On the Zero-divisor Cayley Graph of a Finite Commutative Ring

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Abstract. Let $R$ be a finite commutative ring. Let $Z(R)$ and $J(R)$ be the set of all zero-divisor elements and the Jacobson radical of $R$, respectively. The zero-divisor Cayley graph of $R$, denoted by $Z\text{CAY}(R)$, is the graph obtained by setting all the elements of $Z(R)$ to be the vertices and defining distinct vertices $x$ and $y$ to be adjacent if and only if $x - y \in Z(R)$. The induced subgraph of $Z\text{CAY}(R)$ on the vertex set $Z(R) \setminus J(R)$ is denoted by $Z\text{CAY}^*(R)$. In this paper, the basic properties of $Z\text{CAY}(R)$ and $Z\text{CAY}^*(R)$ are investigated and some characterization results regarding connectedness, girth and planarity of $Z\text{CAY}(R)$ and $Z\text{CAY}^*(R)$ are given. Finally, we study the clique number of $Z\text{CAY}(R)$.

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1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in the last decade, see for example [1, 2, 4, 10]. The Cayley graph introduced by Arthur Cayley in 1878 is a useful tool for connection between group theory and the theory of algebraic graphs. Let $G$ be an abelian group and $S$ be a subset of $G$. The Cayley graph $\text{CAY}(G, S)$ is
A graph whose vertices are elements of $G$ and in which two distinct vertices $x$ and $y$ are joined by an edge if and only if $x - y \in S$. We refer the reader to [8] for general properties of Cayley graphs. Let $R$ be a commutative ring with identity and $R^+$ and $Z(R)$ be the additive group and the set of all zero-divisors of $R$, respectively. The authors in [1] have studied $\text{CAY}(R^+, Z(R))$ and its subgraph $\text{RegCAY}(R)$ the induced subgraph on the regular elements of $R$. We denote by $Z\text{CAY}(R)$ the induced subgraph on zero-divisor elements of $R$. In this paper, following [4], we are interested in studying $Z\text{CAY}(R)$.

Let $J(R)$ denote the Jacobson radical of $R$. It is easy to see that every $x \in J(R)$ is adjacent to each vertex of $Z\text{CAY}(R)$. Thus the main part of the graph $Z\text{CAY}(R)$ is the induced subgraph of $Z\text{CAY}(R)$ on the vertex set $Z(R) \setminus J(R)$. We denote it by $Z\text{CAY}^*(R)$. The graphs in Figure 1 are the zero-divisor Cayley graphs of the rings indicated. In the figures of this paper, the vertices in Jacobson radical are shown by circle.

![Graph Diagram](image)

Figure 1. The zero-divisor Cayley graphs of some specific rings.

The plan of the paper is as follows: In Section 2 of this paper, we bring some preliminaries and notations about graph and ring theory. In Section 3, we state some basic properties of $Z\text{CAY}(R)$. In Section 4, we study the connectivity, diameter and girth of $Z\text{CAY}(R)$ and $Z\text{CAY}^*(R)$. In Section 5, the planarity of $Z\text{CAY}(R)$ and $Z\text{CAY}^*(R)$ are investigated. In the final section, we study the clique number of $Z\text{CAY}(R)$.

2. Preliminaries and Notations

The graphs in this paper are simple, that is they have no loops or multiple edges. For a graph $G$, let $V(G)$ denote the set of vertices, and let $E(G)$ denote the set of edges. For $x \in V(G)$ we denote by $N(x)$ the set of all vertices of $G$...
adjacent to $x$. Also, the \textit{degree} of $x$, denoted $d_G(x)$, is the size of $N(x)$. The maximum and minimum degree of vertices of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The \textit{union} of two simple graphs $G$ and $H$ is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If $V(G)$ and $V(H)$ are disjoint, we refer to their union as a \textit{disjoint union}, and denote it by $G + H$. The \textit{join} of simple graphs $G$ and $H$, written $G \ast H$, is the graph obtained from the disjoint union $G + H$ by adding edges joining every vertex of $G$ to every vertex of $H$.

