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n-submodules

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Abstract. Let R be a commutative ring with identity. A proper submodule N of an R-module M is an n-submodule if $rm \in N$ $(r \in R, m \in M)$ with $r \notin \sqrt{Ann_R(M)}$, then $m \in N$. A number of results concerning n-submodules are given. For example, we give other characterizations of n-submodules. Also various properties of n-submodules are considered.

Keywords: n-ideal, n-submodule.

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1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. Also $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote, respectively, the natural numbers, the ring of integers, and the field of rational numbers. If N is an R-submodule of M, annihilator of R-module $\dfrac{M}{N}$ is defined to be $Ann_R(\frac{M}{N}) = (N :_R M) = \{r \in R : rM \subseteq N\}$. Also the annihilator of M, denoted by $Ann_R(M)$, is $(0 :_R M)$. Suppose that I is an ideal of R. We denote the radical of I by $\sqrt{I} = \{a \in R : a^n \in I \ \ for \ \ some \ \ n \in \mathbb{N}\}$.
A proper submodule N of M is called prime ($primary$) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N :_R M)$ ($r^n \in (N :_R M)$, for

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some $n \in \mathbb{N}$)(see [1], [6], [9], [11]).
An R-module M is said to be a multiplication module, if for each submodule N of M, there is an ideal I of R, such that $N = IM$. Equivalently, M is a multiplication module if and only if $N = (N :_R M)M$, for each submodule N of M [2],[3].
The concepts of n-ideals and n-submodules were introduced in [12]. A proper ideal I of R is said to be an n-ideal if the condition $ab \in I$ with $a \notin \sqrt{0} = \{a \in R : a^n = 0 \ \ for \ \ some \ \ n \in \mathbb{N}\}$ implies $b \in I$, for every $a, b \in R$. Also a proper submodule N of M is called an n-submodule if for $a \in R$, $x \in M$, $ax \in N$ with $a \notin \sqrt{Ann_R(M)}$, then $x \in N$.
In Section 2, we investigate some properties of n-submodules analogous with n-ideals and also obtain some basic results. Among many results in this article, it is shown in Theorem 2.2, that a proper submodule N of M is an n-submodule if and only if $N = (N :_M a)$ for every $a \notin \sqrt{Ann_R(M)}$. In Theorem 2.22, we show that every n-submodule is a primary submodule. Furthermore, in Theorem 2.27, we characterize torsion-free modules in terms of n-submodules.

2. n-Submodules

Recall that a proper submodule N of a module M over a commutative ring R is said to be an n-submodule, if for $a \in R$, $x \in M$, $ax \in N$ with $a \notin \sqrt{Ann_R(M)}$, then $x \in N$.

Example 2.1. (i) Suppose that R is a ring that has only one prime ideal. Then every proper submodule of R- module R is an n-submodule.
(ii) $\mathbb{Z}_6$ as $\mathbb{Z}$-module has not any n-submodule.

Theorem 2.2. *Let M be an R-module and N be a proper submodule of M. Then the following statements are equivalent:*
(i) N is an n-submodule of M;
(ii) $N = (N :_M a)$, for every $a \notin \sqrt{Ann_R(M)}$;
(iii) For any ideal I of R and submodule K of M, $IK \subseteq N$ with $I \nsubseteq \sqrt{Ann_R(M)}$ implies $K \subseteq N$.

Proof. $(i) \Rightarrow (ii)$ Let N be an n-submodule of M. For every $a \in R$, the inclusion $N \subseteq (N :_M a)$ always holds. Let $a \notin \sqrt{Ann_R(M)}$ and $x \in (N :_M a)$. Then we have $ax \in N$. Since N is an n-submodule, we conclude that $x \in N$ and thus $N = (N :_M a)$.
$(ii) \Rightarrow (iii)$ Suppose that $IK \subseteq N$ where $I \nsubseteq \sqrt{Ann_R(M)}$, for ideal I of R and submodule K of M. Since $I \nsubseteq \sqrt{Ann_R(M)}$, there exists $a \in I$ such that $a \notin \sqrt{Ann_R(M)}$. Then we have $aK \subseteq N$, and so $K \subseteq (N :_M a) = N$ by (ii).

$(iii) \Rightarrow (i)$ Let $ax \in N$ with $a \notin \sqrt{Ann_R(M)}$ for $a \in R$ and $x \in M$. It is sufficient to take $I := Ra$ and $K := Rx$ to prove the result. □

Proposition 2.3. *i) If N is an n-submodule of M, then $(N :_R M) \subseteq \sqrt{Ann_R(M)}$.*
ii) Let $\{N_i\}_{i \in I}$ be a nonempty set of n-submodules of an R-module M. Then $\bigcap_{i \in I} N_i$ is an n-submodule.
iii) Let $\{N_i\}_{i \in I}$ be a chain of n-submodules of a finitely generated R-module M. Then $\bigcup_{i \in I} N_i$ is an n-submodule of M.

