$
\begin{aligned}
\underline{N}_1(t) =& (-\overline{\lambda}_1\overline{N}_1(0)+\underline{r})\frac{1}{\sqrt{\underline{\lambda}_1\overline{\lambda}_1}}\sinh(\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t)+\underline{N}_1(0)\cosh(\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t)\\
&-\overline{\lambda}_1\overline{r}\left(\frac{\cosh(\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t)-1}{\underline{\lambda}_1\overline{\lambda}_1}\right),\\
\overline{N}_1(t) =& (-\underline{\lambda}_1\underline{N}_1(0)+\overline{r})\frac{1}{\sqrt{\underline{\lambda}_1\overline{\lambda}_1}}\sinh(\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t)+\overline{N}_1(0)\cosh(\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t)\\
&-\underline{\lambda}_1\underline{r}\left(\frac{\cosh(\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t)-1}{\underline{\lambda}_1\overline{\lambda}_1}\right),\\
\underline{N}_2(t) =& \underline{r}\underline{\lambda}_1\overline{\lambda}_1\overline{\lambda}_2\left(\frac{e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}+1)}{2\underline{\lambda}_1\overline{\lambda}_1(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}+\frac{e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}+1)}{2\underline{\lambda}_2\overline{\lambda}_2(\underline{\lambda}_2\overline{\lambda}_2-\underline{\lambda}_1\overline{\lambda}_1)}\right.\\
&\left.+\frac{1}{\underline{\lambda}_1\overline{\lambda}_1\underline{\lambda}_2\overline{\lambda}_2}\right)+\underline{N}_2(0)\left(\frac{\underline{\lambda}_1\overline{\lambda}_1e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right.\\
&\left.-\frac{\underline{\lambda}_2\overline{\lambda}_2e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right)+(\underline{\lambda}_1\overline{\lambda}_1\overline{\lambda}_2\overline{N}_2(0)+\underline{\lambda}_1\overline{\lambda}_1\overline{\lambda}_2\underline{N}_1(0)
\end{aligned}
$$

$$- \overline{r}\underline{\lambda}_1\overline{\lambda}_1 - \overline{r}\overline{\lambda}_2\overline{\lambda}_1)\left(\frac{e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}-1)}{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)} + \frac{e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}-1)}{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}(\underline{\lambda}_2\overline{\lambda}_2-\underline{\lambda}_1\overline{\lambda}_1)}\right)$$
$$- (\underline{\lambda}_1\overline{\lambda}_1\underline{N}_2(0) - \underline{r}\underline{\lambda}_1 + \overline{N}_1(0)\underline{\lambda}_1\overline{\lambda}_1 + \overline{N}_1(0)\overline{\lambda}_2\overline{\lambda}_1)$$
$$\left(\frac{e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)} - \frac{e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right)$$
$$- (\overline{\lambda}_2\overline{N}_2(0) - \underline{N}_1(0)\underline{\lambda}_1)\left(\frac{\sqrt{\underline{\lambda}_1\overline{\lambda}_1}e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}-1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right.$$
$$\left.-\frac{\sqrt{\underline{\lambda}_2\overline{\lambda}_2}e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}-1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right),$$

$$\overline{N}_2(t) = \overline{r}\underline{\lambda}_1\overline{\lambda}_1\underline{\lambda}_2\left(\frac{e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}+1)}{2\underline{\lambda}_1\overline{\lambda}_1(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)} + \frac{e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}+1)}{2\underline{\lambda}_2\overline{\lambda}_2(\underline{\lambda}_2\overline{\lambda}_2-\underline{\lambda}_1\overline{\lambda}_1)}\right.$$
$$\left.+\frac{1}{\underline{\lambda}_1\overline{\lambda}_1\underline{\lambda}_2\overline{\lambda}_2}\right) + \overline{N}_2(0)\left(\frac{\underline{\lambda}_1\overline{\lambda}_1e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right.$$
$$\left.-\frac{\underline{\lambda}_2\overline{\lambda}_2e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right) + (\underline{\lambda}_1\overline{\lambda}_1\underline{\lambda}_2\underline{N}_2(0) + \underline{\lambda}_1\overline{\lambda}_1\underline{\lambda}_2\overline{N}_1(0)$$