Let $G$ be a graph. For two vertices $x$ and $y$ of $G$, a \textit{walk (path)} of length $n$ between $x$ and $y$ is an ordered list of (distinct) vertices $x = x_0, x_1, \ldots, x_n = y$ such that $x_{i-1}$ is adjacent to $x_i$ for $i = 1, \ldots, n$. The \textit{distance} between $x$ and $y$, denoted by $d(x, y)$, is the length of shortest path between $x$ and $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path between $x$ and $y$). The largest distance among all distances between pairs of the vertices of a graph $G$ is called the \textit{diameter} of $G$ and is denoted by $\text{diam}(G)$. A \textit{cycle} in $G$ is a path that begins and ends at the same vertex. The \textit{girth} of $G$, denoted by $\text{gr}(G)$, is the length of a shortest cycle in $G$ ($\text{gr}(G) = \infty$ if $G$ has no cycle). A graph $G$ is called \textit{connected} if for any vertices $x$ and $y$ of $G$ there is a path between $x$ and $y$. Otherwise, $G$ is called \textit{disconnected} (a singleton graph is connected with zero diameter). The \textit{null graph} is the graph whose vertex set and edge set are empty. A graph in which each pair of distinct vertices is joined by an edge is called a \textit{complete graph}. We denote the complete graph on $n$ vertices by $K_n$. A \textit{clique} of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph $G$, denoted by $\omega(G)$, is called the \textit{clique number} of $G$. The complement of a graph $G$, denoted by $\overline{G}$, is the graph with the same vertex set as $G$ such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$.

For a set $X$, $|X|$ denotes the cardinal number of $X$. Also, $\mathbb{F}_{p^n}$ denotes the field with $p^n$ elements and $\mathbb{Z}_n$ denotes for the ring of integers modulo $n$. Following the literature, we write

$$D_2(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

In this paper, for convenience, we denote the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $D_2(R)$ by $A$ and $B$, respectively.

It is well known that every finite commutative ring can be expressed as a direct product of finite local rings, and this decomposition is unique up to permutations of such local rings (see [5, Theorem 8.7]). In this paper, we assume that $R$ is a finite commutative ring with identity and we accept the following notations: Let $R = R_1 \times \cdots \times R_n$ be a ring, where $R_i$s are local
rings. We set $e_0 := (0,0,\ldots,0)$, $e_1 := (1,0,\ldots,0)$, $e_2 := (0,1,0,\ldots,0)$, ..., $e_n := (0,\ldots,0,1)$.

For a ring $R$, we denote by $U(R)$ the set of all unit elements of $R$. We note that if $R$ is a finite commutative ring, then $U(R) = R \setminus Z(R)$. In other words, the set of all zero-divisors and the set of all nonunit elements of $R$ coincide. If $R = R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $m_i$, then

$$Z(R) = \{(x_1,x_2,\ldots,x_n) \in R | x_i \in m_i \text{ for some } 1 \leq i \leq n\}.$$

This fact is frequently used in this paper.

3. Some Basic Properties of $ZCAY(R)$

In this section, we study the basic properties of $ZCAY(R)$. We begin with the following proposition.

Proposition 3.1. The following are equivalent:
(1) $ZCAY^*(R)$ is a null graph,
(2) $Z(R) = J(R)$,
(3) $ZCAY(R)$ is a complete graph,
(4) $R$ is a local ring.

Proof. (1)$\Leftrightarrow$(2) is trivial.
(2)$\Rightarrow$(3) Suppose that $J(R) = Z(R)$. Then $Z(R)$ is an ideal and hence $ZCAY(R)$ is a complete graph.
(3)$\Rightarrow$(4) By [9, Lemma 3.13], it is enough to show that $Z(R)$ is an ideal of $R$. It is easy to see that $Z(R)$ is closed under scalar multiplication. Now let $x, y \in Z(R)$. Since $ZCAY(R)$ is a complete graph, we have $x - y \in Z(R)$. So $Z(R)$ is an ideal of $R$.
(4)$\Rightarrow$(2) Let $R$ be a local ring with maximal ideal $m$. Then $J(R) = Z(R) = m$. \qed

In the rest of this section, we study the maximum and minimum degree of $ZCAY(R)$.