Proof. i) Assume that N is an n-submodule; but $(N :_R M) \not\subseteq \sqrt{Ann_R(M)}$. Then there exists $r \in (N :_R M)$ such that $r \notin \sqrt{Ann_R(M)}$. Thus $rM \subseteq N$ and since N is an n-submodule, we conclude that $N = M$, a contradiction. Hence $(N :_R M) \subseteq \sqrt{Ann_R(M)}$.
ii) Let $rx \in \bigcap_{i \in I} N_i$ with $r \notin \sqrt{Ann_R(M)}$, for $r \in R$ and $x \in M$. Then $rx \in N_i$, for every $i \in I$. Since for every $i \in I$, N_i is an n-submodule of M, we get $x \in N_i$ and so $x \in \bigcap_{i \in I} N_i$.
iii) Let $rx \in \bigcup_{i \in I} N_i$ where $r \notin \sqrt{Ann_R(M)}$ for $r \in R$ and $x \in M$. Then $rx \in N_k$ for some $k \in \mathbb{N}$. Since N_k is an n-submodule, we conclude that $x \in N_k \subseteq \bigcup_{i \in I} N_i$ and so $\bigcup_{i \in I} N_i$ is an n-submodule. □

Proposition 2.4. *Let I be an ideal of R such that $I \not\subseteq \sqrt{Ann_R(M)}$. Then the followings hold:*
(i) If K_1 and K_2 are n-submodules of M with $IK_1 = IK_2$, then $K_1 = K_2$.
(ii) If IK is an n-submodule of M, then $IK = K$.

Proof. (i) Since K_1 is an n-submodule and $IK_2 \subseteq K_1$, by Theorem 2.2, we get that $K_2 \subseteq K_1$. Likewise, $K_1 \subseteq K_2$.
(ii) Since IK is an n-submodule and $IK \subseteq IK$, we conclude that $K \subseteq IK$, so this completes the proof. □

The next lemma provides a useful characterization of modules that have n-submodule.

Lemma 2.5. *Let M be a torsion-free R-module. Then zero submodule is an n-submodule of M.*

Proof. Let $ax = 0$ with $a \notin \sqrt{Ann_R(M)}$, for $a \in R$ and $x \in M$. Since M is torsion-free, $x = 0$. Thus zero submodule of M is an n-submodule. □

Lemma 2.6. *If M is a torsion-free multiplication R-module, then zero submodule is the only n-submodule of M.*

Proof. Suppose that N is an n-submodule of M. Then by Proposition 2.3(i), we have $(N :_R M) \subseteq \sqrt{Ann_R(M)} = 0$ and so $(N :_R M) = 0$. As M is multiplication, then $N = 0$. So by Lemma 2.5, the zero submodule is the only n-submodule. □

Proposition 2.7. *Let M be an R-module and I be an ideal of R. If N is an n-submodule of M such that $I \nsubseteq (N :_R M)$, then $(N :_M I)$ is an n-submodule of M.*

Proof. Let $ax \in (N :_M I)$ with $a \notin \sqrt{Ann_R(M)}$, for $a \in R$ and $x \in M$. So $aIx \subseteq N$ and as N is an n-submodule, $Ix \subseteq N$. Hence $x \in (N :_M I)$. □

Proposition 2.8. *Let N be a proper submodule of M. Then N is an n-submodule if and only if for every $x \in M$, $(N :_R x) = R$ or $(N :_R x) \subseteq \sqrt{Ann_R(M)}$.*

Proof. Assume that N is an n-submodule. If $(N :_R x) \nsubseteq \sqrt{Ann_R(M)}$, then there exists $r \in (N :_R x) - \sqrt{Ann_R(M)}$. So $rx \in N$ where $r \notin \sqrt{Ann_R(M)}$. Since N is an n-submodule, $x \in N$. Hence $(N :_R x) = R$. Conversely, let $rx \in N$ where $r \notin \sqrt{Ann_R(M)}$, for $r \in R$ and $x \in M$. So $r \in (N :_R x) - \sqrt{Ann_R(M)}$. By assumption, we have $(N :_R x) = R$ and therefore $x \in N$. □

Corollary 2.9. *Let N be a proper submodule of M. Then N is an n-submodule if and only if for every $x \in M - N$, $(N :_R x) \subseteq \sqrt{Ann_R(M)}$.*

Recall that, $r \in R$ is said to be a zero divisor of an R-module M, if there exists a non-zero element $x \in M$ such that $rx = 0$.

Theorem 2.10. *Let M be an R-module and N be a submodule of M. Then N is an n-submodule if and only if every zero divisor of an R-module $\frac{M}{N}$ is in $\sqrt{Ann_R(M)}$.*

Proof. Let N be an n-submodule and r be a zero divisor of $\frac{M}{N}$. Then there exists $x \in M - N$ such that $rx \in N$. Since N is an n-submodule, we have $r \in \sqrt{Ann_R(M)}$. For the converse, assume that $rx \in N$ where $x \notin N$, for $r \in R$ and $x \in M$. Then r is a zero divisor of $\frac{M}{N}$ and so $r \in \sqrt{Ann_R(M)}$. □

