Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 1 (2025), pp 101-110
DOI: 10.61186/ijmsi.20.1.101

On Graded Semi J_{gr}-2-absorbing and Graded Weakly Semi J_{gr}-2-absorbing Submodules

Shatha Alghueiri, Khaldoun Al-Zoubi*

Department of Mathematics and Statistics Jordan University of Science and Technology P.O.Box 3030, Irbid 22110, Jordan

E-mail: ghweiri64@gmail.com
E-mail: kfzoubi@just.edu.jo

Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concepts of graded semi J_{gr}-2-absorbing and graded weakly semi J_{gr}-2-absorbing submodules of M and study the behavior of these notions under several constructions. A proper graded submodule N of M is said to be a graded semi J_{gr}-2-absorbing (resp. graded weakly semi J_{gr}-2-absorbing) submodule of M if whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $r_g^2 m_h \in N$ (resp. $0 \neq r_g^2 m_h \in N$), then either $r_g m_h \in N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$, where $J_{gr}(M)$ is the graded Jacobson radical.

Keywords: Graded semi J_{gr}-2-absorbing submodules, Graded weakly semi J_{gr}-2-absorbing submodules, Graded J_{gr}-2-absorbing submodules.

2000 Mathematics subject classification: 13A02, 16W50.

1. Introduction and preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary. Graded prime submodules of graded modules over graded commutative rings, have been introduced and studied by many authors, (see [1-3, 6, 9, 10, 17]). The concepts of graded 2-absorbing submodules and graded

*Corresponding Author

Received 07 April 2020; Accepted 22 February 2022

weakly 2-absorbing submodules were introduced by Al-Zoubi and Abu-Dawwas in [2] and studied in [4-5] as generalizations of graded prime submodules and graded weakly prime submodules, respectively. Recently, Al-Zoubi and Al-ghueiri, in [7] studied graded (weakly) J_{gr}-2-absorbing submodules. Here, we introduce the concept of graded (weakly) semi J_{gr}-2-absorbing submodules as a new generalization of graded (weakly) J_{gr}-2-absorbing submodules and study the behavior of these notions under several constructions.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [13-16] for these basic properties and more information on graded rings and modules. Let G be a multiplicative group with identity element e. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $r \in R$, then r can be written uniquely as $\sum_{g\in G} r_g$, where r_g is called a homogeneous component of r in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \sum_{g\in G}(I \cap R_g) := \sum_{g\in G} I_g$. Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A Left R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called a homogeneous. Note that M_g is an R_e-module for every $g \in G$. Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A submodule N of M is said to be a graded submodule of M if $N = \oplus_{g\in G}(N \cap M_g) := \oplus_{g\in G} N_g$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g := (M_g + N)/N$ for $g \in G$.

Let R be a G-graded ring and M be a graded R-module. It is shown in [12, Lemma 2.11] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.

A proper graded submodule N of M is said to be a graded maximal submodule if there is a graded submodule K of M such that $N \subseteq K \subseteq M$, then either $N = K$ or $K = M$ (see [16]).

The graded Jacobson radical of a graded module M, denoted by $J_{gr}(M)$, is defined to be the intersection of all graded maximal submodules of M (if M has no graded maximal submodule then we shall take, by definition, $J_{gr}(M) = M$) (see [16]).

2. Graded semi J_{gr}-2-absorbing submodules

Definition 2.1. Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be *a graded semi J_{gr}-2-absorbing* submodule of M if whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $r_g^2 m_h \in N$, implies either $r_g^2 \in (N + J_{gr}(M) :_R M)$ or $r_g m_h \in N + J_{gr}(M)$.

A proper graded submodule N of a graded R-module M is said to be a graded J_{gr}-2-absorbing submodule if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g s_h \in (N+J_{gr}(M) :_R M)$ or $r_g m_\lambda \in N+J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M)$ (see[7]).

It is easy to see that every graded J_{gr}-2-absorbing submodule is a graded semi J_{gr}-2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.2. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z} \times \mathbb{Z}$ and $M_1 = \{(0, 0)\}$. Now, consider the graded submodule $N = 15\mathbb{Z} \times \{0\}$ of M. Then N is not a graded J_{gr}-2-absorbing submodule of M since $0 \neq 3 \cdot 5 \cdot (1, 0) \in N$ and neither $3 \cdot (1, 0) \in N$ nor $5 \cdot (1, 0) \in N$ nor $3 \cdot 5 \in (N :_R M) = \{0\}$. However, easy computations show that N is a graded semi J_{gr}-2-absorbing submodule of M.