Proposition 3.2. Let $R = R_1 \times \cdots \times R_n$ be a ring, where $R_i$ is a local ring with maximal ideal $m_i$. Let $(a_1,\ldots,a_n) \in Z(R)$ and $G = ZCAY(R)$. Then $d_G(a_1,\ldots,a_n) = d_G(\delta_1,\ldots,\delta_n)$, where

$$\delta_i = \begin{cases} 1 & \text{if } a_i \in U(R_i), \\ 0 & \text{if } a_i \in m_i. \end{cases}$$
Proof. Let $G = ZCA(Y)(R)$ and let $a = (a_1, ..., a_k, \alpha, \beta, a_{k+3}, ..., a_n)$ be an element of $Z(R)$, where $(\alpha, \beta) \in (m_{k+1}, U(R_{k+2}))$. Then

\[
\begin{align*}
d_{\mathcal{C}}(a) &= d_{\mathcal{C}}(a_1, ..., a_k, \alpha, \beta, a_{k+3}, ..., a_n) \\
&= |\{(d_1, ..., d_k, x, y, d_{k+3}, ..., d_n) \in Z(R) | d_i - a_i \in U(R_i) \text{ for } 1 \leq i \leq n \text{ with } i \neq k+1, k+2, x - \alpha \in U(R_{k+1}), y - \beta \in U(R_{k+2})\}| \\
&= |\{(d_1, ..., d_k, x, y, d_{k+3}, ..., d_n) \in Z(R) | d_i - a_i \in U(R_i) \text{ for } 1 \leq i \leq n \text{ with } i \neq k+1, k+2, x \in U(R_{k+1}), y - 1 \in U(R_{k+2})\}| \\
&= d_{\mathcal{C}}(a_1, ..., a_k, 0, 1, a_{k+3}, ..., a_n).
\end{align*}
\]

A similar argument works if $(\alpha, \beta) \in (U(R_{k+1}), m_{k+2}) \cup (m_{k+1}, m_{k+2}) \cup (U(R_{k+1}), U(R_{k+2}))$. Now the assertion follows by repeating this argument. $\square$

Theorem 3.3. Let $R = R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $m_i$. Let $G = ZCA(Y)(R)$ and $\delta = (\delta_1, ..., \delta_n) \in Z(R)$, where $\delta_i \in \{0, 1\}$ for all $i = 1, 2, ..., n$. Then

\[
d_{\mathcal{C}}(\delta) = |Z(R)| - |U(R)| - 1 + \prod_{1 \leq i \leq n} (|R_i| - |m_i|) \prod_{1 \leq i \leq n} (|R_i| - 2|m_i|).
\]

Proof. Let $N := \{(x_1, x_2, ..., x_n) \in R | x_i - \delta_i \in U(R_i) \text{ for all } 1 \leq i \leq n\}$. Then $N = \{\delta_1 + u_1, \delta_2 + u_2, ..., \delta_n + u_n \in R | u_i \in U(R_i) \text{ for all } 1 \leq i \leq n\}$ and hence $|N| = |U(R)|$. On the other hand,

\[
\begin{align*}
|N \cap U(R)| &= |\{(x_1, x_2, ..., x_n) \in U(R) | x_i - \delta_i \in U(R_i) \text{ for all } 1 \leq i \leq n\}| \\
&= |\{(x_1, x_2, ..., x_n) \in U(R) | x_i - \delta_i / m_i \text{ for all } 1 \leq i \leq n\}| \\
&= |\{(x_1, x_2, ..., x_n) \in R | x_i - \delta_i / m_i \text{ and } x_i \not\in m_i \text{ for all } 1 \leq i \leq n\}| \\
&= \prod_{1 \leq i \leq n} (|R_i/m_i| - 1)|m_i| \prod_{1 \leq i \leq n} (|R_i/m_i| - 2)|m_i| \\
&= \prod_{1 \leq i \leq n} (|R_i| - |m_i|) \prod_{1 \leq i \leq n} (|R_i| - 2|m_i|).
\end{align*}
\]

Since $N = (N \cap U(R)) \cup (N \cap Z(R))$ and $(N \cap U(R)) \cap (N \cap Z(R)) = \emptyset$, we have

\[
\begin{align*}
d_{\mathcal{C}}(\delta) &= |N \cap Z(R)| \\
&= |N \setminus (N \cap U(R))| \\
&= |N| - |(N \cap U(R))| \\
&= |U(R)| - \prod_{1 \leq i \leq n} (|R_i| - |m_i|) \prod_{1 \leq i \leq n} (|R_i| - 2|m_i|).
\end{align*}
\]

Now the assertion follows from the fact that $d_{\mathcal{C}}(\delta) + d_{\mathcal{C}}(\delta) = |Z(R)| - 1$. $\square$
The following result is an immediate consequence of the proof of Proposition 3.2 and Theorem 3.3.

