Iranian Journal of Mathematical Sciences and Informatics
Vol. 17, No. 1 (2022), pp 111-123
DOI: 10.52547/ijmsi.17.1.111

On Bernstein Type Inequalities for Complex Polynomial

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Abstract. In this paper, we establish some Bernstein type inequalities for the complex polynomial. Our results constitute generalizations and refinements of some well-known polynomial inequalities.

Keywords: Inequality, Polynomial, Derivative, Maximum modulus, Restricted zeros.

2000 Mathematics subject classification: 30A10, 30C10, 30D15.

1. Introduction and statement of results

Let p be a polynomial of degree atmost n. Then, according to a famous result known as Bernsteins inequality [8]

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1.1}$$

whereas concerning the maximum modulus of p on a large circle $|z| = R > 1$, we have [20]

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequalities (1.1) and (1.2) can be sharpened. In fact, if $p(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can respectively be replaced by

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Received 11 August 2018; Accepted 16 August 2020

$$\max_{|z|=1}|p'(z)| \le \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.3}$$

and

$$\max_{|z|=R}|p(z)| \le \frac{R^n+1}{2}\max_{|z|=1}|p(z)|,\ R>1. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [19], whereas Ankeny and Rivlin [5] used (1.3) to prove (1.4).

In the literature, there are already various generalizations and refinements of (1.3) and (1.4), for example (see Aziz [6], Bidkham et al. [9, 10, 11], Khojastehnezhad and Bidkham [17], Zireh [21], etc).

Inequalities (1.3) and (1.4) were sharpened by Dewan et.al [12, 13] proving that under the same hypothesis, for every real or complex number β with $|\beta| \le 1,\ R>1$ and $|z|=1$, we have

$$|zp'(z)+\frac{n\beta}{2}p(z)| \le \frac{n}{2}\{(|1+\frac{\beta}{2}|+|\frac{\beta}{2}|)\max_{|z|=1}|p(z)| - (|1+\frac{\beta}{2}|-|\frac{\beta}{2}|)\min_{|z|=1}|p(z)|\}, \tag{1.5}$$

and

$$|p(Rz)+\beta(\frac{R+1}{2})^n p(z)| \le \frac{1}{2}\{(|R^n+\beta(\frac{R+1}{2})^n|+|1+\beta(\frac{R+1}{2})^n|)\max_{|z|=1}|p(z)| - (|R^n+\beta(\frac{R+1}{2})^n|-|1+\beta(\frac{R+1}{2})^n|)\min_{|z|=1}|p(z)|\}. \tag{1.6}$$

Also they [12] proved if p has all its zeros in $|z| \le 1$, then for every real or complex number β with $|\beta| \le 1$, we have

$$\min_{|z|=1}|zp'(z)+\frac{n\beta}{2}p(z)| \ge n|1+\frac{\beta}{2}|\min_{|z|=1}|p(z)|. \tag{1.7}$$

In this paper, we first prove an interesting result which is a compact generalization of inequality (1.7).

Theorem 1.1. *If p is a polynomial of degree n having all its zeros in $|z| \le k,\ k>0$, then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \le 1,\ |\beta| \le 1,\ R \ge r,\ rR \ge k^2$ and $|z|=1$, we have*

$$|p(Rz)-\alpha p(rz)+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}p(rz)| \ge \frac{1}{k^n}|R^n-\alpha r^n+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}r^n|\min_{|z|=k}|p(z)|. \tag{1.8}$$

Assuming $\alpha=1$ in Theorem 1.1, we have the following result.

Corollary 1.2. *Let p be a polynomial of degree n such that does not vanish in $|z| > k,\ k > 0$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\ R > r,\ rR \geq k^2$ and $|z| = 1$, we get*

$$\begin{aligned}|p(Rz) - p(rz) + \beta\{(\frac{R+k}{r+k})^n - 1\}p(rz)| \geq \\ \frac{1}{k^n}|R^n - r^n + \beta\{(\frac{R+k}{r+k})^n - 1\}r^n| \min_{|z|=k} |p(z)|.\end{aligned} \tag{1.9}$$

By dividing the two sides of the inequality (1.9) by $(R-r)$ and letting $R \to r$, we get the following interesting result.