Theorem 2.11. *Every maximal n-submodule is a prime submodule.*

Proof. Let N be a maximal n-submodule of M and $ax \in N$ where $a \notin (N :_R M)$, for $a \in R$ and $x \in M$. By Proposition 2.7, $(N :_M a)$ is an n-submodule. Thus $x \in (N :_M a) = N$, by maximality of N. So N is a prime submodule. □

Theorem 2.12. *Let M be a finitely generated R-module. If M has an n-submodule, then M has a prime submodule.*

Proof. Suppose that N is an n-submodule and $\Omega = \{L : L \;\; is \;\; an \;\; n - submodule \;\; of$ M; $N \subseteq L\}$. By Zorn' s Lemma, Ω has a maximal element $K \in \Omega$. Then by Therorm 2.11, K is a prime submodule of M. □

In ring theory (and so in module theory), the concepts prime ideal and n-ideal are not the same in general. (see Example 3.2 in [12]). In the following, we try to find some relations beetwen them.

Proposition 2.13. *For a prime submodule N of M, N is an n-submodule if and only if $(N :_R M) = \sqrt{Ann_R(M)}$.*

Proof. Suppose that N is a prime submodule of M. It is clear that $\sqrt{Ann_R(M)} \subseteq (N :_R M)$. If N is an n-submodule, then by Proposition 2.3(i), we have $(N :_R M) \subseteq \sqrt{Ann_R(M)}$ and so $(N :_R M) = \sqrt{Ann_R(M)}$. For the converse, assume that $(N :_R M) = \sqrt{Ann_R(M)}$. Now we show that N is an n-submodule. Let $ax \in N$ and $a \notin \sqrt{Ann_R(M)}$, for $a \in R$ and $x \in M$. Since N is a prime submodule and $a \notin (N :_R M)$, we get $x \in N$ and so N is an n-submodule. $\square$

Recall from [11], the intersection of all prime submodules contains N, denoted $rad(N)$, is called the radical of N. If there is no prime submodule containing N, $rad(N) = M$.

Proposition 2.14. *Let M be a finitely generated R-module. Then $rad(0)$ is an n-submodule if and only if $rad(0)$ is a prime submodule.*

Proof. Since M is finitely generated, by Theorem 4.4 in [8], $(rad(0) :_R M) = \sqrt{Ann_R(M)}$. Suppose that $rad(0)$ is an n-submodule. Let $ax \in rad(0)$ with $a \notin (rad(0) :_R M)$, for $a \in R$ and $x \in M$. So $a \notin \sqrt{Ann_R(M)}$ and since $rad(0)$ is an n-submodule, we have $x \in rad(0)$. Thus $rad(0)$ is a prime submodule. Now assume that $rad(0)$ is a prime submodule. By Proposition 2.13, $rad(0)$ is an n-submodule. $\square$

Lemma 2.15. *Let N be an n-submodule of an R-module M such that $(N :_R M) = \sqrt{Ann_R(M)}$. Then N is a prime submodule.*

Proof. It is clear. $\square$

Proposition 2.16. *If zero submodule of an R-module M is an n-submodule, then $\sqrt{Ann_R(M)}$ is a prime ideal of R.*

Proof. Let $ab \in \sqrt{Ann_R(M)}$ for $a, b \in R$. So there exists $n \in \mathbb{N}$ such that $a^n b^n M = 0$. If $a \notin \sqrt{Ann_R(M)}$, then since the zero submodule is a n-submodule, we get $b^n M = 0$; i.e. $b \in \sqrt{Ann_R(M)}$. $\square$

Remember that if N is a prime submodule of an R-module M, then $(N :_R M)$ is a prime ideal of R. Now, we give a similar result for n-submodules.

Lemma 2.17. *If M is a faithful R-module and N is an n-submodule of M, then $(N :_R M)$ is an n-ideal of R.*

Proof. Assume that $ab \in (N :_R M)$ with $a \notin \sqrt{0}$, for $a, b \in R$. Since $Ann_R(M) = 0$ and N is an n-submodule, then $b \in (N :_R M)$. $\square$

Corollary 2.18. *Let M be a faithful R-module and R has no n-ideal. Then M has no n-submodule.*

Lemma 2.19. *Let M be a multiplication R-module and N be a submodule of M such that $(N :_R M)$ is an n-ideal of R. Then N is an n-submodule.*

Proof. Let $IK \subseteq N$ with $I \nsubseteq \sqrt{Ann_R(M)}$, where I is an ideal of R and K is a submodule of M. Since M is multiplication and $(N :_R M)$ is an n-ideal, $I(K :_R M) \subseteq (N :_R M)$ and so $(K :_R M) \subseteq (N :_R M)$, by Theorem 2.7 in [12]. Thus $K \subseteq N$ and by Theorem 2.2, N is an n-submodule. □

Corollary 2.20. *Let M be a cyclic R-module and N be a submodule of M such that $(N :_R M)$ is an n-ideal of R. Then N is an n-submodule of M.*

Recall that a proper submodule N of M is said to be an r-submodule, if for $a \in R$, $m \in M$ and whenever $am \in N$ with $ann_M(a) = 0$, then $m \in N$ [5].

