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# Convergence Theorems for Proximal Point Method Involving Multivalued Nonexpansive Mappings in p-Uniformly Convex Metric Spaces

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Abstract. In this paper, we first extend the class of multivalued nonexpansive mappings to $p$-uniformly convex metric spaces. Furthermore, we propose and study an iterative algorithm involving $p$-resolvent operators of proper, convex and lowersemicontinuous functions for approximating a common solution of a finite family of minimization problems which is also a common fixed points of two multivalued nonexpansive mappings in $p$-uniformly convex metric space. Our proposed algorithm converges to a common element in the intersection of the set of minimizers of a finite family of proper, convex and lower semicontinuous functions and the set of common fixed points of two multivalued nonexpansive mappings. Finally, we demonstrate the applicability of our results with a numerical example. Our results improve many important and recent results in this direction.

**Keywords:** $p$-uniformly convex metric spaces, Multivalued nonexpansive mapping, $p$-resolvent operators.

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## 1. Introduction

Let $(X, d)$ be a metric space. $X$ is called a geodesic space if every two points $x, y \in X$ are joined by a geodesic path $\gamma : [0, d(x, y)] \to X$ such that $\gamma(0) = x$, $\gamma(d(x, y)) = y$ and $d(\gamma(t), \gamma(t') = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. In this case, $\gamma$ is an isometry and the image of $\gamma$ is called a geodesic segment joining $x$ to $y$. The space $X$ is said to be uniquely geodesic if every two points of $X$ are joined by exactly one geodesic segment, see [13, 36, 40] for further details. Let $A$ be a nonempty subset of $X$, then $A$ is said to be convex if $A$ includes every geodesic segment joining any two of its points.

Let $X$ be a geodesic convex metric space and $A$ be a nonempty subset of $X$. A subset $A$ is called proximinal (see [15]), if for each $x \in X$ there exists $a \in A$ such that

$$dist(x, A) = \inf\{d(x, a) : a \in A\}.$$

It is well known that if $A$ is proximinal, then $A$ is closed. We denote the family of all nonempty proximinal subsets of $X$ by $P(X)$ and the family of closed and bounded subsets of $X$ by $CB(X)$ respectively. If $A$ and $B$ are nonempty subsets of $X$, then the Hausdorff metric $H$ on $P(X)$ is defined by

$$H(A, B) = \max\{\sup_{a\in A} \ dist(a, B), \ \sup_{b\in B} \ dist(b, A)\}, \ \forall \ A, B \ \in P(X).$$

Let $T : X \to P(X)$ be a multivalued mapping. A point $x \in X$ is called a fixed point of $T$ if $x \in Tx$. The point $x$ is called a strict fixed point of $T$ if $Tx = \{x\}$ and $T$ is said to satisfy the endpoint condition. The notion of multivalued (or setvalued) mappings is more general than the single-valued mappings and was introduced by Markin [32]. Multivalued fixed point theory has interesting applications in control theory, convex optimization, differential inclusion, graph theory and economics (see [14, 17, 26, 34, 51] and other references therein). Due to this advantage, numerous researchers have extensively studied approximation of fixed points of multivalued mappings in nonlinear spaces like CAT(0) spaces, CAT($k$) spaces ($k > 0$), Hadamard spaces, hyperbolic spaces and $\mathbb{R}$-trees (see for example [6, 41, 42, 47, 50]). However, the notion of multivalued mappings are yet to be studied in $p$-uniformly convex metric spaces, which are more general than the aforementioned nonlinear spaces.

The nonlinearity property of metric spaces often makes extension of known results to nonlinear spaces tedious. Due to this limitation, Takahashi [53] introduced convexity property in metric spaces which provides some sufficient informations that enable the extension of some known results in Hilbert and Banach spaces to metric spaces. In 2011, Noar and Siberman [35] introduced a more general convex metric spaces known as $p$-uniformly convex metric spaces (see Section 2 for details). The concept of $p$-uniformly convex metric spaces is a natural generalization of the $p$-uniformly convex Banach spaces (see [11, 20, 54]).

On the other hand, Minimization Problem (MP) is known to be a very important optimization problem in optimization theory which has been studied extensively by many authors due to its interesting applications to real life problems. MPs are closely related to other optimization problems such as the monotone inclusion problems [1, 24, 44], equilibrium problems [22, 37] (see also [2]), variational inequality problems [3, 4, 23, 25, 38, 56] and many more. For a proper, convex and lower semicontinuous function $f$ on $X$, the MP is given as follows: Find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{1.1}$$

In this case, $x$ is called a minimizer of $f$ and the solution set (set of minimizers) of MP (1.1) is denoted by $\arg\min_{y \in X} f(y)$. To solve MP (1.1), the Proximal Point Algorithm (PPA) remains one of the known effective methods. The PPA was first introduced by Martinet [33] and was further developed by Rockafellar [48]. The latter proved that the PPA converges weakly to a minimizer of a proper convex and lower semicontinuous functional. Since then, numerous researchers have devoted attention to finding solutions of MPs using the PPA in Hilbert spaces, Banach spaces and their generalizations, see [18, 21, 46, 52] and other references therein. In 2013, Bačák [7] introduced and studied the PPA in a complete CAT(0) space. He established the $\Delta$-convergence of PPA with the assumptions that $f$ has a minimizer in $X$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. In an attempt to introduce and generalize the PPA to a more general convex metric space setting, Choi and Ji [11] introduced the notion of $p$-resolvent operator in a $p$-uniformly convex metric space as a generalization of the Moreau-Yosida resolvent in a CAT(0) space as follows:

$$J_\lambda^f(x) \quad = \quad \arg\min_{v \in X} \big(f(v) + \frac{1}{2\lambda} d(v,x)^p\big), \tag{1.2}$$

where $f$ is a convex and lower semicontinuous function not identically $\infty$ and $\lambda > 0$ (see [7]). They introduced the following PPA involving the $p$-resolvent operator (1.2) and proved that it converges to a minimizer of $f$ :

$$x_n \quad = \quad J_{\lambda_n}^f(x_{n-1}), \; n \geq 1. \tag{1.3}$$

Recently, Kuwae [28] defined another version of $p$-resolvent operator which is more general than (1.2) in $p$-uniformly convex metric spaces as follows:

$$J_\lambda^f(x) \quad = \quad \arg\min_{v \in X} \big(f(v) + \frac{1}{p\lambda^{p-1}} d(v,x)^p\big). \tag{1.4}$$