Corollary 1.3. *Let p be a polynomial of degree n such that does not vanish in $|z| > k,\ k > 0$. Then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\ r \geq k$ and $|z| = 1$, we get*

$$|zp'(rz) + \frac{n\beta}{r+k}p(rz)| \geq \frac{n}{k^n}|r^{n-1} + \frac{\beta}{r+k}r^n| \min_{|z|=k} |p(z)|. \tag{1.10}$$

Assuming $k = 1,\ r = 1$ in Corollary (1.3), we have the inequality (1.7). Using Theorem 1.1, we prove the following theorem, which provides a compact generalization of inequalities (1.5), (1.6).

Theorem 1.4. *Let p be a polynomial of degree n such that it does not vanish in $|z| < k,\ k > 0$. Then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq \frac{1}{k^2}$ and $|z| = 1$,*

$$\begin{aligned}&|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| \leq \\ &\frac{1}{2}\{[k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|] \max_{|z|=k} |p(z)| - \\ &[k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| - |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|] \min_{|z|=k} |p(z)|\}.\end{aligned} \tag{1.11}$$

Equality holds for the polynomials $az^n + bk^n$, $|a| = |b|$.

Assuming $\alpha = 1$ in Theorem 1.4, we have the following result.

Corollary 1.5. *Let p be a polynomial of degree n such that does not vanish in $|z| < k,\ k > 0$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1,\ R \geq r,\ rR \geq \frac{1}{k^2}$ and $|z| = 1$, we get*

$$\begin{aligned}&|p(Rk^2z) - p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}p(rk^2z)| \leq \frac{1}{2}\{ \\ &[k^n|R^n - r^n + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}r^n| + |\beta\{(\frac{Rk+1}{rk+1})^n - 1\}|] \max_{|z|=k} |p(z)| - \\ &[k^n|R^n - r^n + \beta\{(\frac{Rk+1}{rk+1})^n - 1\}r^n| - |\beta\{(\frac{Rk+1}{rk+1})^n - 1\}|] \min_{|z|=k} |p(z)|\}.\end{aligned} \tag{1.12}$$

By dividing the two sides of the inequality (1.12) by $(R-r)$ and letting $R \to r$, we get the following interesting result.

Corollary 1.6. *Let p be a polynomial of degree n such that does not vanish in $|z| < k,\ k > 0$. Then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, \quad r \geq \frac{1}{k}$ and $|z| = 1$, we have*

$$|k^2 zp'(rk^2 z) + \frac{n\beta k}{rk+1} p(rk^2 z)| \leq \frac{n}{2}\{[k^n|r^{n-1} + \frac{\beta k}{rk+1} r^n| + |\frac{\beta k}{rk+1}|] \max_{|z|=k} |p(z)| - [k^n|r^{n-1} + \frac{\beta k}{rk+1} r^n| - |\frac{\beta k}{rk+1}|] \min_{|z|=k} |p(z)|\}. \tag{1.13}$$

Remark 1.7. Assuming $k = 1$ and $r = 1$ in Corollary 1.6 we have the inequality (1.5).

2. Lemmas

To prove of these theorems, we need the following lemmas. The first lemma is due to Aziz and Zargar [7].

Lemma 2.1. *Let p be a polynomial of degree n having all its zeros in $|z| \leq k$, $k > 0$. Then for every $R \geq r$ and $rR \geq k^2$, we have*

$$|p(Rz)| \geq (\frac{R+k}{r+k})^n |p(rz)|, \quad |z| = 1. \tag{2.1}$$

Lemma 2.2. *Let p be a polynomial of degree n such that does not vanish in $|z| < k,\ k > 0$, and $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$. Then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq \frac{1}{k^2}$ and $|z| = 1$, we have*

$$|p(Rk^2 z) - \alpha p(rk^2 z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\} p(rk^2 z)| \leq k^n |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\} q(rz)|. \tag{2.2}$$