Recall from [8] that a proper graded submodule N of a graded R-module M is called *a graded J_{gr}-semiprime* submodule if whenever $r_g \in h(R)$, $m_h \in h(M)$ and $k \in \mathbb{Z}^+$ with $r_g^k m_h \in N$, implies $r_g m_h \in N + J_{gr}(M)$. It is easy to see that every graded J_{gr}-semiprime submodule is a graded semi J_{gr}-2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.3. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodule $N = 4\mathbb{Z}$ of M. Then N is not a graded J_{gr}-semiprime submodule of M since $2^2 \cdot 1 \in 4\mathbb{Z}$ but $2 \cdot 1 \notin 4\mathbb{Z}$. However, an easy computation shows that N is a graded semi J_{gr}-2-absorbing submodule of M.

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If $N + J_{gr}(M)$ is a graded semi J_{gr}-2-absorbing submdule of M, then N is a graded semi J_{gr}-2-absorbing submodule of M.*

Proof. Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g^2 m_h \in N$. It follows that $r_g^2 m_h \in N + J_{gr}(M)$, so we get either $r_g m_h \in N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$ as $N + J_{gr}(M)$ is a graded semi J_{gr}-2-absorbing submodule of M. Therefore, N is a graded semi J_{gr}-2-absorbing submodule of M. □

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module and N, K be two graded submodules of M with $N \subsetneq K$. If N is a graded semi J_{gr}-2-absorbing submodule of M with $J_{gr}(M) \subseteq J_{gr}(K)$, then N is a graded semi J_{gr}-2-absorbing submodule of K.*

Proof. Let $r_g \in h(R)$ and $k_h \in K \cap h(M)$ such that $r_g^2 k_h \in N$. Hence, either $r_g k_h \in N + J_{gr}(M)$ or $r_g^2 M \subseteq N + J_{gr}(M)$ as N is a graded semi J_{gr}-2-absorbing submodule of M. But $J_{gr}(M) \subseteq J_{gr}(K)$, so we get either $r_g k_h \in N + J_{gr}(K)$ or $r_g^2 K \subseteq r_g^2 M \subseteq N + J_{gr}(M) \subseteq N + J_{gr}(K)$. Therefore, N is a graded semi J_{gr}-2-absorbing submodule of K. □

The following example shows that the intersection of any two graded semi J_{gr}-2-absorbing submodules need not be a graded semi J_{gr}-2-absorbing.

Example 2.6. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 2\mathbb{Z}$ and $K = 9\mathbb{Z}$ of M. It is easy to see that N and K are graded semi J_{gr}-2-absorbing submodules of M. However, $N \cap K = 18\mathbb{Z}$ is not a graded semi J_{gr}-2-absorbing submodule of M since $3^2 \cdot 2 \in 18\mathbb{Z}$ and neither $3 \cdot 2 \in 18\mathbb{Z}$ nor $3^2 \in (18\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}) = 18\mathbb{Z}$.

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and N, K be two proper graded submodules of M such that $K \nsubseteq N$. If N is a graded semi J_{gr}-2-absorbing submodule of M and $J_{gr}(M) = J_{gr}(K)$, then $N \cap K$ is a graded semi J_{gr}-2-absorbing submodule of K.*

Proof. Since $K \nsubseteq N$, $N \cap K$ is a proper graded submodule of K. Now, let $r_g \in h(R)$ and $k_h \in K \cap h(M)$ such that $r_g^2 k_h \in N \cap K$. It follows that $r_g^2 k_h \in N$ and then either $r_g k_h \in N + J_{gr}(M)$ or $r_g^2 M \subseteq N + J_{gr}(M)$ as N is a graded semi J_{gr}-2-absorbing submodule of M. But $J_{gr}(M) = J_{gr}(K)$, so either $r_g k_h \in N + J_{gr}(K)$ or $r_g^2 M \subseteq N + J_{gr}(K)$. If $r_g k_h \in N + J_{gr}(K)$, then $r_g k_h \in (N + J_{gr}(K)) \cap K$ which yields that $r_g k_h \in N \cap K + J_{gr}(K)$ by Modular law. Also, if $r_g^2 M \subseteq N + J_{gr}(K)$, then $r_g^2 K \subseteq (N + J_{gr}(K)) \cap K$, so by Modular law we get $r_g^2 K \subseteq N \cap K + J_{gr}(K)$. Therefore, $N \cap K$ is a graded semi J_{gr}-2-absorbing submodule of K. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then the following statements are equivalent:*