He established the unique existence of the $p$-resolvent operator (1.4) associated to a coercive proper lower semicontinuous functional. In addition, he applied (1.4) to obtain solutions of initial boundary value problems for $p$-harmonic maps. Izuchukwu et al. [20] adopted (1.4) to approximate the solution of a Split Minimization Problem (SMP) in $p$-uniformly convex metric spaces. They proposed a Backward-Backward Algorithm and an Alternating Proximal Algorithm that both converge to a solution of the SMP under some mild conditions. Very recently, Aremu et al. [5] proposed and studied a multi-step iterative algorithm that comprises of a finite family of asymptotically $k$-strictly pseudo-contractive mappings and a $p$-resolvent operator of form (1.4) associated with a proper, convex and lower semicontinuous function in a $p$-uniformly convex metric space. They established the $\Delta$-convergence of the proposed algorithm to a common fixed point of a finite family of asymptotically $k$-strictly pseudo-contractive mappings which is also a minimizer of the proper, convex and lower semicontinuous function. For more, some researchers have used the $p$-resolvent operator (1.2) to obtain solutions of MPs in $p$-uniformly convex metric spaces. For more recent and interesting results in $p$-uniformly convex metric spaces, see [5, 10] and other references therein.

Motivated by the current research interest in this direction, we extend the class of multivalued nonexpansive mappings to $p$-uniformly convex metric spaces. Furthermore, we propose and study an iterative algorithm involving $p$-resolvent operators (of type (1.4)) of proper, convex and lowersemicontinuous functions for approximating a common solution of a finite family of minimization problems which is also a common fixed points of two multivalued nonexpansive mappings in $p$-uniformly convex metric space. Our proposed algorithm converges to a common element in the intersection of the set of minimizers of a finite family of proper, convex and lower semicontinuous functions and the set of common fixed points of two multivalued nonexpansive mappings. Finally, we demonstrate the applicability of our results with a numerical example. Our results improve many important and recent results in this direction.

## 2. Preliminaries

In this section, we recall some results and definitions that will be needed in the proof of our main results.

Let $\{x_n\}$ be a bounded sequence in a metric space $X$ and $r(\cdot,\{x_n\}) : X \to [0,\infty)$ be a continuous functional defined by $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x,\{x_n\}) : x \in X\}$, while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}$. A sequence $\{x_n\}$ in $X$ is said to be $\Delta$-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we say that $x$ is the $\Delta$-limit of $\{x_n\}$ (see [13, 27]). The notion of $\Delta$-convergence in metric spaces was introduced and studied by Lim [31], and it is known as analogue of the notion of weak convergence in Banach spaces.

**Definition 2.1.** [30] A convex metric space $X$ is called uniformly convex, if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $a, x, y \in X$, we have that $d(x, a) \leq r,\ d(y, a) \leq r$ and $d(x, y) \geq \epsilon r$ imply

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, a) \leq (1 - \delta)r. \tag{2.1}$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \epsilon)$ for any given $r > 0$ and $\epsilon \in (0, 2]$, is called the modulus of uniform convexity.

**Definition 2.2.** Let $1 < p < \infty$, a metric space $X$ is called $p$-uniformly convex with parameter $c > 0$ if and only if $X$ is a geodesic space and

$$d(v, (1-t)x \oplus ty)^p \quad \leq \quad (1-t)d(v, x)^p + td(v, y)^p - \frac{c}{2}t(1-t)d(x, y)^p \tag{2.2}$$

for all $x, y, v \in X,\ t \in [0, 1]$.

*Remark* 2.3. [5] If $X$ is a $p$-uniformly convex metric space for $1 < p < \infty$, with parameter $c > 0$. Then, the modulus of uniform convexity of $X$ is given as

$$\delta_X(\epsilon) = \frac{c\epsilon^p}{8p}.$$

The following are typical examples of $p$-uniformly convex metric spaces:
**1.** Let $X$ be a real Banach space. The modulus of convexity of $X$ with $dim(X) \geq 2$ is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) := \inf \{1 - ||\frac{x+y}{2}|| : ||x|| = ||y|| = 1,\ \epsilon = ||x - y||\}.$$

The Banach space $X$ is called $p$-uniformly convex for $p > 1$[12] (see also [8]), if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. The $L_p$ space with $p \geq 2$, is a $p$-uniformly convex Banach space with $\delta_X(\epsilon) \geq \frac{\epsilon^p}{p2^p}$ [12]. If a Banach space $X$ is $p$-uniformly convex for $p \geq 2$, then $X$ is $p$-uniformly convex metric space (see [10, 29]).
**2.** Let $X$ be a CAT(0) space. For any two elements $x, y \in X$, there exists an element $m \in X$ such that

$$d(z, m)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2 \ \forall\ z \in X.$$

For any $x, y \in X$, there exists a unique geodesic $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Furthermore, for any $z \in X$ and $t \in [0, 1]$,

$$d(z, \gamma(t))^2 \leq (1-t)d(z, x)^2 + td(z, y)^2 - t(1-t)d(x, y)^2.$$

Then a CAT(0) space is 2-uniformly convex metric space with parameter $c = 2$ and $p = 2$ (see [10, 20, 49]).
**3.** Let $X$ be a CAT($k$) space with $diam(X) < \frac{\pi}{2\sqrt{k}}$ for $k > 0$ and parameter $c = (\pi - 2\sqrt{k}\epsilon)\tan(\sqrt{k}\epsilon)$ for any $0 < \epsilon \leq \frac{\pi}{2\sqrt{k}} - diam(X)$. Let $\gamma : [0, 1] \to X$ be any geodesic with $\gamma(0) = x,\ \gamma(1) = y,\ z \in X$ and $t \in [0, 1]$, such that

$$d(z, \gamma(t))^2 \leq (1-t)d(z, x)^2 + td(z, y)^2 - \frac{c}{2}t(1-t)d(x, y)^2$$

holds. Then, CAT($k$) space $X$ with $diam(X) < \frac{\pi}{2\sqrt{k}}$ is a 2-uniformly convex metric space (see [10, 29, 45]).

*Remark* 2.4. [16, 49] Let $X$ be a complete $p$-uniformly convex metric space. Then,

(i) every bounded sequence in X has a unique asymptotic center,
(ii) every bounded sequence in X has a $\Delta$-convergent subsequence.