Proposition 2.21. *Every n-submodule is an r-submodule.*

Proof. Let N be an n-submodule of M. Now, we will show that N is an r-submodule. Let $am \in N$ with $ann_M(a) = 0$, for some $a \in R$, $m \in M$. Assume that $a \in \sqrt{Ann_R(M)}$. Then there exists $n \in \mathbb{N}$ such that $a^n M = 0$. Choose the smallest positive integer n such that $a^n M = 0$. Then we have $a^{n-1}M \neq 0$. Since $a(a^{n-1}M) = a^n M = 0$, we have $a^{n-1}M \subseteq ann_M(a) = 0$ and so $a^{n-1}M = 0$ which is a contradiction. So that $a \notin \sqrt{Ann_R(M)}$. As N is an n-submodule and $am \in N$, we get $m \in N$. Hence, N is an r-submodule of M. □

Theorem 2.22. *Let N be a submodule of M such that $(N :_R M) \subseteq \sqrt{Ann_R(M)}$. Then the following statements are equivalent:*
(i) N is an n-submodule;
(ii) N is a primary submodule of M.

Proof. $(i) \Rightarrow (ii)$ Let $ax \in N$ with $a \notin \sqrt{(N :_R M)}$, for $a \in R$ and $x \in M$. As N is an n-submodule, we have $x \in N$. Thus N is a primary submodule.
$(ii) \Rightarrow (i)$ Let $ax \in N$ with $a \notin \sqrt{Ann_R(M)}$, for $a \in R$ and $x \in M$. As $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$, we have $a \notin \sqrt{Ann_R(M)}$. Since N is a primary submodule, we get $x \in N$. Therefore N is an n-submodule. □

By the proof of previous theorem, every n-submodule is a primary submodule. So it is straightforward to get that if N is an n-submodule of R-module M, then $(N :_R M)$ is a primary ideal of R. Recall if $(N :_R M)$ is a maximal ideal of ring R, then N is a primary submodule of M. So we have:

Corollary 2.23. *Let $Ann_R(M)$ be a maximal ideal of R. Then every proper submodule of M is an n-submodule.*

By using the fact that every irreducible submodule of a Noetherian module is a primary submodule (Proposition 1-17 in [4]), we can get the following corollary:

Corollary 2.24. *Let M be a Noetherian R-module and N be an irreducible submodule of M such that $(N :_R M) \subseteq \sqrt{Ann_R(M)}$. Then N is an n-submodule of M.*

Proposition 2.25. *If N is a primary R-submodule of M such that $(N :_R M)$ is maximal in the set of all n-ideals, then N is an n-submodule of M.*

Proof. Let $ax \in N$ with $a \notin \sqrt{Ann_R(M)}$, for $a \in R$ and $x \in M$. By Theorem 2.11 [12], $\sqrt{0} = \sqrt{(N :_R M)}$. Since N is a primary submodule and $a \notin \sqrt{(N :_R M)}$, $x \in N$. □

Lemma 2.26. *If N is an n-submodule and L is a primary submodule of an R-module M such that $(L :_R M) \subseteq Ann_R(M)$, then $N \cap L$ is an n-submodule of M.*

Proof. Let $rx \in N \cap L$ where $r \notin \sqrt{Ann_R(M)}$, for $r \in R$, $x \in M$. Then $r \notin \sqrt{(L :_R M)}$. Since L is primary, $x \in L$. Also, since N is an n-submodule, $x \in N$. Thus $x \in N \cap L$. □

Recall that a proper ideal I of R is called semiprime, if whenever $a^n \in I$ for $a \in R$ and $n \in \mathbb{N}$, then $a \in I$ [10]. Now, in the following theorem we give a characterization for torsion free modules in terms of n-submodules.

Theorem 2.27. *Let M be an R-module. Then the following statements are equivalent:*
(i) M is a torsionfree R-module;
(ii) M is faithful, zero submodule is an n-submodule of M and zero ideal is a semiprime ideal of R.

Proof. $(i) \Rightarrow (ii)$ It follows from Lemma 2.5.
$(ii) \Rightarrow (i)$ Let $rx = 0$ and $r \neq 0$, for $r \in R$, $x \in M$. Since (0) is a semiprime ideal of R, $\sqrt{0} = 0$. As M is faithful, it follows that $r \notin \sqrt{Ann_R(M)} = \sqrt{0} = 0$. Since the zero submodule is an n-submodule, $x = 0$. Therefore, M is a torsion-free module. □

Theorem 2.28. *Let $f : M \longrightarrow M'$ be an R-homomorphism. Then the followings hold:*
(i) If f is an epimorphism and N is an n-submodule of M containing $ker(f)$, then $f(N)$ is an n-submodule of M'.
(ii) If f is a monomorphism and L' is an n-submodule of M', then $f^{-1}(L') = M$ or $f^{-1}(L')$ is an n-submodule of M.

Proof. (i) Let $rx' \in f(N)$ where $r \notin \sqrt{Ann_R(M')}$, for $r \in R$, $x' \in M'$. Since f is epimorphism, there exists $x \in M$ such that $x' = f(x)$. Then $rx' = rf(x) = f(rx) \in f(N)$. As $ker(f) \subseteq N$, we conclude that $rx \in N$. Also, note that $r \notin \sqrt{Ann_R(M)}$. Since N is an n-submodule of M, we get the result that $x \in N$ and so $x' = f(x) \in f(N)$.