Proof. Based on the hypotheses that the polynomial p has no zeros in $|z| < k$, therefore the polynomial $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$ has all its zeros in $|z| \leq \frac{1}{k}$. Since $\frac{1}{k^n}|p(k^2 z)| = |q(z)|$ for $|z| = \frac{1}{k}$, therefore the function $\phi(z) = \frac{p(k^2 z)}{k^n q(z)}$ is analytic in the disc $|z| \geq \frac{1}{k}$ and $|\phi(z)| = 1$ on $|z| = \frac{1}{k}$. Hence based on the the maximum modulus principle $|\phi(z)| < 1$ for $|z| > \frac{1}{k}$, or equivalently

$$|p(k^2 z)| \leq k^n |q(z)|,\ |z| \geq \frac{1}{k}. \tag{2.3}$$

Since $\frac{1}{k^n}|p(k^2 z)| = |q(z)|$ for $|z| = \frac{1}{k}$, therefore for every real or complex number δ with $|\delta| < 1$ and $|z| = \frac{1}{k}$, $|\delta p(k^2 z)| < |k^n q(z)|$. Now using Rouche's theorem it follows that all the zeros of $H(z) := k^n q(z) + \delta p(k^2 z)$ lie in $|z| \leq \frac{1}{k}$. While applying Lemma 2.1, we have

$$|H(Rz)| \geq (\frac{Rk+1}{rk+1})^n |H(rz)| > |H(rz)|, \quad |z| = 1, \tag{2.4}$$

where $R > r,\ rR \geq \frac{1}{k^2}$.
It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, we get

$$|H(Rz)-\alpha H(rz)| \geq |H(Rz)|-|\alpha||H(rz)| \geq \{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|H(rz)|,\ |z|=1$$

i.e.

$$|H(Rz)-\alpha H(rz)| \geq \{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|H(rz)| \text{ for } |z|=1. \tag{2.5}$$

Since $H(Rz)$ has all its zeros in $|z| \leq \frac{1}{Rk} < 1$, and $|H(rz)| < |H(Rz)|$ for $|z| = 1$, a direct application of Rouche's theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Using Rouche's theorem again, it follows that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $R > r,\ rR \geq \frac{1}{k^2}$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}H(rz)$$

lie in $|z| < 1$.
Replacing $H(z)$ by $k^n q(z) + \delta p(k^2 z)$, we conclude that all the zeros of

$$\begin{aligned} T(z) = k^n[q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)]+ \\ \delta\{p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)\} \end{aligned} \tag{2.6}$$

lie in $|z| < 1$, for every $R > r,\ rR \geq \frac{1}{k^2},\ |\alpha| \leq 1,\ |\beta| < 1$ and $|\delta| < 1$. We now show that (2.6) implies (2.2). Indeed, suppose otherwise. Then, there is a point $z = z_0$ with $|z_0| = 1$ such that

$$\begin{aligned} |p(Rk^2z_0) - \alpha p(rk^2z_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z_0)| > \\ k^n|q(Rz_0) - \alpha q(rz_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz_0)|. \end{aligned}$$

We take

$$\delta = -\frac{k^n[q(Rz_0) - \alpha q(rz_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz_0)]}{p(Rk^2z_0) - \alpha p(rk^2z_0) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z_0)},$$

then $|\delta| < 1$ and with this choice of δ, we have, $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts that T has all its zeros in $|z| < 1$. For the case β, with $|\beta| = 1$, (2.2) follows by continuity. For $R = r$ inequality (2.2) follows by inequality (2.3). This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let p be a polynomial of degree n. Then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ rR \geq k^2,\ k > 0$ and $|z| = 1$, we have*

$$\begin{aligned} |p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)| \leq \\ \frac{1}{k^n}|R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n|\max_{|z|=k}|p(z)|. \end{aligned} \tag{2.7}$$