**Definition 2.5.** Let $X$ be a complete convex metric space. A multivalued nonlinear mapping $T : X \to 2^X$ is said to be demiclosed if for any bounded sequence $\{x_n\}$ in $X$ such that $\Delta - \lim_{n\to\infty} x_n = v$ and $\lim_{n\to\infty} d(x_n, z_n) = 0$, (where $z_n \in Tx_n$) we have that $v \in F(T)$.

**Lemma 2.6.** [19] *Let $X$ be a metric space and $A, B$ are nonempty subsets in $P(X)$. Then for all $a \in A$, there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

**Definition 2.7.** Let $X$ be a $p$-uniformly convex metric space. A function $f : X \to (-\infty, \infty]$ is said to be

(i) convex, if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\ \forall x, y \in X,\ \lambda \in (0,1),$$

(ii) proper, if $\mathbb{D} := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where $\mathbb{D}$ denotes the domain of $f$,
(iii) lower semicontinuous at a point $x \in \mathbb{D}$, if

$$f(x) \leq \liminf_{n\to\infty} f(x_n),$$

for each sequence $\{x_n\}$ in $\mathbb{D}$ such that $\lim_{n\to\infty} x_n = x$,
(iv) lower semicontinuous on $\mathbb{D}$, if it is lower semicontinuous at every point in $\mathbb{D}$.

**Lemma 2.8.** [20] *For $1 < p < \infty$, let $X$ be a $p$-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Then, for any $\lambda > 0$ and $x \in X$, there exists a unique point, say $J_\lambda^f(x) \in X$ such that*

$$J_\lambda^f(x) + \frac{1}{p\lambda^{p-1}} d(J_\lambda^f(x), x)^p = \inf_{v\in X} \big(f(v) + d(v, x)^p\big).$$

**Lemma 2.9.** [20] *For $1 < p < \infty$, let $X$ be a $p$-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Then, the $p$-resolvent operator $J_\lambda^f$ of $f$ is nonexpansive.*

**Lemma 2.10.** [55] *For $1 < p < \infty$, let $X$ be a $p$-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Let $J_\lambda^f$ be the $p$-resolvent mapping of $f$ such that $F(J_\lambda^f) \neq \emptyset$, then for $\lambda > 0$, we have the following:*

(i) $v \in F(J_\lambda^f)$ *if and only if* $v$ *is a minimizer of* $f$;
(ii) $d(v, J_\lambda^f x)^p + d(J_\lambda^f x, x)^p \leq d(v, x)^p$ *for all* $x \in X$ *and* $v \in F(J_\lambda^f)$;
(iii) $d(J_\lambda x, x)^p \leq d(J_\mu x, x)^p for \lambda < \mu$ *and* $x \in X$.

## 3. Main results

In this section, we prove some lemmas which are crucial in establishing our main results. We begin with the following definition of multivalued nonexpansive mapping and example in $p$-uniformly convex metric spaces.

**Definition 3.1.** Let $X$ be a $p$-uniformly convex metric space. A mapping $T : X \to P(X)$ is said to be multivalued nonexpansive, if

$$H(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in X.$$

Example 3.2. Let $Y := \{(x, e^x) : x \in \mathbb{R}\}$ and $X_n := \{(n, y) : y \geq e^n\}$ for each $n \in \mathbb{Z}$. Set $X := Y \cup \bigcup_{n \in \mathbb{Z}} X_n$ equipped with a metric $d : X \times X \to [0, \infty)$, defined for all $x = (x_1, x_2), y = (y_1, y_2) \in X$ by

$$d(x, y) = \begin{cases} \int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt + |x_2 - e^{x_1}| + |y_2 - e^{y_1}|, & \text{if } x_1 \neq y_1, \\ |x_2 - y_2|, & \text{if } x_1 = y_1, \end{cases} \tag{3.1}$$

where $\dot{\gamma}$ is the derivative of the curve $\gamma : \mathbb{R} \to X$ given as $\gamma(t) := (t, e^t)$ for each $t \in \mathbb{R}$ (see [9]). Then $(X, d)$ is a complete $p$-uniformly convex metric space with $p = 2$ and parameter $c = 2$.

Now, let $T : X \to P(X)$ be defined by $Tx = \{(x_1, e^{x_1}), (0, 0)\}$ for all $x = (x_1, x_2) \in X$. Clearly $F(T) = \{(0, 0)\}$. We check that $T$ is nonexpansive. Indeed, for each $(x_1, x_2), (y_1, y_2) \in X$, we have

$$dist((x_1, e^{x_1}), Ty) = \inf\{d((x_1, e^{x_1}), (y_1, e^{y_1})), d((0, 0), (x_1, e^{x_1}))\}.$$

But,

$$d\big((x_1, e^{x_1}), (y_1, e^{y_1})\big) = \begin{cases} \int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt + |e^{x_1} - e^{x_1}| + |e^{y_1} - e^{y_1}| & \text{if } x_1 \neq y_1, \\ |e^{x_1} - e^{y_1}| & \text{if } x_1 = y_1, \end{cases}$$

$$= \begin{cases} \int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt & \text{if } x_1 \neq y_1, \\ |e^{x_1} - e^{y_1}| & \text{if } x_1 = y_1, \end{cases}$$

and

$$d\big((0,0),(x_1,e^{x_1})\big) = \begin{cases} \displaystyle\int_0^{x_1} ||\dot{\gamma}(t)||_2 dt + |0-e^0| + |e^{x_1}-e^{x_1}| & \text{if } x_1 \neq 0, \\ e^{x_1} & \text{if } x_1 = 0, \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^{x_1} ||\dot{\gamma}(t)||_2 dt + 1 & \text{if } x_1 \neq 0, \\ 1 & \text{if } x_1 = 0. \end{cases}$$

Therefore,

$$dist((x_1,e^{x_1}),Ty) = d((x_1,e^{x_1}),(y_1,e^{y_1})).$$

Also,

$$\begin{aligned} dist((0,0),Ty) &= \inf\{d((0,0),(y_1,e^{y_1})), d((0,0),(0,0))\} \\ &= d((0,0),(0,0)). \end{aligned}$$

Similarly,

$$\begin{aligned} dist((y_1,e^{y_1}),Tx) &= \inf\{d((x_1,e^{x_1}),(y_1,e^{y_1})), d((0,0),(y_1,e^{y_1}))\} \\ &= d((x_1,e^{x_1}),(y_1,e^{y_1})) \end{aligned}$$

and

$$\begin{aligned} dist((0,0),Ty) &= \inf\{d((0,0),(x_1,e^{x_1})), d((0,0),(0,0))\} \\ &= d((0,0),(0,0)). \end{aligned}$$