(i) *$N + J_{gr}(M)$ is a graded semi J_{gr}-2-absorbing submodule of M.*
(ii) *For each $r_g \in h(R)$, either $(N + J_{gr}(M) :_R r_g m_h) = (N + J_{gr}(M) :_R r_g^2 m_h)$ for any $m_h \in h(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $N + J_{gr}(M)$ is a graded semi J_{gr}-2-absorbing submodule of M and let $r_g \in h(R)$ such that $r_g^2 \notin (N + J_{gr}(M) :_R M)$. Now, let $m_h \in h(M)$, then it is clear that $(N + J_{gr}(M) :_R r_g m_h) \subseteq (N + J_{gr}(M) :_R r_g^2 m_h)$, so let $s_\lambda \in (N + J_{gr}(M) :_R r_g^2 m_h) \cap h(R)$, then $r_g^2 s_\lambda m_h \in N + J_{gr}(M)$ which yields that $r_g s_\lambda m_h \in N + J_{gr}(M)$ as $N + J_{gr}(M)$ is a graded semi J_{gr}-2-absorbing submodule of M and $r_g^2 \notin (N + J_{gr}(M) :_R M)$. Hence, $s_\lambda \in (N + J_{gr}(M) :_R r_g m_h)$. Therefore, $(N + J_{gr}(M) :_R r_g m_h) = (N + J_{gr}(M) :_R r_g^2 m_h)$.

$(ii) \Rightarrow (i)$ Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g^2 m_h \in N + J_{gr}(M)$ and $r_g^2 \notin (N + J_{gr}(M) :_R M)$. So by (ii) we get $(N + J_{gr}(M) :_R r_g m_h) = (N + J_{gr}(M) :_R r_g^2 m_h)$. Since $r_g^2 m_h \in N + J_{gr}(M)$, $(N + J_{gr}(M) :_R r_g^2 m_h) = R$. This follows that $(N + J_{gr}(M) :_R r_g m_h) = R$, and then $r_g m_h \in N + J_{gr}(M)$. Therefore, $N + J_{gr}(M)$ is a graded semi J_{gr}-2-absorbing submodule of M. □

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module and $K = \oplus_{g \in G} K_g$ a proper graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded semi J_{gr}-2-absorbing submodule of M.*
(ii) *For each $r_g \in h(R)$ with $r_g^2 K \subseteq N$, implies either $r_g K \subseteq N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $r_g^2 K \subseteq N$ and $r_g K \nsubseteq N + J_{gr}(M)$, so there exists $k_h \in K \cap h(M)$ such that $r_g k_h \notin N + J_{gr}(M)$. But $r_g^2 k_h \in N$, this yields that $r_g^2 \in (N + J_{gr}(M) :_R M)$ as N is a graded semi J_{gr}-2-absorbing submodule of M.

$(ii) \Rightarrow (i)$ Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g^2 m_h \in N$. Let $K = \langle m_h \rangle$ be a graded submodule of M generated by m_h, hence $r_g^2 K \subseteq N$. By (ii) we have either $r_g K \subseteq N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$ and then either $r_g m_h \in N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$. Therefore, N is a graded semi J_{gr}-2-absorbing submodule of M. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $f : M \to M'$ is a homomorphism of R-modules verifying $f(M_g) \subseteq M'_g$ for every $g \in G$ (see [16]).

Recall from [6] that a proper graded submodule N of a graded R-module M is said to be a gr-small submodule of M (for short $N <<_g M$) if for every proper graded submodule L of M, we have $N + L \neq M$.

The following result studies the behavior of graded semi J_{gr}-2-absorbing submodules under graded homomorphism.

Theorem 2.10. *Let R be a G-graded ring, M and M' be two graded R-modules and $f : M \to M'$ be a graded epimorphism.*

(i) *If N is a graded semi J_{gr}-2-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded semi J_{gr}-2-absorbing submodule of M'.*
(ii) *If N' is a graded semi J_{gr}-2-absorbing submodule of M' with $ker(f) <<_g M$, then $f^{-1}(N')$ is a graded semi J_{gr}-2-absorbing submodule of M.*

Proof. (i) Suppose that N is a graded semi J_{gr}-2-absorbing submodule of M. It is easy to see that $f(N)$ is a proper graded submodule of M'. Now, let $r_g \in h(R)$ and $m'_h \in h(M')$ such that $r_g^2 m'_h \in f(N)$. Since f is a graded epimorphism, there exists $m_h \in h(M)$ such that $f(m_h) = m'_h$. Hence $r_g^2 f(m_h) = f(r_g^2 m_h) \in f(N)$, so there exists $n_{g^2h} \in N \cap h(M)$ such that $f(r_g^2 m_h) = f(n_{g^2h})$. Thus,