$$- \underline{r}\underline{\lambda}_1\overline{\lambda}_1 - \underline{r}\underline{\lambda}_2\underline{\lambda}_1)\left(\frac{e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}-1)}{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)} + \frac{e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}-1)}{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}(\underline{\lambda}_2\overline{\lambda}_2-\underline{\lambda}_1\overline{\lambda}_1)}\right)$$
$$- (\underline{\lambda}_1\overline{\lambda}_1\overline{N}_2(0) - \overline{r}\overline{\lambda}_1 + \underline{N}_1(0)\underline{\lambda}_1\overline{\lambda}_1 + \underline{N}_1(0)\underline{\lambda}_2\underline{\lambda}_1)$$
$$\left(\frac{e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)} - \frac{e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}+1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right)$$
$$- (\underline{\lambda}_2\underline{N}_2(0) - \overline{N}_1(0)\overline{\lambda}_1)\left(\frac{\sqrt{\underline{\lambda}_1\overline{\lambda}_1}e^{-\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}(e^{2\sqrt{\underline{\lambda}_1\overline{\lambda}_1}t}-1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right.$$
$$\left.-\frac{\sqrt{\underline{\lambda}_2\overline{\lambda}_2}e^{-\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}(e^{2\sqrt{\underline{\lambda}_2\overline{\lambda}_2}t}-1)}{2(\underline{\lambda}_1\overline{\lambda}_1-\underline{\lambda}_2\overline{\lambda}_2)}\right).$$

Remark 6.2. The results in Case 1 are similar to results obtained by the authors in [15] and [29]. However, the authors in [15] and [29] only considered the condition when both $N_1(t)$ and $N_2(t)$ are (i)-differentiable. To the best of our knowledge, this is the first time the solution for Case 2 ($N_1(t)$ and $N_2(t)$ to be (ii)-differentiable) for problem adapted from fuzzy radioactive decay problem is provided in details.

Case 2: Next, consider that both $N_1(t)$ and $N_2(t)$ to be (ii)-differentiable. So,

$$\begin{aligned}
\frac{s[\underline{N}_1(t)] - \underline{N}_1(0)}{u} &= -\underline{\lambda}_1 s[\underline{N}_1(t)] + \overline{r}, \\
\frac{s[\overline{N}_1(t)] - \overline{N}_1(0)}{u} &= -\overline{\lambda}_1 s[\overline{N}_1(t)] + \underline{r}, \\
\frac{s[\underline{N}_2(t)] - \underline{N}_2(0)}{u} &= \overline{\lambda}_1 s[\overline{N}_1(t)] - \underline{\lambda}_2 s[\underline{N}_2(t)], \\
\frac{s[\overline{N}_2(t)] - \overline{N}_2(0)}{u} &= \underline{\lambda}_1 s[\underline{N}_1(t)] - \overline{\lambda}_2 s[\overline{N}_2(t)].
\end{aligned} \tag{6.6}$$

Rearranging Eq. (6.6),

$$\begin{aligned}
&(1 + \underline{\lambda}_1 u)s[\underline{N}_1(t)] = \underline{N}_1(0) + \overline{r}u, \\
&(1 + \overline{\lambda}_1 u)s[\overline{N}_1(t)] = \overline{N}_1(0) + \underline{r}u, \\
&(1 + \underline{\lambda}_2 u)s[\underline{N}_2(t)] - \overline{\lambda}_1 s[\overline{N}_1(t)]u = \underline{N}_2(0), \\
&(1 + \overline{\lambda}_2 u)s[\overline{N}_2(t)] - \underline{\lambda}_1 s[\underline{N}_1(t)]u = \overline{N}_2(0).
\end{aligned} \tag{6.7}$$