Hence,

$$\begin{aligned} H(Tx,Ty) &= \max\Big\{ \sup_{a\in Tx} dist(a,Ty), \sup_{b\in Ty} dist(b,Tx)\Big\} \\ &= \max\Big\{ \sup\{d((x_1,e^{x_1}),(y_1,e^{y_1})), d((0,0),(0,0))\}, \\ &\quad \sup\{d((x_1,e^{x_1}),(y_1,e^{y_1})), d((0,0),(0,0))\}\Big\} \\ &= d((x_1,e^{x_1}),(y_1,e^{y_1})) \\ &= \begin{cases} \displaystyle\int_{x_1}^{y_1} ||\dot{\gamma}(t)||_2 dt & \text{if } x_1 \neq y_1, \\ |e^{x_1}-e^{y_1}| & \text{if } x_1 = y_1, \end{cases} \\ &\leq d(x,y). \end{aligned}$$

Therefore, $T$ is a multivalued nonexpansive mapping.

**Lemma 3.3.** *(Demiclosedness). Let $X$ be a $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$. Let $T : X \to P(X)$ be a multivalued nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in $X$ such that*

*$\{x_n\}$ $\Delta$-converges to $v \in X$ and $\lim_{n\to\infty} d(x_n, z_n) = 0$ (where $z_n \in Tx_n$). Then $v \in F(T)$.*

*Proof.* Since $\{x_n\}$ is bounded, we get from Remark 2.4(i) that the asymptotic center of the sequence $\{x_n\}$ is unique. Also, since $\{x_n\}$ $\Delta$-converges to a point $v$, it then follows that $A(\{x_n\}) = \{v\}$. Let $\varphi(x) = \limsup_{n\to\infty} d(x_n, x)$, since $\lim_{n\to\infty} d(x_n, z_n) = 0$, we obtain that

$$\varphi(x) = \limsup_{n\to\infty} d(z_n, x).$$

If $v^* \in Tv$, then

$$\begin{aligned} \varphi(v^*) &= \limsup_{n\to\infty} d(z_n, v^*) \\ &\leq \limsup_{n\to\infty} H(Tx_n, Tv) \\ &\leq \limsup_{n\to\infty} d(x_n, v) \\ &= \varphi(v). \end{aligned} \tag{3.2}$$

If we let $t = \frac{1}{2}$ in (2.2), we obtain that

$$d(x_n, \frac{1}{2}(v \oplus v^*))^p \leq \frac{1}{2} d(x_n, v)^p + \frac{1}{2} d(x_n, v^*) - \frac{c}{8} d(v, v^*)^p. \tag{3.3}$$

Taking the $\limsup$ of (3.3) as $n \to \infty$, we have that

$$\varphi(v)^p \leq \varphi(\frac{1}{2}(v \oplus v^*))^p \leq \frac{1}{2}\varphi(v)^p + \frac{1}{2}\varphi(v^*)^p - \frac{c}{8} d(v, v^*)^p,$$

which implies

$$cd(v, v^*)^p \leq 4(\varphi(v^*)^p - \varphi(v)^p). \tag{3.4}$$

It is easy to see from (3.2), (3.4) and $c > 0$ that $d(v, v^*) = 0$, hence $v \in Tv$. This completes the proof. □

**Lemma 3.4.** *For $1 < p < \infty$, let $X$ be a $p$-uniformly convex metric space with parameter $c > 0$. Then for all $v, w, x, y, z \in X$ and $\alpha, \beta, \gamma \in (0, 1)$, with $\alpha + \beta + \gamma = 1$, we have*

$$d(\alpha x \oplus \beta y \oplus \gamma z, v)^p \leq \alpha d(x, v)^p + \beta d(y, v)^p + \gamma d(z, v)^p.$$

*Proof.* Let $w = \alpha x \oplus \beta y \oplus \gamma z$. We may rewrite $w$ as $w = \alpha x \oplus (1-\alpha)\big(\frac{\beta}{1-\alpha} y \oplus \frac{\gamma}{1-\alpha} z\big)$. Then from (2.2), we have

$$\begin{aligned} d(w,v)^p &= d\Big(\alpha x \oplus (1-\alpha) d\big(\frac{\beta}{1-\alpha} y \oplus \frac{\gamma}{1-\alpha} z\big), v\Big)^p \\ &\leq \alpha d(x,v)^p + (1-\alpha) d\Big(\big(\frac{\beta}{1-\alpha} y \oplus \frac{\gamma}{1-\alpha} z\big), v\Big)^p \\ &- \frac{c}{2}\alpha(1-\alpha) d\Big(x, \big(\frac{\beta}{1-\alpha} y \oplus \frac{\gamma}{1-\alpha} z\big)\Big)^p \\ &\leq \alpha d(x,v)^p \\ &+ (1-\alpha)\Big[\frac{\beta}{1-\alpha} d(y,v)^p + \frac{\gamma}{1-\alpha} d(z,v)^p - \frac{c\beta\gamma}{2(1-\alpha)^2} d(y,z)^p\Big] \\ &\leq \alpha d(x,v)^p + \beta d(y,v)^p + \gamma d(z,v)^p. \end{aligned}$$

□

We are now ready to present the main results of this paper. Henceforth, for each $i = 1, 2 \ldots, N$, we denote by $J_{\lambda_n{}^{(i)}}$ the $p$-resolvent operators of the form (1.4) of finite family of proper, convex and lower semicontinuous functions $f_i$.

**Lemma 3.5.** *Let $X$ be a $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$ be a finite family of proper convex and lower semicontinuous functions. Let $T_j : X \to P(X)$, $j = 1, 2$ be two multivalued nonexpansive mappings such that $T_j v = \{v\}$ and $\Gamma := \bigcap_{j=1}^{2} F(T_j) \cap \bigcap_{i=1}^{N} \arg\min_{y \in X} f_i(y) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n, \\ x_{n+1} = \alpha_n x_n \oplus \beta_n z_{n,1} \oplus \gamma_n z_{n,2} \end{cases} \tag{3.5}$$

*where $z_{n,1} \in T_1 y_n$, $z_{n,2} \in T_2 y_n$ and $\{\lambda_n^{(i)}\}$ is a sequence for each $i = 1, 2, \cdots, N$, such that the following conditions are satisfied:*

(C1) $\lambda_n^{(i)} > \lambda^{(i)} > 0$,
(C2) $\alpha_n \in [a,b] \subset (0,1)$, $\beta_n \in [d,e] \subset (0,1)$, $\gamma_n \in [g,h] \subset (0,1)$, $\alpha_n + \beta_n + \gamma_n = 1$ *for* $n \geq 1$.