Proof. Let $M = \max_{|z|=k} |p(z)|$, then for δ with $|\delta| > 1$, we can conclude from Rouche's theorem that all zeros of polynomial $H(z) = p(z) - \delta M(\frac{z}{k})^n$ lie in the closed disk $|z| \leq k,\ k > 0$. Using Lemma 2.1, we have

$$|H(Rz)| \geq (\frac{R+k}{r+k})^n |H(rz)| > |H(rz)|, \quad |z| = 1, \tag{2.8}$$

where $R > r,\ rR \geq k^2$.
It follows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, we get

$$|H(Rz) - \alpha H(rz)| \geq |H(Rz)| - |\alpha||H(rz)| \geq \{(\frac{R+k}{r+k})^n - |\alpha|\}|H(rz)|,\ |z| = 1,$$

i.e.

$$|H(Rz) - \alpha H(rz)| \geq \{(\frac{R+k}{r+k})^n - |\alpha|\}|H(rz)|,\ |z| = 1. \tag{2.9}$$

Since $H(Rz)$ has all its zeros in $|z| \leq \frac{k}{R} < 1$, and $|H(rz)| < |H(Rz)|$, a direct application of Rouche's theorem shows that the polynomial $H(Rz) - \alpha H(rz)$ has all its zeros in $|z| < 1$. Using Rouche's theorem again ,implies that for every $\beta \in \mathbb{C}$ with $|\beta| < 1$ and $R > r,\ rR \geq k^2$, all the zeros of the polynomial

$$T(z) = H(Rz) - \alpha H(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}H(rz)$$

lie in $|z| < 1$.
Replacing $H(z)$ by $p(z) - \delta M(\frac{z}{k})^n$, we conclude that all the zeros of

$$\begin{aligned} T(z) = [p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)] + \\ \delta \frac{Mz^n}{k^n}\{R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n\} \end{aligned} \tag{2.10}$$

lie in $|z| < 1$, for every $R > r,\ rR \geq k^2,\ |\alpha| \leq 1,\ |\beta| < 1$ and $|\delta| > 1$. This implies

$$\begin{aligned} |p(Rz) - \alpha p(rz) + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}p(rz)| \leq \\ |R^n - \alpha r^n + \beta\{(\frac{R+k}{r+k})^n - |\alpha|\}r^n|\frac{M}{k^n}, \end{aligned} \tag{2.11}$$

where $|z| = 1$.
For β, with $|\beta| = 1$, (2.11) follows by continuity. For $R = r$ inequality (2.11) reduces to $|p(rz)| \leq \frac{r^n}{k^n} \max_{|z|=k} |p(z)|$ which it follows by taking $p(kz)$ and $|z| = \frac{r}{k}$ where $\frac{r}{k} \geq 1$ in inequality (1.2). This completes the proof of Lemma 2.3. □

Lemma 2.4. *If p is a polynomial of degree n, then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ \ rR \geq \frac{1}{k^2}$ and $|z| = 1$,*

$$|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)| +$$
$$k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq$$
$$\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\} \max_{|z|=k} |p(z)|, \tag{2.12}$$

where $q(z) = z^n\overline{p(1/\overline{z})}$.

Proof. Assume that $M = \max_{|z|=k} |p(z)|$. Then, for δ with $|\delta| > 1$, we can conclude from Rouche's theorem that the polynomial $G(z) = p(z) - \delta M$ does not vanish in $|z| < k$. If we take $H(z) = z^n\overline{G(1/\overline{z})}$, then $|G(k^2z)| = k^n|H(z)|$ for $|z| = \frac{1}{k}$. Using Lemma 2.2, for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r\ \ rR \geq \frac{1}{k^2}$ and $|z| = 1$, we have

$$\begin{aligned} &|G(Rk^2z) - \alpha G(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}G(rk^2z)| \leq \\ &k^n|H(Rz) - \alpha H(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}H(rz)|. \end{aligned} \tag{2.13}$$