(ii) Let $f^{-1}(L^{'}) \neq M$ and $rx \in f^{-1}(L^{'})$ where $r \notin \sqrt{Ann_R(M)}$, for $r \in R$, $x \in M$. Then $f(rx) = rf(x) \in L^{'}$. Since f is a monomorphism and $r \notin \sqrt{Ann_R(M)}$, we get $r \notin \sqrt{Ann_R(M')}$. Since $L^{'}$ is an n-submodule of $M^{'}$, $f(x) \in L^{'}$ and so $x \in f^{-1}(L^{'})$. Consequently, $f^{-1}(L^{'})$ is an n-submodule of M. □

Corollary 2.29. *Let M be an R-module and $L \subseteq N$ be two submodules of M. Then the followings hold:*
(i) If N is an n-submodule of M, then $\frac{N}{L}$ is an n-submodule of $\frac{M}{L}$.
(ii) If $\frac{N}{L}$ is an n-submodule of $\frac{M}{L}$ and $(L :_R M) \subseteq \sqrt{Ann_R(M)}$, then N is an n-submodule of M.
(iii) If $\frac{N}{L}$ is an n-submodule of $\frac{M}{L}$ and L is an n-submodule of M, then N is an n-submodule of M.

Proof. (i) Assume that N is an n-submodule of M and $L \subseteq N$. Let $\pi : M \longrightarrow \frac{M}{L}$ be the natural homomorphism. Note that $ker(\pi) = L \subseteq N$, and so by Theorem 2.28(i), $\frac{N}{L}$ is an n-submodule of $\frac{M}{L}$.
(ii) Let $rx \in N$ where $r \notin \sqrt{Ann_R(M)}$ for $r \in R$, $x \in M$. Then we have $(r + I)(x + L) = rx + L \in \frac{N}{L}$ and $r + I \notin \sqrt{Ann_{\frac{R}{I}}(\frac{M}{L})}$, where $I = (L :_R M)$. Since $\frac{N}{L}$ is an n-submodule of $\frac{M}{L}$, we conclude that $x + L \in \frac{N}{L}$ and so $x \in N$. Consequently, N is an n-submodule of M.
(iii) It follows from (ii) and Proposition 2.3(i). □

Corollary 2.30. *Let M be an R-module and N be a submodule of M. If L is an n-submodule of M such that $N \nsubseteq L$, then $L \cap N$ is an n-submodule of N.*

Proof. Consider the injection $i : N \longrightarrow M$. Note that $i^{-1}(L) = L \cap N$, so by Theorem 2.28(ii), $L \cap N$ is an n-submodule of N. □

Let M be an R-module and S be a multiplicative closed subset of R. Consider the natural homomorphsim π from M to M_S as $\pi(m) = \dfrac{m}{1}$, for any $m \in M$. Then for each submodule L of M_S, we define L^c as an inverse image of L under this natural homomorphism.

Proposition 2.31. *Let M be an R-module and S a multiplicative closed subset of R.*
(i) If N is an n-submodule of M, then $N_S = M_S$ or N_S is an n-submodule of M_S.
(ii) If M is finitely generated, L is an n-submodule of M_S and $S \cap (Ann_R(M) :_R a) = \emptyset$ for every $a \notin Ann_R(M)$, then $L^c = M$ or L^c is an n-submodule of M.

Proof. (i) Let $N_S \neq M_S$ and $\frac{a}{s}\frac{m}{t} \in N_S$ where $\frac{a}{s} \notin \sqrt{Ann_{R_S}(M_S)}$, for $a \in R$, $s, t \in S$, $m \in M$. Then we have $uam \in N$, for some $u \in S$. It is clear that $a \notin \sqrt{Ann_R(M)}$. Since N is an n-submodule of M, we conclude that $um \in N$ and so $\frac{m}{t} = \frac{um}{ut} \in N_S$. Therefore N_S is an n-submodule of M_S.

(ii) Let $L^c \neq M$ and $am \in L^c$ where $a \notin \sqrt{Ann_R(M)}$ for $a \in R$, $m \in M$. Then we have $\frac{a}{1}\frac{m}{1} \in L$. Now we show that $\frac{a}{1} \notin \sqrt{Ann_{R_S}(M_S)}$. Suppose $\frac{a}{1} \in \sqrt{Ann_{R_S}(M_S)}$. There exists a positive integer k such that $(\frac{a}{1})^k M_S = 0$. Then we get $ua^k M = 0$ for some $u \in S$, as M is finitely generated. Since $a \notin \sqrt{Ann_R(M)}$, $a^k M \neq 0$ and so $u \in (Ann_R(M) :_R a^k) \cap S$, which is a contradiction. Thus we have $\frac{a}{1} \notin \sqrt{Ann_{R_S}(M_S)}$. As L is an n-submodule of M_S, we conclude that $\frac{m}{1} \in L$ and so $m \in L^c$. $\square$