*Then,*
*(a)* $\lim_{n\to\infty} d(x_n, v)^p$ *exists* $\forall\ v \in \Gamma$,
*(b)* $\lim_{n\to\infty} d(c_n^{(i)}, J_{\lambda^{(i)}} c_n^{(i)}) = 0$, $\forall\ i = 1, 2, \cdots, N$, *where* $c_n^{(i+1)} = J_{\lambda_n^{(i)}} c_n^{(i)}$, *and* $c_n^{(1)} = x_n$ *for each* $i = 1, 2, \cdots, N$, $n \geq 1$ *and*
*(c)* $\lim_{n\to\infty} dist(T_j y_n, y_n) = 0$, *for each* $j = 1, 2$.

*Proof.* $(a)$ Since $c_n^{(i+1)} = J_{\lambda_n^{(i)}} c_n^{(i)}$, for all $n \geq 1$, and $c_n^{(1)} = x_n$, then

$$c_n^{(N+1)} = J_{\lambda_n^{(N)}} c_n^{(N)} = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(3)}} \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n.$$

Let $v \in \Gamma$, we get from Lemma 2.10(i) that $v = J_{\lambda_n^{(i)}} v$ for all $i = 1, 2, \cdots, N$. Then from (3.5), Lemma 2.6, Lemma 2.9 and Lemma 3.4, we have that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(x_n, v)^p + \beta_n d(z_{n,1}, v)^p + \gamma_n d(z_{n,2}, v)^p \\ &\leq \alpha_n d(x_n, v)^p + \beta_n H(T_1 y_n, T_1 v)^p + \gamma_n H(T_2 y_n, T_2 v)^p \\ &\leq \alpha_n d(x_n, v)^p + \beta_n d(y_n, v)^p + \gamma_n d(y_n, v)^p \\ &= \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(y_n, v)^p \\ &= \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(c_n^{(N+1)}, v)^p \qquad (3.6) \\ &\leq \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(c_n^{(N)}, v)^p \\ &\vdots \\ &\leq \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(c_n^{(1)}, v)^p \\ &= d(x_n, v)^p. \end{aligned}$$

Hence, $\lim_{n \to \infty} d(x_n, v)^p$ exists.

$(b)$ From Lemma 2.10(ii), for all $i = 1, 2, \ldots, N$, we have that

$$d(c_n^{(i+1)}, v)^p \leq d(c_n^{(i)}, v)^p - d(c_n^{(i)}, c_n^{(i+1)})^p. \tag{3.7}$$

Setting $i = N$ in (3.7), we obtain from (3.6) that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(c^{(N+1)}, v)^p \\ &\leq \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) \big[ d(c_n^{(N)}, v)^p - d(c_n^{(N)}, c_n^{(N+1)})^p \big] \\ &\vdots \\ &\leq \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) \big[ d(c_n^{(1)}, v)^p - d(c_n^{(N)}, c_n^{(N+1)})^p \big] \\ &= \alpha_n d(x_n, v)^p + (1 - \alpha_n) \big[ d(x_n, v)^p - d(c_n^{(N)}, c_n^{(N+1)})^p \big] \\ &= d(x_n, v)^p - (1 - \alpha_n) d(c_n^{(N)}, c_n^{(N+1)})^p, \qquad (3.8) \end{aligned}$$

which implies from condition C2 and (a) that

$$(1 - \alpha_n) d(c_n^{(N)}, c_n^{(N+1)})^p \leq d(x_n, v)^p - d(x_{n+1}, v)^p \to 0, \text{ as } n \to \infty. \tag{3.9}$$

Hence,

$$d(c_n^{(N)}, c_n^{(N+1)}) \to 0, \text{ as } n \to \infty. \tag{3.10}$$

Again, if we set $i = N - 1$ in (3.7) and following the same argument in (3.8), we obtain that

$$d(c_n^{(N-1)}, c_n^{(N)}) \to 0, \text{ as } n \to \infty. \tag{3.11}$$

Continuing in the same manner, we have that

$$d(c_n^{(i)}, c_n^{(i+1)}) \to 0, \text{ as } n \to \infty, \text{ for all } i = 1, 2, \cdots, N-2. \tag{3.12}$$

Then, by (3.10), (3.11) and (3.12), we get

$$d(c_n^{(i)}, c_n^{(i+1)}) \to 0, \text{ as } n \to \infty, \text{ for all } i = 1, 2, \cdots, N. \tag{3.13}$$

Hence, for each $i = 1, 2, \cdots, N$, we obtain by applying the triangle inequality that

$$d(c_n^{(1)}, c_n^{(i)}) = d(x_n, c_n^{(i)}) \to 0, \text{ as } n \to \infty \text{ for all } i = 1, 2, \cdots, N+1. \tag{3.14}$$

By condition C1, we have from Lemma 2.10(iii) and (3.13) that

$$d(c_n^{(i)}, J_{\lambda^{(i)}} c_n^{(i)}) \le d(c_n^{(i)}, J_{\lambda_n^{(i)}} c_n^{(i)}) \to 0, \text{ as } n \to \infty, \text{ for all } i = 1, 2, \cdots N. \tag{3.15}$$