Solving Eq. (6.7),

$$\begin{aligned}
s[\underline{N}_1(t)] =& \underline{N}_1(0)\frac{1}{(1 + \underline{\lambda}_1 u)} + \overline{r}\frac{u}{(1 + \underline{\lambda}_1 u)}, \\
s[\overline{N}_1(t)] =& \overline{N}_1(0)\frac{1}{(1 + \overline{\lambda}_1 u)} + \underline{r}\frac{u}{(1 + \overline{\lambda}_1 u)}, \\
s[\underline{N}_2(t)] =& \underline{r}\overline{\lambda}_1 \frac{u^2}{(1 + u\overline{\lambda}_1)(1 + u\underline{\lambda}_2)} + \underline{N}_2(0)\frac{1}{1 + u\underline{\lambda}_2} + \overline{N}_1(0)\overline{\lambda}_1 \frac{u}{(1 + u\overline{\lambda}_1)(1 + u\underline{\lambda}_2)}, \\
s[\overline{N}_2(t)] =& \overline{r}\underline{\lambda}_1 \frac{u^2}{(1 + u\underline{\lambda}_1)(1 + u\overline{\lambda}_2)} + \overline{N}_2(0)\frac{1}{1 + u\overline{\lambda}_2} + \underline{N}_1(0)\overline{\lambda}_1 \frac{u}{(1 + u\underline{\lambda}_1)(1 + u\overline{\lambda}_2)}.
\end{aligned} \tag{6.8}$$

By using the inverse of FST, the solutions obtained are as follows.

$$\begin{aligned}
\underline{N}_1(t) =& \underline{N}_1(0)e^{-\underline{\lambda}_1 t} + \overline{r}\frac{1 - e^{-\underline{\lambda}_1 t}}{\underline{\lambda}_1}, \\
\overline{N}_1(t) =& \overline{N}_1(0)e^{-\overline{\lambda}_1 t} + \underline{r}\frac{1 - e^{-\overline{\lambda}_1 t}}{\overline{\lambda}_1}, \\
\underline{N}_2(t) =& \underline{r}\overline{\lambda}_1 \frac{1}{\underline{\lambda}_2 - \overline{\lambda}_1}\left(\frac{1 - e^{-\overline{\lambda}_1 t}}{\overline{\lambda}_1} - \frac{1 - e^{-\underline{\lambda}_2 t}}{\underline{\lambda}_2}\right) + \overline{N}_1(0)\overline{\lambda}_1 \frac{1}{\underline{\lambda}_2 - \overline{\lambda}_1}(e^{-\overline{\lambda}_1 t} - e^{-\underline{\lambda}_2 t}) \\
&+ \underline{N}_2(0)e^{-\underline{\lambda}_2 t}, \\
\overline{N}_2(t) =& \overline{r}\underline{\lambda}_1 \frac{1}{\overline{\lambda}_2 - \underline{\lambda}_1}\left(\frac{1 - e^{-\underline{\lambda}_1 t}}{\underline{\lambda}_1} - \frac{1 - e^{-\overline{\lambda}_2 t}}{\overline{\lambda}_2}\right) + \underline{N}_1(0)\underline{\lambda}_1 \frac{1}{\overline{\lambda}_2 - \underline{\lambda}_1}(e^{-\underline{\lambda}_1 t} - e^{-\overline{\lambda}_2 t}) \\
&+ \overline{N}_2(0)e^{-\overline{\lambda}_2 t}.
\end{aligned} \tag{6.9}$$

For illustration in tables and graphs, we choose $r = (4.9, 5, 5.1)$, $N_1 = (995, 1000, 1005)$, $N_2 = (0, 0, 0)$, $\widetilde{\lambda}_1 = (0.2, 0.3, 0.4)$ and $\widetilde{\lambda}_2 = (0.02, 0.03, 0.04)$ respectively. Their parametric representations can be calculated as in Definition 2.1. The results of $N_1(t)$ and $N_2(t)$ for Case 1 are illustrated in Figs. 1 and 2, plotted using MATLAB software, and the values are listed in Tables 7 and 7. The results of $N_1(t)$ and $N_2(t)$ for Case 2 are illustrated in Figs. 3 and 4, also using MATLAB software. The numerical values corresponding to this case are in Tables 7 and 7.

It can be seen that for Case 1, the results expand as t increases. This is in accord with the concept of fuzzy derivatives for (i)-differentiability when using the strongly generalized differentiability to interpret SLFDEs. While for Case 2, the results implicate that the solutions contract as the values of t increases. Similarly, this condition is in accord with the concept of fuzzy derivatives for (ii)-differentiability.