Lemma 2.32. *Let M be a finitely generated R-module such that for every multiplicative closed set $S \subseteq R$, the kernel of $\varphi : M \longrightarrow M_S$ is either (0) or M. Then (0) is an n-submodule of M.*

Proof. Let $rx = 0$ where $r \in R - \sqrt{Ann_R(M)}$ and $x \in M$. So $r^n \neq 0$, for every $n \in \mathbb{N}$. We put $S = \{r^n : n \in \mathbb{N} \cup \{0\}\}$. Clearly S is a multiplicative closed set in R. If $ker(\varphi) = 0$, then as $\varphi(x) = \frac{x}{1} = \frac{rx}{r} = 0$ we have $x = 0$. Let $ker(\varphi) = M$. Since M is finitely generated, we can write $M = Rx_1 + Rx_2 + ... + Rx_t$, for some $x_1, x_2, ..., x_t \in M$. Then $\varphi(x_i) = \frac{x_i}{1} = 0$ for any $1 \leq i \leq t$. Thus for any i, there exists $l_i \in \mathbb{N}$ such that $r^{l_i} x_i = 0$. Put $j := max\{l_1, l_2, ..., l_t\}$. Thus we have $r^j M = 0$ and so $r \in \sqrt{Ann_R(M)}$, which is a contradiction. $\square$

We recall that a nonempty subset S of R where $R - \sqrt{0} \subseteq S$ is said to be an n-multiplicatively closed subset of R, if $xy \in S$ for all $x \in R - \sqrt{0}$ and all $y \in S$ (see [12]).

Theorem 2.33. *Let M be a finitely generated R-module and N be a proper submodule of M such that $(N :_R M) \cap S = \emptyset$, where S is an n-multiplicatively closed set in R. Then there exists an n-submodule L of M cotaining N such that $(L :_R M) \cap S = \emptyset$.*

Proof. Consider that set $\Omega = \{L : \ L \textit{ is a submodule of } M;\ (L :_R M) \cap S = \emptyset\}$. Since $N \in \Omega$, we have $\Omega \neq \emptyset$. Since M is finitely generated, by using Zorn' s lemma, we get a maximal element K of Ω. Now we show that K is an n-submodule of M. Suppose that $rx \in K$, for some $r \notin \sqrt{Ann_R(M)}$ and $x \notin K$. Thus we get $x \in (K :_M r)$ and $K \subset (K :_M r)$. By maximality of K , we have $((K :_M r) :_R M) \cap S \neq \emptyset$ and thus there exists $t \in S$ such that $tM \subseteq (K :_M r)$. Also $rt \in S$, because $r \in R - \sqrt{0}$ and $t \in S$ and S is an n-multiplicatively closed subset of R. We get $(K :_R M) \cap S \neq \emptyset$, which is a contradictions. Hence K is an n-submodule of M. $\square$

Proposition 2.34. *Suppose that $N \subseteq \bigcup_{i=1}^n N_i$, where N, N_i $(1 \leq i \leq n)$, are R-submodules of M. If there exists N_j such that $N \not\subseteq \bigcup_{i \neq j} N_i$, N_j is an n-submodule and $(\bigcap_{i \neq j} N_i :_R M) \not\subseteq \sqrt{Ann_R(M)}$, then $N \subseteq N_j$*

Proof. We may assume that $j = 1$. Since $N \not\subseteq \bigcup_{i \geq 2} N_i$, there exists $x \in N - \bigcup_{i=2}^n N_i$. Thus we have $x \in N_1$. Let $y \in N \cap (\bigcap_{i=2}^n N_i)$. Since $x \notin N_k$ and $y \in N_k$ for every $2 \leq k \leq n$, we have $x + y \notin N_k$. Thus $x + y \in N - \bigcup_{i=2}^n N_i$

and so $x + y \in N_1$. As $x + y \in N_1$ and $x \in N_1$, it follows that $y \in N_1$ and so $N \cap (\bigcap_{i=2}^{n} N_i) \subseteq N_1$. Also we have $(\bigcap_{i=2}^{n} N_i :_R M)N \subseteq N \cap (\bigcap_{i=2}^{n} N_i)$. Now since $(\bigcap_{i=2}^{n} N_i :_R M)N \subseteq N_1$, $(\bigcap_{i=2}^{n} N_i :_R M) \nsubseteq \sqrt{Ann_R(M)}$ and N_1 is an n-submodule of M, we have $N \subseteq N_1$. $\square$

Following Lemma 1.1 in [9], a sbmodule K of an R-module M is prime if and only if $p = (K :_R M)$ is a prime ideal of R and the $\frac{R}{p}$-module $\frac{M}{K}$ is torsion-free. Now, we give a similar result for n-submodules.