**Corollary 3.4.** Let \( R = F_1 \times \cdots \times F_n \), where \( F_i \)'s are fields and \( |F_i| = p_i^{\alpha_i} \) and \( p_1^{\alpha_1} \leq p_2^{\alpha_2} \leq \cdots \leq p_n^{\alpha_n} \). If \( G = \ZCA(R) \), then
\[
\Delta(G) = d_G(0,0,\ldots,0) = |Z(R)| - 1,
\]
\[
\delta(G) = d_G(1,1,\ldots,1,0) = |Z(R)| - 1 - \left[ p_n^{\alpha_n} - 1 \right] \prod_{i=1}^{n-1} (p_i^{\alpha_i} - 1) - \prod_{i=1}^{n-1} (p_i^{\alpha_i} - 2).
\]

4. Connectivity, Diameter and Girth

The following theorem determines the diameter of \( \ZCA(R) \).

**Theorem 4.1.** Let \( R \) be a ring. Then \( \ZCA(R) \) is connected and
\[
diam(\ZCA(R)) = \begin{cases} 0 & \text{if } R \text{ is a field,} \\ 1 & \text{if } R \text{ is local which is not a field,} \\ 2 & \text{otherwise.} \end{cases}
\]

**Proof.** Let \( R = R_1 \times \cdots \times R_n \), where \( R_i \)'s are local rings. First suppose that \( n = 1 \). In this case we may assume \( R \) is a local ring with maximal ideal \( m \). If \( R \) is a field, then \( \ZCA(R) \) has only one vertex and hence \( diam(\ZCA(R)) = 0 \). If \( n = 1 \) and \( R \) is not a field, then \( \ZCA(R) \cong K_m \). Hence \( diam(\ZCA(R)) = 1 \).

Second suppose that \( n \geq 2 \). Let \( a, b \) be two distinct elements of \( Z(R) \setminus J(R) \) and let \( c \in J(R) \). Then \( a, b \in N(c) \). It follows that \( diam(\ZCA(R)) \leq 2 \). On the other hand, if \( x = (1,0,0,\ldots,0) \) and \( y = (0,1,1,\ldots,1) \), then \( x \) and \( y \) are not adjacent, and so \( d(x,y) \geq 2 \). Therefore \( diam(\ZCA(R)) = 2 \) and the proof is complete. \( \Box \)

In the following theorem, we completely characterize the girth of \( \ZCA(R) \).

**Theorem 4.2.** Let \( R \) be a ring. Then \( gr(\ZCA(R)) \in \{3, \infty\} \) and \( gr(\ZCA(R)) = \infty \) if and only if \( R \) is isomorphic to one of the following rings:
\[
F_{p^\alpha}, Z_4, D_2(Z_2), Z_2 \times Z_2.
\]

**Proof.** Let \( R = R_1 \times \cdots \times R_n \), where \( R_i \)'s are local rings. We consider the following cases:

Case 1: \( n = 1 \). In this case we may assume \( R \) is a local ring with maximal ideal \( m \). If \( |m| = 1 \), then \( R \) is a field and hence \( \ZCA(R) \) has only one vertex and hence \( gr(\ZCA(R)) = \infty \) and \( R = F_{p^\alpha} \). If \( |m| = 2 \), then \( \ZCA(R) \cong K_2 \) and hence \( gr(\ZCA(R)) = \infty \). Since \( m \) is a nonzero finite dimensional vector space over the field \( R/m \), we must have \( |R/m| \leq |m| \) and hence \( |R| = 4 \). Therefore \( R = Z_4 \) or \( R = D_2(Z_2) \) (see [6, Page 687]). If \( |m| \geq 3 \), then every three distinct elements of \( m \) form a triangle and so \( gr(\ZCA(R)) = 3 \).
Case 2: \( n = 2 \). If \( |R_1| \geq 3 \), then \((r_1, 0), (r_2, 0), (r_3, 0)\), where \( r_1, r_2, r_3 \) are distinct elements of \( R_1 \), form a triangle and so \( \text{gr}(\text{ZCA\!Y}(R)) = 3 \). A similar argument shows that if \( |R_2| \geq 3 \), then \( \text{gr}(\text{ZCA\!Y}(R)) = 3 \). Otherwise \( R = R_1 \times R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) and hence \( \text{gr}(\text{ZCA\!Y}(R)) = \infty \).