$(c)$ From (2.2), (3.5) and Lemma 2.6, we have

$$\begin{aligned}
d(x_{n+1}, v)^p &= d\Big((1-\gamma_n)\big(\frac{\alpha_n}{1-\gamma_n} x_n \oplus \frac{\beta_n}{1-\gamma_n} z_{n,1}\big) \oplus \gamma_n z_{n,2}, v\Big)^p \\
&\le (1-\gamma_n)\Big[\frac{\alpha_n}{1-\gamma_n} d(x_n, v)^p + \frac{\beta_n}{1-\gamma_n} d(z_{n,1}, v)^p \\
&- \frac{c\alpha_n\beta_n}{2(1-\gamma_n)^2} d(x_n, z_{n,1})^p\Big] + \gamma_n d(z_{n,2}, v)^p \\
&= \alpha_n d(x_n, v)^p + \beta_n d(z_{n,1}, v)^p - \frac{c\alpha_n\beta_n}{2(1-\gamma_n)} d(x_n, z_{n,1})^p \\
&+ \gamma_n d(z_{n,2}, v)^p \\
&\le \alpha_n d(x_n, v)^p + \beta_n H(T_1 y_n, T_1 v)^p \\
&- \frac{c\alpha_n\beta_n}{2(1-\gamma_n)} d(x_n, z_{n,1})^p + \gamma_n H(T_2 y_n, T_2 v)^p \\
&\le \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(c_n^{(N)}, v)^p - \frac{c\alpha_n\beta_n}{2(1-\gamma_n)} d(x_n, z_{n,1})^p \\
&\vdots \\
&\le \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n) d(c_n^{(1)}, v)^p - \frac{c\alpha_n\beta_n}{2(1-\gamma_n)} d(x_n, z_{n,1})^p \\
&= d(x_n, v)^p - \frac{c\alpha_n\beta_n}{2(1-\gamma_n)} d(x_n, z_{n,1})^p,
\end{aligned}$$

which implies that

$$\frac{c\alpha_n\beta_n}{2(1-\gamma_n)} d(x_n, z_{n,1})^p \le d(x_n, v)^p - d(x_{n+1}, v)^p.$$

Therefore, by condition C2 and the fact that $c > 0$, we obtain that

$$d(x_n, z_{n,1}) \to 0, \text{ as } n \to \infty. \tag{3.16}$$

Similarly, from (3.5), (2.2) and Lemma 2.6, we have

$$\begin{aligned}
d(x_{n+1}, v)^p &= d\Big((1-\beta_n)\big(\frac{\alpha_n}{1-\beta_n}x_n \oplus \frac{\gamma_n}{1-\beta_n}z_{n,2}\big) \oplus \beta_n z_{n,1}, v\Big)^p \\
&\leq (1-\beta_n)\Big[\frac{\alpha_n}{1-\beta_n}d(x_n, v)^p + \frac{\gamma_n}{1-\beta_n}d(z_{n,2}, v)^p \\
&- \frac{c\alpha_n\gamma_n}{2(1-\beta_n)^2}d(x_n, z_{n,2})^p\Big] + \beta_n d(z_{n,1}, v)^p \\
&= \alpha_n d(x_n, v)^p + \gamma_n d(z_{n,2}, v)^p - \frac{c\alpha_n\gamma_n}{2(1-\beta_n)}d(x_n, z_{n,2})^p \\
&+ \beta_n d(z_{n,1}, v)^p \\
&\leq \alpha_n d(x_n, v)^p + \gamma_n H(T_2 y_n, T_2 v)^p - \frac{c\alpha_n\gamma_n}{2(1-\beta_n)}d(x_n, z_{n,2})^p \\
&+ \beta_n H(T_1 y_n, T_1 v)^p \\
&\leq \alpha_n d(x_n, v)^p + (\beta_n + \gamma_n)d(c_n^{(N)}, v)^p - \frac{c\alpha_n\gamma_n}{2(1-\beta_n)}d(x_n, z_{n,2})^p \\
&\vdots \\
&\leq \alpha_n d(x_n, v)^p + (\gamma_n + \beta_n)d(c_n^{(1)}, v)^p - \frac{c\alpha_n\beta_n}{2(1-\beta_n)}d(x_n, z_{n,2})^p \\
&= d(x_n, v)^p - \frac{c\alpha_n\gamma_n}{2(1-\beta_n)}d(x_n, z_{n,2})^p,
\end{aligned}$$

which implies that

$$\frac{c\alpha_n\gamma_n}{2(1-\beta_n)}d(x_n, z_{n,2})^p \leq d(x_n, v)^p - d(x_{n+1}, v)^p$$

and hence, by condition C2 and the fact that $c > 0$, we obtain that

$$d(x_n, z_{n,2}) \to 0, \text{ as } n \to \infty. \tag{3.17}$$

Thus, from (3.16) and (3.17), we have that

$$d(z_{n,j}, x_n) \to 0, \text{ as } n \to \infty, \text{ for } j = 1, 2. \tag{3.18}$$

Therefore, we conclude from (3.14) (when $i = N+1$) and (3.18) that

$$dist(y_n, T_j y_n) \leq d(y_n, x_n) + d(x_n, z_{n,j}) \to 0, \text{ as } n \to \infty, \text{ for } j = 1, 2. \tag{3.19}$$

Finally, from (3.14) (when $i = N+1$) and (3.19), we obtain

$$\begin{aligned}
dist(x_n, T_j x_n) &\leq d(x_n, y_n) + d(y_n, T_j y_n) + d(T_j y_n, T_j x_n) \\
&\leq d(x_n, y_n) + d(y_n, T_j y_n) + d(y_n, x_n) \to 0, \text{ as } n \to \infty. \quad (3.20)
\end{aligned}$$

□

**Theorem 3.6.** *Let $X$ be a complete $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$ be a finite family of proper convex and lower semicontinuous functions. Let $T_j : X \to P(X)$,*

*$j = 1, 2$ be two multivalued nonexpansive mappings such that $T_j v = \{v\}$ and $\Gamma := \bigcap_{j=1}^{2} F(T_j) \cap \bigcap_{i=1}^{N} \arg\min_{y\in X} f_i(y) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by (3.5) such that conditions (C1) and (C2) in Lemma 3.5 are satisfied. Then, the sequence $\{x_n\}$ $\Delta$-converges to an element of $\Gamma$.*

*Proof.* Since $\{x_n\}$ is bounded and $X$ is a complete $p$-uniformly convex metric space, then by Remark 2.4(i) the sequence $\{x_n\}$ has a unique asymptotic center (that is $A(\{x_n\}) = \{v\}$). Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{u\}$. Then by (3.20), we have $\lim_{k\to\infty} dist(T_j x_{n_k}, x_{n_k}) = 0,\ j = 1, 2$. Thus, by Remark 2.4(ii) and Lemma 3.3, we obtain that $u \in \Gamma$. Also, since $J_{\lambda^{(i)}}$ is nonexpansive mapping for each $i = 1, 2, \cdots, N$, it then follows from Remark 2.4(ii), (3.14) and (3.15) that $u \in \Gamma$.