Therefore, by using the equality

$$\begin{aligned} H(z) &= z^n\overline{G(\frac{1}{\overline{z}})} = z^n\overline{p(\frac{1}{\overline{z}})} - \overline{\delta}Mz^n \\ &= q(z) - \overline{\delta}Mz^n, \end{aligned}$$

we get

$$\begin{aligned} &|\{p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)\} - \\ &\delta\{1 - \alpha + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}\}M| \leq \\ &k^n|\{q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)\} - \\ &\overline{\delta}\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}M|. \end{aligned} \tag{2.14}$$

Since $\frac{1}{k^n}|p(k^2z)| = |q(z)|$ for $|z| = \frac{1}{k}$, therefore

$$\begin{aligned} \max_{|z|=\frac{1}{k}} |q(z)| &= \frac{1}{k^n} \max_{|z|=k} |p(z)|, \\ \max_{|z|=\frac{1}{k}} |q(z)| &= \frac{M}{k^n}. \end{aligned}$$

Now by applying Lemma 2.3 to $q(z)$ for $\frac{1}{k} > 0$, we have

$$|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq$$
$$\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}k^n \max_{|z|=\frac{1}{k}} |q(z)|.$$

i.e.

$$|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq$$
$$\{R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n\}M.$$

Now by suitable choice of argument of δ, we get

$$\begin{aligned}|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)\}-\\ \overline{\delta}\{R^n - \alpha r^n + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}r^n\}M| =\\ |\delta||R^n - \alpha r^n + \beta\{(\frac{R+1}{r+1})^n - |\alpha|\}r^n|M-\\ |q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)|.\end{aligned} \tag{2.15}$$

Combining right hand sides of (2.14) and (2.15) we can obtain

$$\begin{aligned}|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)|-\\ |\delta||1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|M \leq\\ |\delta|k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n|M||-\\ k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)|,\end{aligned}$$

which implies

$$\begin{aligned}&|p(Rk^2z) - \alpha p(rk^2z) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}p(rk^2z)|+\\ &k^n|q(Rz) - \alpha q(rz) + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}q(rz)| \leq\\ &|\delta|\{k^n|R^n - \alpha r^n + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}r^n| + |1 - \alpha + \beta\{(\frac{Rk+1}{rk+1})^n - |\alpha|\}|\}M.\end{aligned}$$

Making $|\delta| \to 1$, we have the result. □

Lemma 2.5. *Let p be a polynomial of degree n having no zeros in $|z| < k,\ k > 0$. Then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R \geq r,\ \ Rr \geq \frac{1}{k^2}$ and $|z| = 1$,* *we have*

$$|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\leq\frac{1}{2}$$
$$\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|+|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}\max_{|z|=k}|p(z)| \tag{2.16}$$

Proof. Since p does not vanish in $|z|<k,\ k>0$, Lemma 2.2, yields

$$\begin{aligned}|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\leq\\ k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|,\end{aligned} \tag{2.17}$$

Now by combining the inequalities (2.12) and (2.17), we have

$$\begin{aligned}&2|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\leq\\ &|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|+\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|\leq\\ &\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|+|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}\max_{|z|=k}|p(z)|.\end{aligned} \tag{2.18}$$

This gives the result. □

3. Proofs of the theorems

Proof of Theorem 1.1. If p has a zero on $|z|=k$, then inequality is trivial. Therefore, we assume that $p(z)$ has all its zeros in $|z|<k$. If $m=\min_{|z|=k}|p(z)|$, then $m>0$ and $|p(z)|\geq m$ for $|z|=k$. If $|\lambda|<1$, then it follows by Rouche's theorem that the polynomial $p(z)-\lambda m(\frac{z}{k})^n$, has all its zeros in $|z|<k,\ k>0$. Proceeding similarly as in the proof of Lemma 2.3, it follows that all the zeros of

$$\begin{aligned}p(Rz)-\alpha p(rz)+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}p(rz)+\\ \lambda m(\frac{z}{k})^n\{R^n-\alpha r^n+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}r^n\}\end{aligned} \tag{3.1}$$