Case 3: \( n \geq 3 \). In this case, \( e_1, e_2, e_3 \) form a triangle in \( \text{ZCA\!Y}(R) \) and so \( \text{gr}(\text{ZCA\!Y}(R)) = 3 \). □

**Proposition 4.3.** Let \( R \) be a ring such that \( \text{ZCA\!Y}^*(R) \) is not a null graph. Then the following are equivalent:

1. \( \text{ZCA\!Y}^*(R) \) is disconnected,
2. \( R \) is a direct product of two local rings,
3. \( \text{ZCA\!Y}^*(R) \) is disjoint union of two complete graphs.

**Proof.** Let \( R = R_1 \times \cdots \times R_n \), where \( R_i \) is a local ring with maximal ideal \( m_i \).

(1)\(\Rightarrow\)(2) By Proposition 3.1, it is enough to show that \( n \leq 2 \). Suppose on the contrary that \( n \geq 3 \). Now let \( a = (a_1, a_2, a_3, \ldots, a_n) \) and \( b = (b_1, b_2, b_3, \ldots, b_n) \) be two arbitrary distinct vertices of \( \text{ZCA\!Y}^*(R) \). Set \( c = (1, 0, a_3, \ldots, a_n) \) and \( d = (1, 0, b_3, \ldots, b_n) \). Then \( a, c, d, b \) is a walk and hence \( \text{ZCA\!Y}^*(R) \) is connected, which is a contradiction.

(2)\(\Rightarrow\)(3) Let \( R = R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are local rings with maximal ideals \( m_1 \) and \( m_2 \), respectively. Then \( \text{ZCA\!Y}^*(R) \) is disjoint union of two complete graphs with vertex sets \( m_1 \times U(R_2) \) and \( U(R_1) \times m_2 \).

(3)\(\Rightarrow\)(1) is trivial. □

In the following figures the graphs \( \text{ZCA\!Y}(\mathbb{F}_m \times \mathbb{F}_n), \text{ZCA\!Y}(\mathbb{F}_n \times \mathbb{Z}_4) \) and \( \text{ZCA\!Y}(\mathbb{F}_n \times D_2(\mathbb{Z}_2)) \) are presented.

![Graph](image)

**Figure 2.** The graph \( \text{ZCA\!Y}(\mathbb{F}_m \times \mathbb{F}_n) = (K_{m-1} + K_{n-1}) \lor K_1 \).
Figure 3. The graph $\text{ZCAY}(\mathbb{F}_n \times \mathbb{Z}_4)$, if $x = 2$ and $y = 3$; and the graph $\text{ZCAY}(\mathbb{F}_n \times D_2(\mathbb{Z}_2))$, if $x = A$ and $y = B$.

We note that $\text{ZCAY}(\mathbb{F}_n \times \mathbb{Z}_4) = \text{ZCAY}(\mathbb{F}_n \times D_2(\mathbb{Z}_2)) = (K_{2(n-1)} + K_2) \cup K_2$, by Figure 3.

The following theorem characterizes the diameter of $\text{ZCAY}^*(R)$.

**Theorem 4.4.** Let $R$ be a ring such that $\text{ZCAY}^*(R)$ is not a null graph. Then $\text{diam}(\text{ZCAY}^*(R)) \in \{2, \infty\}$ and $\text{diam}(\text{ZCAY}^*(R)) = \infty$ if and only if $R$ is a direct product of two local rings.

**Proof.** Let $R = R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $m_i$.

Since $\text{ZCAY}^*(R)$ is not a null graph, by Proposition 3.1, we have two following cases:

Case 1: $n = 2$. In this case, Proposition 4.3 implies that $\text{diam}(\text{ZCAY}^*(R)) = \infty$.

Case 2: $n \geq 3$. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two arbitrary distinct vertices of $\text{ZCAY}^*(R)$. Then there are $i, j \in \{1, 2, \ldots, n\}$ such that $a_i \in m_i$ and $b_j \in m_j$. Let $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$. Then $a, c_k, b$ form a path in $\text{ZCAY}^*(R)$ and so $\text{diam}(\text{ZCAY}^*(R)) \leq 2$. On the other hand, if $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 1, \ldots, 1)$, then $x$ and $y$ are not adjacent, and so $d(x, y) \geq 2$. Therefore $\text{diam}(\text{ZCAY}^*(R)) = 2$ and the proof is complete. \(\square\)

In the following theorem, we completely characterize the girth of $\text{ZCAY}^*(R)$.