$r_g^2 m_h - n_{g^2h} \in ker(f) \subseteq N$, then $r_g^2 m_h \in N$. Hence either $r_g m_h \in N + J_{gr}(M)$ or $r_g^2 M \subseteq N + J_{gr}(M)$ as N is a graded semi J_{gr}-2-absorbing submodule of M. This yields that either $r_g m'_h \in f(N) + f(J_{gr}(M)) \subseteq f(N) + J_{gr}(M')$ or $r_g^2 M' \subseteq f(N) + f(J_{gr}(M)) \subseteq f(N) + J_{gr}(M')$ by [8]. Therefore, $f(N)$ is a graded semi J_{gr}-2-absorbing submodule of M'.

(ii) Suppose that N' is a graded semi J_{gr}-2-absorbing submodule of M'. It is easy to see $f^{-1}(N')$ is a proper graded submodule of M. Now, let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g^2 m_h \in f^{-1}(N')$, hence $r_g^2 f(m_h) \in N'$. Then either $r_g f(m_h) \in N' + J_{gr}(M')$ or $r_g^2 M' \subseteq N' + J_{gr}(M')$ as N' is a graded semi J_{gr}-2-absorbing submodule of M'. We want to show that $f^{-1}(f(J_{gr}(M)) \subseteq J_{gr}(M)$. Since f is a graded epimorphism and $ker(f) <<_g M$, $f(J_{gr}(M)) = J_{gr}(M')$ by [8, Theorem 2.12(ii)]. Now, by by [8, Lemma 2.11(ii)] we get $J_{gr}(M') <<_g M'$, so by [8, Lemma 2.11(iv)] we get $f^{-1}(J_{gr}(M') <<_g M$ which yields that $f^{-1}(J_{gr}(M') \subseteq J_{gr}(M)$. Thus, either $f(r_g m_h) \in N' + f(J_{gr}(M))$ or $f(r_g^2 M) \subseteq N' + f(J_{gr}(M))$ which yields that either $r_g m_h \in f^{-1}(N') + f^{-1}(f(J_{gr}(M)) \subseteq f^{-1}(N') + J_{gr}(M)$ or $r_g^2 M \in f^{-1}(N') + f^{-1}(f(J_{gr}(M)) \subseteq f^{-1}(N') + J_{gr}(M)$. Therefore, $f^{-1}(N')$ is a graded semi J_{gr}-2-absorbing submodule of M. □

The following result studies the behavior of graded semi J_{gr}-2-absorbing submodules under localization.

Theorem 2.11. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded semi J_{gr}-2-absorbing submodule of M with $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a graded semi J_{gr}-2-absorbing submodule of $S^{-1}M$.*

Proof. Since $(N :_R M) \cap S = \emptyset$, $S^{-1}N$ is a proper graded submodule of $S^{-1}M$. Let $\frac{r_{g_1}}{s_{h_1}} \in h(S^{-1}R)$ and $\frac{m_{g_2}}{s_{h_2}} \in h(S^{-1}M)$ such that $(\frac{r_{g_1}}{s_{h_1}})^2 \frac{m_{g_2}}{s_{h_2}} = \frac{r_{g_1}^2 m_{g_2}}{s_{h_1}^2 s_{h_2}} \in S^{-1}N$. Then there exists $s_{h_3} \in S$ such that $s_{h_3} r_{g_1}^2 m_{g_2} \in N$ which yields that either $s_{h_3} r_{g_1} m_{g_2} \in N + J_{gr}(M)$ or $r_{g_1}^2 \in (N + J_{gr}(M) :_R M)$. Hence, either $\frac{r_{g_1}}{s_{h_1}} \frac{m_{g_2}}{s_{h_2}} = \frac{s_{h_3} r_{g_1} m_{g_2}}{s_{h_3} s_{h_1} s_{h_2}} \in S^{-1}N + J_{gr}(S^{-1}M)$ or $(\frac{r_{g_1}}{s_{h_1}})^2 \in (S^{-1}N + J_{gr}(S^{-1}M) :_{S^{-1}R} S^{-1}M)$. Therefore, $S^{-1}N$ is a graded semi J_{gr}-2-absorbing submodule of $S^{-1}M$. □

Theorem 2.12. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules and let N_1 and N_2 be two proper graded submodules of M_1 and M_2, respectively. Let $M = M_1 \times M_2$.*