For the sake of comparison, we would like to emphasized that the results obtained when $\alpha = 1$ are similar to the solutions for system of ordinary differential equations. Thus, it can be concluded that SLFDE is an extension of the system of ordinary differential equations. Furthermore, we can also said that SLFDE is a generalized version of system of ordinary differential equations since the set of real numbers $\mathbb{R}$ are subset of the set of fuzzy numbers or fuzzy real numbers $\mathcal{F}(\mathbb{R})$. The advantage of the solutions that we provided in this paper, is that this system are able to cope with uncertainties at initial values as well as having FCCs. This is very common when dealing with real life problems.

7. Conclusions

In this work, we have studied FST for solving SLFDEs with FCCs. One of the advantage of FST, the scale preserving property has been highlighted in Subsection 3.1. This is done to justify why we chose to utilize FST in this paper. Then, procedures for obtaining the solutions of SLFDEs using FST are constructed in Section 5. The process for finding the solutions is then demonstrated on a numerical example adapted from fuzzy radioactive decay model and this can be seen in Section 6. We believe that this research will be beneficial for scientists and engineers when they face problems that are surrounded with fuzziness. For future research, we intend to explore other real life problems. Besides, we will also focus on integrating FST for solving system of fuzzy nonlinear problems including fuzzy differential equation of fractional order.

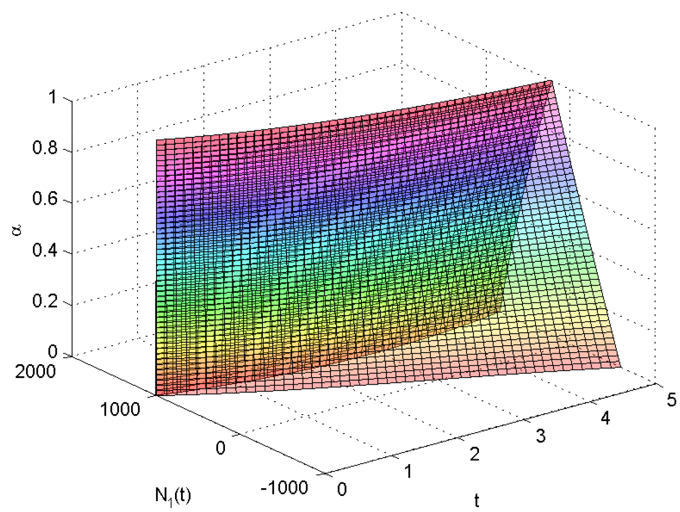

FIGURE 1. The graphical solution of $N_1(t)$ in Eq. (6.1) for Case 1.

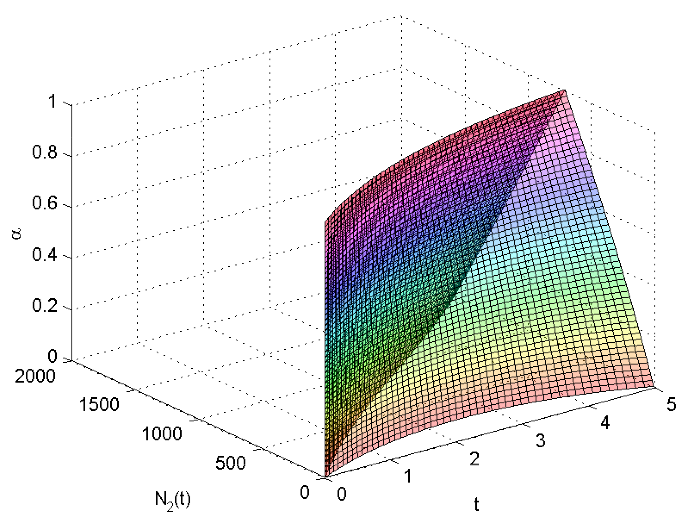

FIGURE 2. The graphical solution of $N_2(t)$ in Eq. (6.1) for Case 1.