**Theorem 4.5.** Let $R$ be a ring such that $\text{ZCAY}^*(R)$ is not a null graph. Then $\text{gr}(\text{ZCAY}^*(R)) \in \{3, \infty\}$ and $\text{gr}(\text{ZCAY}^*(R)) = \infty$ if and only if $R \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times D_2(\mathbb{Z}_2), \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times D_2(\mathbb{Z}_2)\}$.

**Proof.** Let $R = R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $m_i$.

Since $\text{ZCAY}^*(R)$ is not a null graph, we only have two following cases:

Case 1: $n = 2$. If $|U(R_1)| = |R_1 \setminus m_1| \geq 3$ or $|U(R_2)| = |R_2 \setminus m_2| \geq 3$, then $\text{ZCAY}^*(R)$ has a triangle and so $\text{gr}(\text{ZCAY}^*(R)) = 3$. Now suppose that $|U(R_1)| \leq 2$ and $|U(R_2)| \leq 2$. Then [7, Corollary 4.5] implies that $R = R_1 \times R_2$, where $R_1, R_2 \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{D}_2(\mathbb{Z}_2)\}$. In view of Figure 2, we have $\text{gr}(\text{ZCAY}^*(R)) = \infty$ if $R \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{D}_2(\mathbb{Z}_2), \mathbb{Z}_3 \times \mathbb{D}_2(\mathbb{Z}_2)\}$. If $R = \mathbb{Z}_4 \times \mathbb{Z}_4$, then $(1, 0), (3, 0), (1, 2)$ form a triangle in $\text{ZCAY}^*(R)$. If $R = D_2(\mathbb{Z}_2) \times D_2(\mathbb{Z}_2)$, then $(1, 0), (B, 0), (1, A)$ form a triangle in $\text{ZCAY}^*(R)$ and if $R = \mathbb{Z}_4 \times D_2(\mathbb{Z}_2)$, then $(1, 0), (3, 0), (1, A)$ form a triangle in $\text{ZCAY}^*(R)$.

Case 2: $n \geq 3$. In this case, $e_1, e_2, e_3$ form a triangle in $\text{ZCAY}^*(R)$ and so $\text{gr}(\text{ZCAY}^*(R)) = 3$. \(\square\)
5. Planarity

A graph is said to be planar if it can be drawn in the plane such that its edges intersect only at their ends. A subdivision of an edge is obtained by inserting some new vertices of degree two into this edge. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$ (see [11, Theorem 6.2.2]). We recall the following proposition.

**Proposition 5.1.** ([3, Proposition 2.1]) If $R$ is a local ring with maximal ideal $m$, then there exists a prime number $p$ such that $|R/m|$, $|R|$ and $|m|$ are all powers of $p$.

The following theorem gives a necessary and sufficient condition for the planarity of $\mathcal{ZCAY}^*(R)$.

**Theorem 5.2.** Let $R = R_1 \times \cdots \times R_n$, where $R_i$ is a local ring with maximal ideal $m_i$. Then $\mathcal{ZCAY}^*(R)$ is planar if and only if $\mathcal{ZCAY}^*(R)$ is a null graph or $R$ is isomorphic to one of the following rings:

1. $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
2. $R = R_1 \times R_2$, where $R_1, R_2$ are fields with $|R_1| \leq 5$ and $|R_2| \leq 5$.

**Proof.** By Proposition 3.1, $n \geq 2$. We consider the following cases:

Case 1: $n \geq 4$. In this case the set of vertices $\{e_1, e_2, e_3, e_4, e_1 + e_2\}$ is a clique and hence $\mathcal{ZCAY}^*(R)$ is not planar, by Kuratowski’s Theorem.