Theorem 2.35. *Let N be an R-submodule of M such that $I = \sqrt{Ann_R(M)} \subseteq (N :_R M)$. Then N is an n-submodule of M if and only if $\frac{M}{N}$ is a torsion-free $\frac{R}{I}$-module.*

Proof. Let N be an n-submodule and $(r + I)(x + N) = 0_{\frac{M}{N}}$, for $r \in R$ and $x \in M$. Then we have $rx \in N$. If $r \in I$, then $r + I = 0$. Otherwise, since N is an n-submodule, we conclude that $x \in N$ and so $x + N = 0$. For the converse, assume that $\frac{M}{N}$ is a torsion-free $\frac{R}{I}$-module and $rx \in N$, for $x \in M$ and $r \in R - \sqrt{Ann_R(M)}$. Then $(r+I)(x+N) = rx+N = N = 0_{\frac{M}{N}}$. Now as $\frac{M}{N}$ is a torsion-free $\frac{R}{I}$-module and $r \notin I$, we have $x \in N$. So N is an n-submodule of M. $\square$

Lemma 2.36. *Let $\{L_i\}_{i\in I}$ be a family of R-submodules of $\{M_i\}_{i\in I}$. If $\Pi_{i\in I} Li$ is an n-submodule of $\Pi_{i\in I} M_i$, then for every $i \in I$, L_i is an n-submodule of M_i.*

Proof. Let $\Pi_{i\in I} Li$ be an n-submodule of $\Pi_{i\in I} M_i$ and i be an arbitrary in I. We will prove L_i is an n-submodule of M_i. Suppose that $rx \in L_i$ where $r \notin \sqrt{Ann_R(M_i)}$, for $r \in R$ and $x \in M_i$. Put $x_i := x$ and $x_j := 0$ for all $j \neq i$. Then we have $r(x_j)_{j\in I} \in \Pi_{j\in I} L_j$ and $r \notin \sqrt{Ann_R(\Pi_{j\in I} M_j)}$. Since $\Pi_{j\in I} L_j$ is an n-submodule of $\Pi_{j\in I} M_j$, so $(x_j)_{j\in I} \in \Pi_{j\in I} L_j$. Hence $x_i \in L_i$. $\square$

Corollary 2.37. *Let M_1 and M_2 be R-module and $M = M_1 \times M_2$. Then the following are satisfied:*
(i) If $L_1 \times M_2$ is an n-submodule of M, then L_1 is an n-submodule of M_1.
(ii) If $M_1 \times L_2$ is an n-submodule of M, then L_2 is an n-submodule of M_2.

Theorem 2.38. *Let N be a proper R-submodule of M. Then N is an n-submodule of M if and only if for each $a \in R - \sqrt{Ann_R(M)}$, the homothety $\lambda_a : \frac{M}{N} \longrightarrow \frac{M}{N}$ is an injective.*

Proof. Suppose that N is an n-submodule and $\lambda_a(x + N) = 0_{\frac{M}{N}}$ for $a \in R - \sqrt{Ann_R(M)}$, $x \in M$. Then $ax \in N$ and since N is an n-submodule, so $x \in N$ and $x + N = 0$. Hence λ_a is injective. Conversely, suppose that $rx \in N$ where $r \notin \sqrt{Ann_R(M)}$, for $r \in R$, $x \in M$. It follows that $\lambda_r(x + N) = 0$. Since λ_r is injective, $x + N = 0$ and so $x \in N$. $\square$

In [7], I.G. Macdonald introduced the notion of secondary modules. A nonzero R-module M is said to be secondary, if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent.

Proposition 2.39. *If M is a secondary R-module such that every ascending chain of cyclic submodules of it stops, then every proper submodule of M is an n-submodule.*

Proof. Let N be a proper submodule of M and $rx \in N$, for $r \in R$ and $x \in M$. Assume that φ_r is the homothety $M \to M$ for $r \in R$. If φ_r is nilpotent, then there exists $n \in \mathbb{N}$ such that $(\varphi_r)^n = 0$. It follows that $r^n \in Ann_R(M)$ and so $r \in \sqrt{Ann_R(M)}$. If φ_r is surjective, then we have

$$x = rx_1$$

$$x_1 = rx_2$$

$$x_2 = rx_3$$

$$...$$

$$x_n = rx_{n+1}$$

$$...$$

for some $x_i \in M$. Then $< x > \subseteq < x_1 > \subseteq < x_2 > ... \subseteq < x_n > \subseteq ....$ Since M is complete, there exists $n \in \mathbb{N}$ such that $< x_n > = < x_i >$, for every $i \geq n$. Hence there exists $s \in R$ such that $x_{n+1} = sx_n$. It follows that $x_n = rsx_n$. So $(1 - rs)x = 0$ and we have $x = srx$. As $rx \in N$, so $x \in N$. $\square$

Corollary 2.40. *Let M be a Noetherian secondary module. Then every proper submodule is an n-submodule.*

Proposition 2.41. *If N is an n-R-submodule of M, then $N[x]$ an n-submodule of $M[x]$.*

Proof. Let r be a zero divisor of an R-module $\frac{M[x]}{N[x]}$. Since $\frac{M[x]}{N[x]} \cong \frac{M}{N}[x]$, then there exists $f(x) = a_0 + a_1x + ... + a_tx^t \in M[x]$ such that $0 \notin \overline{f(x)} \in \frac{M}{N}[x]$ and $r\overline{f(x)} = \overline{0}$. Hence $ra_i \in N$, for $1 \leq i \leq t$. If for every i, $a_i \in N$, then $\overline{f(x)} = \overline{0}$, which is a contradiction. Thus there exists $1 \leq i \leq t$ such that $a_i \notin N$ with $ra_i \in N$. On the other hand, as N is an n-submodule, so $r \in \sqrt{Ann_R(M)}$. Since $M \subseteq M[x]$, so $r \in \sqrt{Ann_R(M[x])}$. Then by Theorem 2.10, $N[x]$ is an n-submodule of $M[x]$. $\square$

3. EXAMPLES

EXAMPLE 3.1. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then every proper submodule of M is an n-submodule. It is clear that every proper submodule of M is prime and the colon ideal of M into submodules are equal $2\mathbb{Z}$. Now according to Proposition 2.13, every proper submodule of M is an n-submodule.