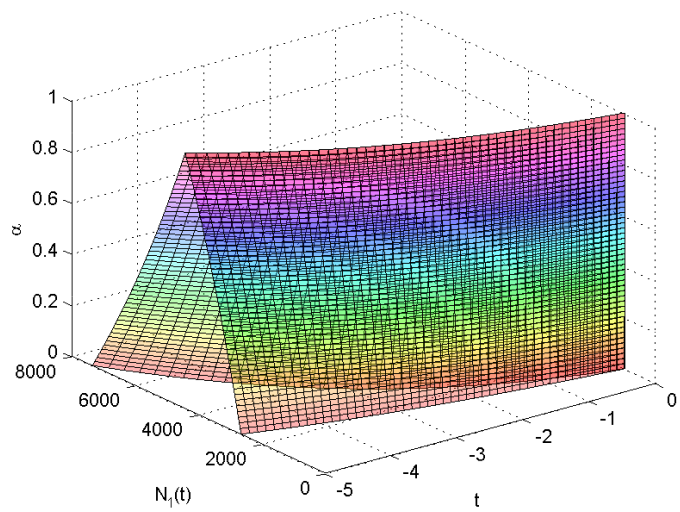

FIGURE 3. The graphical solution of $N_1(t)$ in Eq. (6.1) for Case 2.

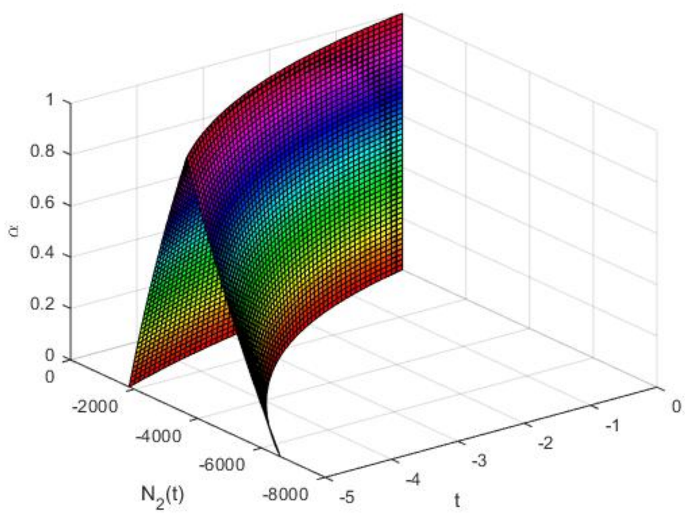

FIGURE 4. The graphical solution of $N_2(t)$ in Eq. (6.1) for Case 2.

α	$t = 0.00$		$t = 2.00$		$t = 4.00$	
	$\underline{N}_1(t)$	$\overline{N}_1(t)$	$\underline{N}_1(t)$	$\overline{N}_1(t)$	$\underline{N}_1(t)$	$\overline{N}_1(t)$
0.0	995.0000	1005.0000	317.0607	759.3051	-265.0646	759.1110
0.1	995.5000	1004.5000	342.3528	740.8885	-203.7037	722.4528
0.2	996.0000	1004.0000	367.3685	722.0330	-142.8925	683.8660
0.3	996.5000	1003.5000	392.0974	702.7460	-82.7248	643.4041
0.4	997.0000	1003.0000	416.5293	683.0353	-23.2936	601.1243
0.5	997.5000	1002.5000	440.6543	662.9089	35.3101	557.0864
0.6	998.0000	1002.0000	464.4623	642.3748	92.9964	511.3538
0.7	998.5000	1001.5000	487.9435	621.4415	149.6776	463.9925
0.8	999.0000	1001.0000	511.0884	600.1174	205.2677	415.0716
0.9	999.5000	1000.5000	533.8875	578.4111	259.6827	364.6629
1.0	1000.0000	1000.0000	556.3314	556.3314	312.8410	312.8410

TABLE 1. The solutions of $N_1(t)$ in Eq. (6.1) for Case 1.