Case 2: $n = 3$. If $|R_i| \leq 2$ for all $i = 1, 2, 3$, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and Figure 4 shows $\mathcal{ZCAY}^*(R)$ is planar. Now suppose that $|R_i| \geq 3$ for some $1 \leq i \leq 3$, say $i = 1$. Therefore $|U(R_1)| \geq 2$, by Proposition 5.1. Let $H$ be the subgraph induced by the vertices $\{(1, 0, 0), (a, 0, 0), (0, 1, 1), (0, 0, 1), (0, 1, 0), (1, 1, 0)\}$, where $a \in U(R_1) \setminus \{1\}$. Then it is easy to see that $K_{3,3}$ is a subgraph of $H$ and hence $\mathcal{ZCAY}^*(R)$ is not planar, by Kuratowski’s Theorem.

Case 3: $n = 2$ and $R = R_1 \times R_2$ with $|R_i| > 5$ for some $1 \leq i \leq 2$. Without loss of generality, we may assume that $|R_1| > 5$. By Proposition 5.1, we have $|R_1 \setminus m_1| \geq 6$. Now the set $U(R_1) \times \{0\}$ has a clique of order 6 and hence $\mathcal{ZCAY}^*(R)$ is not planar, by Kuratowski’s Theorem.

So, if $\mathcal{ZCAY}^*(R)$ is planar, then $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R = R_1 \times R_2$, where $R_1, R_2$ are local rings with $|R_1| \leq 5$ and $|R_2| \leq 5$.

On the other hand, in view of Figures 2, 3 and 4, it is easy to see that if $R$ is isomorphic to one of the above rings, then $\mathcal{ZCAY}^*(R)$ is planar. This completes the proof. □
This completes the proof.

in view of Figures 2 and 3, it is easy to see that, if $Z$ appears in the statement of the theorem, then $Z$ is one of the rings that $R$ is isomorphic to one of the following rings: $\mathbb{F}_p$, $\mathbb{Z}_4$, $D_2(\mathbb{Z}_2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times D_2(\mathbb{Z}_2)$.

Theorem 5.3. Let $R$ be a ring. Then $ZCA\mathcal{Y}(R)$ is planar if and only if $R$ is isomorphic to one of the following rings: $\mathbb{F}_p$, $\mathbb{Z}_4$, $D_2(\mathbb{Z}_2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 + x + 1)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times D_2(\mathbb{Z}_2)$.

Proof. Let $R = R_1 \times \cdots \times R_n$, where $R_i$'s are local rings. Let $ZCA\mathcal{Y}(R)$ be a planar graph. We consider the following cases:

Case 1: $n = 1$. In this case, we may assume $R$ is a local ring with maximal ideal $m$. If $m = 0$, then $R$ is a field and $ZCA\mathcal{Y}(R)$ has only one vertex and hence $ZCA\mathcal{Y}(R)$ is planar. Now let $m \neq 0$. Since $m$ is a nonzero finite dimensional vector space over the field $R/m$, we must have $|R/m| \leq |m|$. Since $ZCA\mathcal{Y}(R)$ is planar, Kuratowski’s Theorem implies that $|m| \leq 4$ and hence $|R| \leq 16$. In view of [6, Page 687], we have $R \in \{\mathbb{Z}_4, \mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 + x + 1)\}$.

Case 2: $n = 2$. If $|R_1| \geq 5$, then the set $R_1 \times \{0\}$ contains a subgraph isomorphic to $K_5$. Thus by Kuratowski’s Theorem, $ZCA\mathcal{Y}(R)$ is not planar. A similar argument shows that $|R_2| \leq 4$. Therefore [7, Corollary 4.5] implies that $R = R_1 \times R_2$, where $R_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{F}_4, D_2(\mathbb{Z}_2)\}$. We note that $ZCA\mathcal{Y}(R)$ is not planar if $R \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times D_2(\mathbb{Z}_2)\}$, since $ZCA\mathcal{Y}(R) = (K_4 + K_4) \cup K_3$.

Case 3: $n \geq 3$. In this case the elements $e_0, e_1, e_2, e_3, e_1 + e_2$ and $e_2 + e_3$ obtain a graph $K_{3,3}$ in the structure of $ZCA\mathcal{Y}(R)$. Thus by Kuratowski’s Theorem, $ZCA\mathcal{Y}(R)$ is not planar.