Furthermore, since we have from Lemma 3.5(a) that $\lim_{n\to\infty} d(x_n, u)$ exists. It follows from the uniqueness of asymptotic center that

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, u) &\leq \limsup_{k\to\infty} d(x_{n_k}, v) \\
&\leq \limsup_{n\to\infty} d(x_n, v) \\
&\leq \limsup_{n\to\infty} d(x_n, u) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, u)
\end{aligned}$$

which implies that $v = u$. Therefore, $\{x_n\}$ $\Delta$-converges to an element of $\Gamma$. □

The following is consequence of Theorem 3.6.

**Corollary 3.7.** *Let $X$ be a complete $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to P(X)$, be a multivalued nonexpansive mapping such that $Tv = \{v\}$ and $\Gamma := F(T) \cap \bigcap_{i=1}^{N} \arg\min_{y\in X} f_i(y) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) z_n, \end{cases} \tag{3.21}$$

*where $z_n \in Ty_n$, $\{\lambda_n^{(i)}\}$ is a sequence for each $i = 1, 2, \cdots, N$ such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ and $\{\alpha_n\} \in [a, b] \subset (0, 1)$. Then, the sequence $\{x_n\}$ $\Delta$-convergence to an element of $\Gamma$.*

*Proof.* If we set $\gamma_n = 0$ and $\beta_n = (1 - \alpha_n)$ in (3.5), and applying (2.2) in place of Lemma 3.4, then the proof follows as the proofs of Lemma 3.5 and Theorem 3.6. □

Now, we present some strong convergence results.

**Theorem 3.8.** *Let $X$ be a complete $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$ be a finite family of proper convex and lower semicontinuous functions. Let $T_j : X \to P(X)$, $j = 1, 2$ be two multivalued nonexpansive mappings such that $T_j v = \{v\}$ and $\Gamma := \bigcap_{j=1}^{2} F(T_j) \cap \bigcap_{i=1}^{N} \arg\min_{y \in X} f_i(y) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by* (3.5) *such that conditions (C1) and (C2) in Lemma 3.5 are satisfied. Then, the sequence $\{x_n\}$ strongly converges to an element of $\Gamma$ if and only if $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$.*

*Proof.* Suppose that $\{x_n\}$ converges to a point $v \in \Gamma$. Then, $\lim_{n\to\infty} d(x_n, v) = 0$ and since $0 \leq dist(x_n, \Gamma) \leq d(x_n, v)$, it follows that $\lim_{n\to\infty} dist(x_n, \Gamma) = 0$. Thus, $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$. Conversely, suppose $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$, we arbitrarily choose $\xi > 0$, for a positive integer $m_0$ such that

$$dist(x_n, \Gamma) < \frac{\xi}{4}, \ \forall\, n \geq m_0.$$

In particular,

$$\inf\{d(x_{m_0}, v) : v \in \Gamma\} < \frac{\xi}{4}.$$

Then, there exists $v^* \in \Gamma$ such that $d(x_{m_0}, v^*) < \frac{\xi}{2}$. Then, for all $m, n \geq m_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, v^*) + d(v^*, x_n) \\ &\leq 2d(x_{m_0}, v^*) \\ &\leq \xi. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, is implies that the sequence $\{x_n\}$ converges to some point $v^*$ in $X$. Also, since $\Gamma$ is closed and $\lim_{n\to\infty} dist(x_n, \Gamma) = 0$, we have $v^* \in \Gamma$. This completes the proof. □

**Theorem 3.9.** *Let $X$ be a complete $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$ be a finite family of proper convex and lower semicontinuous functions. Let $T_j : X \to P(X)$, $j = 1, 2$ be two multivalued nonexpansive mappings such that $T_j v = \{v\}$ and $\Gamma := \bigcap_{j=1}^{2} F(T_j) \cap \bigcap_{i=1}^{N} \arg\min_{y \in X} f_i(y) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by* (3.5) *such that conditions (C1) and (C2) in Lemma 3.5 are satisfied. Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function with $\Phi(0) = 0$ and $\Phi(r) > 0$ for all $r > 0$ such that*

$$\Phi(dist(x, \Gamma)) \leq dist(x, T_j x) \ \textit{for}\ j = 1, 2 \tag{3.22}$$

*or*

$$\Phi(dist(x, \Gamma)) \leq dist(x, J_{\lambda^{(i)}} x) \ \textit{for}\ i = 1, 2, \cdots, N, \tag{3.23}$$

*for all $x \in D$. Then, the sequence $\{x_n\}$ strongly converges to an element of $\Gamma$.*

*Proof.* It follows from Lemma 3.5(a) that $\lim_{n\to\infty} d(x_n, v) = 0$. This implies that $\lim_{n\to\infty} d(x_n, \Gamma) = 0$. Then, we have from (3.15) and (3.20) that

$$\lim_{n\to\infty} \Phi(dist(x,\Gamma)) \leq \lim_{n\to\infty} dist(x, T_j x) = 0 \text{ for } j = 1, 2$$

or

$$\lim_{n\to\infty} \Phi(dist(x,\Gamma)) \leq \lim_{n\to\infty} dist(x, J_{\lambda^{(i)}} x) = 0 \text{ for } i = 1, 2, \cdots, N.$$

Thus, we obtain that

$$\lim_{n\to\infty} \Phi(dist(x_n,\Gamma)) = 0.$$

Since $\Phi$ is nondecreasing, it implies that

$$\lim_{n\to\infty} dist(x_n,\Gamma) = 0.$$

Following the same line of argument in the proof of Theorem 3.8, we conclude that $\{x_n\}$ converges strongly to a point in $\Gamma$. This completes the proof. □

In what follows, we give some consequences of our main results.
By setting $N = 1$ in Theorem 3.8, we obtain the following result:

**Corollary 3.10.** *Let $X$ be a complete $p$-uniformly convex metric space $X$ with $1 < p < \infty$ and parameter $c \geq 2$ and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Let $T_j : X \to P(X)$, $j = 1, 2$ be two multivalued nonexpansive mappings such that $T_j v = \{v\}$ and $\Gamma := \bigcap_{j=1}^{2} F(T_j) \cap \arg\min_{y\in X} f(y) \neq \emptyset$. For arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = J_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n x_n \oplus \beta_n z_{n,1} \oplus \gamma_n z_{n,2} \end{cases} \tag{3.24}$$