α	$t = 0.00$		$t = 2.00$		$t = 4.00$	
	$\underline{N}_2(t)$	$\overline{N}_2(t)$	$\underline{N}_2(t)$	$\overline{N}_2(t)$	$\underline{N}_2(t)$	$\overline{N}_2(t)$
0.0	0.0000	0.0000	227.2468	682.1144	159.7583	1265.3728
0.1	0.0000	0.0000	246.7859	656.6515	202.8278	1202.1398
0.2	0.0000	0.0000	266.7095	631.4204	247.5739	1139.2478
0.3	0.0000	0.0000	287.0105	606.4306	293.9422	1076.7858
0.4	0.0000	0.0000	307.6815	581.6917	341.8578	1014.8416
0.5	0.0000	0.0000	328.7148	557.2130	391.3146	953.5020
0.6	0.0000	0.0000	350.1029	533.0038	442.1961	892.8524
0.7	0.0000	0.0000	371.8379	509.0733	494.4550	832.9766
0.8	0.0000	0.0000	393.9118	485.4305	548.0236	773.9570
0.9	0.0000	0.0000	416.3164	462.0842	602.8317	715.8741
1.0	0.0000	0.0000	439.0433	439.0433	658.8067	658.8067

TABLE 2. The solutions of $N_2(t)$ in Eq. (6.1) for Case 1.

α	$t=-4.00$		$t=-2.00$		$t=0.00$	
	$\underline{N}_1(t)$	$\overline{N}_1(t)$	$\underline{N}_1(t)$	$\overline{N}_1(t)$	$\underline{N}_1(t)$	$\overline{N}_1(t)$
0.0	2183.2000	4929.4000	1471.8000	2221.7000	995.0000	1005.0000
0.1	2274.0000	4732.9000	1502.5000	2176.4000	995.5000	1004.5000
0.2	2368.7000	4544.3000	1533.7000	2132.1000	996.0000	1004.0000
0.3	2467.2000	4363.1000	1565.7000	2088.7000	996.5000	1003.5000
0.4	2569.9000	4189.2000	1598.2000	2046.1000	997.0000	1003.0000
0.5	2676.8000	4022.1000	1631.5000	2004.4000	997.5000	1002.5000
0.6	2788.1000	3861.7000	1665.4000	1963.6000	998.0000	1002.0000
0.7	2904.0000	3707.7000	1700.1000	1923.6000	998.5000	1001.5000
0.8	3024.8000	3559.8000	1735.5000	1884.4000	999.0000	1001.0000
0.9	3150.5000	3417.8000	1771.6000	1846.0000	999.5000	1000.5000
1.0	3281.4000	3281.4000	1808.4000	1808.4000	1000.0000	1000.0000

TABLE 3. The solutions of $N_1(t)$ in Eq. (6.1) for Case 2.

α	$t=-4.00$		$t=-2.00$		$t=0.00$	
	$\underline{N}_2(t)$	$\overline{N}_2(t)$	$\underline{N}_2(t)$	$\overline{N}_2(t)$	$\underline{N}_2(t)$	$\overline{N}_2(t)$
0.0	-4064.2892	-1297.0516	-1248.0417	-505.7156	0.0000	0.0000
0.1	-3869.2213	-1390.7951	-1203.6690	-536.4104	0.0000	0.0000
0.2	-3681.6257	-1488.1661	-1160.1560	-567.7006	0.0000	0.0000
0.3	-3501.2085	-1589.3129	-1117.4851	-599.5985	0.0000	0.0000
0.4	-3327.6868	-1694.3904	-1075.6394	-632.1166	0.0000	0.0000
0.5	-3160.7894	-1803.5594	-1034.6023	-665.2678	0.0000	0.0000
0.6	-3000.2554	-1916.9876	-994.3574	-699.0650	0.0000	0.0000
0.7	-2845.8343	-2034.8494	-954.8887	-733.5216	0.0000	0.0000
0.8	-2697.2856	-2157.3263	-916.1806	-768.6511	0.0000	0.0000
0.9	-2554.3782	-2284.6076	-878.2176	-804.4675	0.0000	0.0000
1.0	-2416.8899	-2416.8899	-840.9848	-840.9848	0.0000	0.0000

TABLE 4. The solutions of $N_2(t)$ in Eq. (6.1) for Case 2.

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