Let $R = \mathbb{F}_4[x]/I$ and $S = \mathbb{Z}_4[x]/J$, where $I = (x^2)$ and $J = (x^2 + x + 1)$. Since $Z(R) = \{1, x+I, ax+I, bx+I\}$, where $a, b$ are distinct elements of $\mathbb{F}_4 \setminus \{0, 1\}$ and $Z(S) = \{J, 2+J, 2x+J, 2+2x+J\}$, we must have $ZCA\mathcal{Y}(R) = ZCA\mathcal{Y}(S) = K_4$. We also have $ZCA\mathcal{Y}(\mathbb{F}_p) = K_1$ and $ZCA\mathcal{Y}(\mathbb{Z}_4) = ZCA\mathcal{Y}(D_2(\mathbb{Z}_2)) = K_2$. So, in view of Figures 2 and 3, it is easy to see that, if $R$ is one of the rings that appears in the statement of the theorem, then $ZCA\mathcal{Y}(R)$ is a planar graph.

This completes the proof.

\[\square\]
6. Clique Number

In the following, we denote the set of maximal ideals of a ring $R$ by $\text{max}(R)$.

**Lemma 6.1.** Let $R$ be a ring and $m \in \text{max}(R)$. Then $m$ is a clique of $\text{ZCA}(R)$.

**Proof.** Every two vertices of $m$ are adjacent. If $x$ is a vertex of $\text{ZCA}(R)$ such that $x$ is adjacent to every vertex of $m$. Then

$$x + m \subseteq \bigcup_{m \in \text{max}(R)} m.$$ 

It follows from [9, Theorem 3.64] that

$$Rx + m \subseteq \bigcup_{m \in \text{max}(R)} m.$$ 

By Prime Avoidance Theorem (see for example [9, Theorem 3.61]) there exists maximal ideal $m_0$ of $R$ such that $x + m \subset m_0$. Therefore $x \in m_0 = m$. Hence $m$ is a clique of $\text{ZCA}(R)$. □

**Theorem 6.2.** Let $G = \text{ZCA}(F_1 \times \cdots \times F_n)$, where $F_i$s are finite fields and $|F_1| \leq |F_2| \leq \cdots \leq |F_n|$. Then $m = 0 \times F_2 \times \cdots \times F_n$ is a largest clique of $G$. In particular $\omega(G) = |R|/|F_1|$.

**Proof.** Let $f_i : F_i \setminus \{0\} \rightarrow F_i \setminus \{0\}$ be an injective function for all $i \in \{2, 3, \ldots, n\}$ (note that $|F_1| \leq |F_i|$). Let $K$ be an arbitrary clique of $G$. We define $\varphi : K \rightarrow m$ as follows: If $x \in K \cap m$, then $\varphi(x) = x$. If $x = (x_1, x_2, \ldots, x_n) \in K \setminus m$, then $\varphi(x_1, x_2, \ldots, x_n) = (0, x_2 + f_2(x_1), \ldots, x_n + f_n(x_1))$. We claim that $\varphi$ is injective. Suppose that $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n) \in K$ and $\varphi(a) = \varphi(b)$. We consider the following cases:

Case 1: $a, b \in K \cap m$. In this case, we have $a = \varphi(a) = \varphi(b) = b$.

Case 2: $a \in K \cap m$ and $b \notin K \cap m$. Since $a, b$ are adjacent and $0 = a_1 \neq b_1$, there exists $2 \leq i \leq n$ such that $a_i = b_i$. Then $a_i = b_i + f_i(b_1)$, since $\varphi(a) = \varphi(b)$. It follows that $f_i(b_1) = 0$, a contradiction.

Case 3: $a \notin K \cap m$ and $b \in K \cap m$. Then by an argument symmetric to that of Case 2, we get a contradiction.

Case 4: $a \notin K \cap m$ and $b \notin K \cap m$. Since $a, b$ are adjacent, there exists $1 \leq i \leq n$ such that $a_i = b_i$. If $i = 1$, then $\varphi(a) = \varphi(b)$ implies that $a = b$. Now, suppose that $a_1 \neq b_1$ and $a_i = b_i$ for some $i \geq 2$. Then $\varphi(a) = \varphi(b)$ implies $a_i + f_i(a_1) = b_i + f_i(b_1)$. It follows that $f_i(a_1) = f_i(b_1)$. Since $f_i$ is injective we must have $a_1 = b_1$, a contradiction.

So $\varphi$ is injective and hence $|K| \leq |m|$. It follows from Lemma 6.1 that $m$ is a largest clique of $G$. □
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