*where $z_{n,1} \in T_1 y_n, z_{n,2} \in T_2 y_n$ and $\{\lambda_n\}$, is a sequence for each $i = 1, 2, \cdots, N$, such that $\lambda_n > \lambda > 0$ and condition (C2) in Lemma 3.5 holds. Then, the sequence $\{x_n\}$ strongly converges to an element of $\Gamma$ if and only if $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$.*

By setting $J_{\lambda_n} \equiv I$ in Corollary 3.10, we obtain the following result:

**Corollary 3.11.** *Let $X$ be a complete $p$-uniformly convex metric space with $1 < p < \infty$ and parameter $c \geq 2$ and $T_j : X \to P(X)$, $j = 1, 2$ be two multivalued nonexpansive mappings such that $T_j v = \{v\}$ and $\Gamma := \bigcap_{j=1}^{2} F(T_j) \neq \emptyset$. For arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n x_n \oplus \beta_n z_{n,1} \oplus \gamma_n z_{n,2},$$

*where* $z_{n,1} \in T_1 x_n, z_{n,2} \in T_2 x_n$, *such that condition (C2) in Lemma 3.5 holds. Then, the sequence* $\{x_n\}$ *strongly converges to an element of* $\Gamma$ *if and only if* $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$.

If $T_1$ and $T_2$ are singlevalued nonexpansive mappings in Theorem 3.8, we obtain the following result:

**Corollary 3.12.** *Let* $X$ *be a complete* $p$*-uniformly convex metric space with* $1 < p < \infty$ *and parameter* $c \geq 2$ *and* $f_i : X \to (-\infty, \infty]$ *be a finite family of proper convex and lower semicontinuous functions. Let* $T_j : X \to X$, $j = 1, 2$ *be two nonexpansive mappings such that* $\Gamma := \bigcap_{j=1}^{2} F(T_j) \cap \bigcap_{i=1}^{N} \arg\min_{y\in D} f_i(y) \neq \emptyset$. *For arbitrary* $x_1 \in X$, *let the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n, \\ x_{n+1} = \alpha_n x_n \oplus \beta_n T_1 y_n \oplus \gamma_n T_2 y_n, \end{cases} \tag{3.25}$$

*such that conditions (C1) and (C2) in Lemma 3.5 are satisfied. Then, the sequence* $\{x_n\}$ *strongly converges to an element of* $\Gamma$ *if and only if* $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$.

## 4. Numerical example

In this section, we give a numerical example to demonstrate the applicability of Algorithm (3.5).

Let $(X, d)$ be a complete $p$-uniformly convex metric space and $T_1 : X \to P(X)$ be a multivalued noexpansive mapping both defined as in Example 3.2. Similarly, let $T_2 : X \to P(X)$ be defined by $T_2 x = \{(-x_1, e^{-x_1}), (0, 0)\}$ for all $x = (x_1, x_2) \in X$. Clearly $F(T_2) = \{(0, 0)\}$. Indeed $T_2$ is also multivalued nonexpansive following the same line of argument for $T_1$ in Example 3.2. Now, define $f_i := ||.||_2^2 : X \to \mathbb{R}$ for $i = 1, 2, 3$. Then, $f_i$ are proper, convex and lower semicontinuous in $(X, d)$ for each $i = 1, 2, 3$ (see [9]).

Take $\alpha_n = \frac{3n+1}{6n+9}$, $\beta_n = \frac{2n+3}{6n+9}$, $\gamma_n = \frac{n+5}{6n+9}$ and $\lambda_{ni} = \frac{ni}{2n+1}$ for $i = 1, 2, 3$ for all $n \geq 1$. Hence, Algorithm (3.5) becomes

$$\begin{cases} h_n = \arg\min_{v\in X} \left(f_1(v) + \frac{1}{2\lambda_n^{p-1}} d(v, x_n)^p\right), \\ g_n = \arg\min_{v\in X} \left(f_2(v) + \frac{1}{2\lambda_n^{p-1}} d(v, h_n)^p\right), \\ y_n = \arg\min_{v\in X} \left(f_3(v) + \frac{1}{2\lambda_n^{p-1}} d(v, g_n)^p\right), \\ x_{n+1} = \frac{3n+1}{6n+9} x_n \oplus \frac{2n+3}{6n+9} z_{n,1} \oplus \frac{n+5}{6n+9} z_{n,2}, \; n \geq 1. \end{cases} \tag{4.1}$$

We now consider the following 4 cases for our numerical experiments below.
**Case 1:** $x_1 = (0.25,\ e^{0.25})^T$ and **Case 2:** $x_1 = (0.5,\ 1)^T$,
**Case 3:** $x_1 = (-0.25,\ e^{0.25})^T$ and **Case 4:** $x_1 = (-1, 1)^T$.

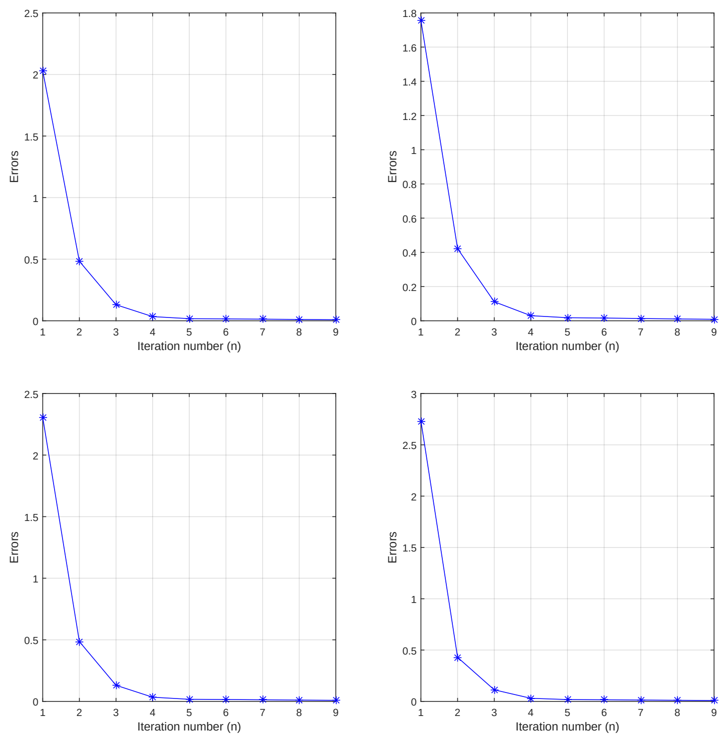


FIGURE 1. Errors vs Iteration numbers(n): **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

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