lie in $|z|<1$, for every $R\geq r,\ Rr\geq k^2,\ |\alpha|\leq 1,\ |\beta|<1$ and $|\lambda|<1$. This implies

$$\begin{aligned}\frac{m}{k^n}|R^n-\alpha r^n+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}r^n|\leq\\ |p(Rz)-\alpha p(rz)+\beta\{(\frac{R+k}{r+k})^n-|\alpha|\}p(rz)|,\end{aligned} \tag{3.2}$$

where $|z|=1$. This completes the proof. $\square$

Proof of Theorem 1.4. If $p(z)$ has a zero on $|z|=k$, then $\min_{|z|=k}|p(z)|=0$ and in this case the result follows from Lemma 2.5. Hence we assume that $p(z)\neq 0$ in $|z|\leq k$. In this case we have $m=\min_{|z|=k}|p(z)|>0$ and for γ with $|\gamma|<1$, we get $|\gamma m|<m\leq|p(z)|$, where $|z|=k$. Now we conclude from Rouche's theorem that the polynomial $G(z)=p(z)-\gamma m$ does not vanish in $|z|<k$. If we take $H(z)=z^n\overline{G(1/\overline{z})}$, then by using the polynomials $G(z)$ and $H(z)$ in Lemma 2.2, we have

$$\begin{aligned}&|G(Rk^2z)-\alpha G(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}G(rk^2z)|\leq\\&k^n|H(Rz)-\alpha H(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}H(rz)|.\end{aligned}\tag{3.3}$$

Using the fact that

$$H(z)=z^n\overline{G(\frac{1}{\overline{z}})}=z^n\overline{p(\frac{1}{\overline{z}})}-\overline{\gamma}mz^n=q(z)-\overline{\gamma}mz^n,$$

or

$$H(z)=q(z)-\overline{\gamma}mz^n,$$

and substituting $G(z)$ and $H(z)$ in (3.3), we get

$$\begin{aligned}&|\{p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)\}-\\&\gamma\{1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}\}m|\leq\\&k^n|\{q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)\}-\\&\overline{\gamma}\{R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n\}mz^n|.\end{aligned}\tag{3.4}$$

Since the polynomial $q(z)=z^n\overline{p(\frac{1}{\overline{z}})}$ has all zeros in $|z|\leq\frac{1}{k}$ and $m=\min_{|z|=k}|p(z)|=k^n\min_{|z|=\frac{1}{k}}|q(z)|$, hence by applying Theorem 1.1 for the polynomial $q(z)$ with $\frac{1}{k}$, we obtain

$$\begin{aligned}&|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|m\leq\\&|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|.\end{aligned}$$

Therefore, by suitable choice of argument of γ, we get

$$\begin{aligned}|\{q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)\}-\\ \overline{\gamma}\{R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n m|=\\ |q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ |\gamma||R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|m.\end{aligned} \tag{3.5}$$

Now combining (3.4) and (3.5), we get

$$\begin{aligned}|p(Rk^2z)-\alpha p(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|-\\ |\gamma||1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|m\le\\ k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ |\gamma|k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|m.\end{aligned}$$

This implies

$$\begin{aligned}&|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\le\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ &|\gamma|\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|z^n|-|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}m.\end{aligned}$$

Letting $|\gamma|\to 1$, we have

$$\begin{aligned}&|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|\le\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|-\\ &\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|-|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}m.\end{aligned} \tag{3.6}$$

On the other hand, based on Lemma 2.4, we have

$$\begin{aligned}&|p(Rk^2z)-\alpha p(rk^2z)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}p(rk^2z)|+\\ &k^n|q(Rz)-\alpha q(rz)+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}q(rz)|\le\\ &\{k^n|R^n-\alpha r^n+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}r^n|+|1-\alpha+\beta\{(\frac{Rk+1}{rk+1})^n-|\alpha|\}|\}\max_{|z|=k}|p(z)|.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we get (1.11) and this completes the proof of Theorem 1.4. □

Acknowledgments

The authors wish to thank the referees for their comments and sugestions.

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