Now we have an example which shows that there exists an R-module that does not have an n-submodule.

EXAMPLE 3.2. Let p be any prime number. Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$ and $R = \mathbb{Z}$. Then every proper submodule of M is not an n-submodule. Let N be an n-submodule of M. By Proposition 2.3(i), $(N :_R M) \subseteq \sqrt{Ann_R(M)} = \sqrt{0} = 0$. It follows that $(N :_R M) = \sqrt{Ann_R(M)}$. Then by Lemma 2.15, N is a prime submodule. On the other hand, pM is the only prime submodule of M. So $N = pM$ and $(N :_R M) = (pM :_R M) = p\mathbb{Z}$, which is a contradiction.

Remark 3.3. (i) By Theorem 2.22, every n-submodule of a module is a primary submodule. However, the converse is not true in general. Since for example: if $R = \mathbb{Z}$, $M = \mathbb{Z}$ and $N = 4\mathbb{Z}$, then N is a primary submodule of M, however it is not n-submodule, as $2.2 \in N$, but $2 \notin \sqrt{Ann_R(M)}$ and $2 \notin N$.
(ii) It is well known that if N is a prime submodule of M, then $(N :_R M)$ is a prime ideal of R. Contrary to what happens for a prime submodules, if N is an n-submodule, the ideal $(N :_R M)$ is not in general an n-ideal of R. For example: Let $M = \mathbb{Z}_4$, $R = \mathbb{Z}$. Take $N = (\bar{0})$. Certainly N is an n-submodule of M, but $(N :_R M) = 4\mathbb{Z}$ is not an n-ideal of R.

The following example shows that the converse of Lemma 2.5, is not necessarily true.

EXAMPLE 3.4. Consider the $\mathbb{Z}$-module $\mathbb{Z}_4$ and $N = (\bar{0})$. Clearly N is an n-submodule, but M is not a torsion-free module.

In the next example, we show that zero submodule is not always the only n-submodule of torsion-free modules.

EXAMPLE 3.5. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $R = \mathbb{Z}$. consider the submodule $N = 0 \oplus \mathbb{Z}$. Let $a(m, n) = (am, an) \in N$ with $a \notin \sqrt{Ann_R(M)} = 0$ for some $a, m, n, \in \mathbb{Z}$. Then we have $am = 0$ and so $m = 0$. This implies that $(m, n) = (0, n) \in N$. Thus N is a nonzero n-submodule of M.

The next example shows that the sum of two n-submodule is not an n-submodule in general.

EXAMPLE 3.6. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $R = \mathbb{Z}$. Consider the submodules $N = 0 \oplus \mathbb{Z}$ and $K = \mathbb{Z} \oplus 0$. One can easily see that K and N are n-submodules. Since $N + K = M$, $N + K$ is not an n-submodule of M.

Proposition 3.7. *$\mathbb{Q}$ as $\mathbb{Z}$-module has only one n-submodule.*

Proof. By Lemma 2.5, zero submodule is an n-submodule of $\mathbb{Q}$. Let N be an n-submodule. It follows that $(N :_{\mathbb{Z}} \mathbb{Q}) = 0$. Then by Lemma 2.15, N is an prime submodule of $\mathbb{Q}$, which is zero. $\square$

Now we give an example to show that in Theorem 2.27, it is necessary that zero submodule be an n-submodule.

EXAMPLE 3.8. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}$. M is faithful and zero ideal is a semiprime ideal. By Example 3.2, zero submodule is not n-submodule of M and M is not torsion-free.

In the following examples we show that the condition $ker(f) \subseteq N$ in Theorem 2.28(i) and the condition monomorphism in Theorem 2.28(ii), are necessary.

EXAMPLE 3.9. Consider the $\mathbb{Z}$-epimorphism

$$\psi : \mathbb{Z} \longrightarrow \mathbb{Z}_6; \qquad a \longmapsto \bar{a}$$

Clearly $\psi(0) = \bar{0}$ and $ker(\psi) = 6\mathbb{Z} \nsubseteq (0)$. By Example 2.1(ii), $(\bar{0})$ is not n-submodule of $\mathbb{Z}_6$.

EXAMPLE 3.10. Consider the zero homomorphism

$$g : \mathbb{Q} \longrightarrow \mathbb{Z};$$

clearly $ker(g) = \mathbb{Q}$. So g is not monomorphism. By Proposition 3.7, $g^{-1}(0)$ is not an n-submodule.

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