(i) *N_1 is a graded semi J_{gr}-2-absorbing submodule of M_1 if and only if $N_1 \times M_2$ is a graded semi J_{gr}-2-absorbing submodule of M.*
(ii) *N_2 is a graded semi J_{gr}-2-absorbing submodule of M_2 if and only if $M_1 \times N_2$ is a graded semi J_{gr}-2-absorbing submodule of M.*

Proof. (i) Suppose that N_1 is a graded semi J_{gr}-2-absorbing submodule of M_1. Let $r_g \in h(R)$ and $(m_{1_h}, m_{2_h}) \in h(M) = h(M_1) \times h(M_2)$ such that

$r_g^2(m_{1_h}, m_{2_h}) = (r_g^2 m_{1_h}, r^2 m_{2_h}) \in N_1 \times M_2$. It follows that $r_g^2 m_{1_h} \in N_1$, and then either $r_g m_{1_h} \in N_1 + J_{gr}(M_1)$ or $r_g^2 M_1 \subseteq N_1 + J_{gr}(M_1)$. Thus, either $r_g(m_{1_h}, m_{2_h}) = (r_g m_{1_h}, r_g m_{2_h}) \in N_1 \times M_2 + J_{gr}(M_1) \times J_{gr}(M_2) = N_1 \times M_2 + J_{gr}(M)$ or $r_g(M_1 \times M_2) \subseteq N_1 \times M_2 + J_{gr}(M)$. Therefore, $N_1 \times M_2$ is a graded semi J_{gr}-2-absorbing submodule of M. Conversely, suppose that $N_1 \times M_2$ is a graded semi J_{gr}-2-absorbing submodule of M. Let $r_g \in h(R)$ and $m_h \in h(M_1)$ such that $r_g^2 m_h \in N_1$. Then $r_g^2(m_h, 0) \in N_1 \times M_2$, hence either $r_g(m_h, 0) \in N_1 \times M_2 + J_{gr}(M)$ or $r_g^2 M \subseteq N_1 \times M_2 + J_{gr}(M)$. This yields that either $r_g m_h \in N_1 + J_{gr}(M_1)$ or $r_g^2 M_1 \subseteq N_1 + J_{gr}(M_1)$. Therefore, N_1 is a graded semi J_{gr}-2-absorbing submodule of M_1.

(*ii*) The proof similar to that in (i). □

Theorem 2.13. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules and let N_1 and N_2 be two graded semi J_{gr}-2-absorbing submodules of M_1 and M_2, respectively. Let $M = M_1 \times M_2$. If $(N_1 + J_{gr}(M_1) :_R M_1) = (N_2 + J_{gr}(M_2) :_R M_2)$, then $N_1 \times N_2$ is a graded semi J_{gr}-2-absorbing submodule of M.*

Proof. Let $r_g \in h(R)$ and $(m_{1_h}, m_{2_h}) \in h(M)$ such that $r_g^2(m_{1_h}, m_{2_h}) = (r_g^2 m_{1_h}, r_g^2 m_{2_h}) \in N_1 \times N_2$. It follows that $r_g^2 m_{1_h} \in N_1$ and $r_g^2 m_{2_h} \in N_2$, hence, we get either $r_g m_{1_h} \in N_1 + J_{gr}(M_1)$ or $r_g^2 \in (N_1 + J_{gr}(M_1) :_R M_1)$ and either $r_g m_{2_h} \in N_2 + J_{gr}(M_2)$ or $r_g^2 \in (N_2 + J_{gr}(M_2) :_R M_2)$. But $(N_1 + J_{gr}(M_1) :_R M_1) = (N_2 + J_{gr}(M_2) :_R M_2)$, so either $r_g^2 \in (N_1 + J_{gr}(M_1) :_R M_1)$ or $r_g m_{1_h} \in N_1 + J_{gr}(M_1)$ and $r_g m_{2_h} \in N_2 + J_{gr}(M_2)$. If $r^2 \in (N_1 + J_{gr}(M_1) :_R M_1)$, then $r^2 \in (N_1 \times N_2 + J_{gr}(M) :_R M)$. Also, if $r_g m_{1_h} \in N_1 + J_{gr}(M_1)$ and $r_g m_{2_h} \in N_2 + J_{gr}(M_2)$, then $r_g(m_{1_h}, m_{2_h}) \in N_1 \times N_2 + J_{gr}(M)$. Therefore, $N_1 \times N_2$ is a graded semi J_{gr}-2-absorbing submodule of M. □

Recall from [16] that a graded R-module M is said to be a graded cyclic module, if there exists $m_g \in h(M)$ such that $M = Rm_g$.

Theorem 2.14. *Let R be a G-graded ring, M a graded cyclic R-module and N a graded submodule of M. If $(N :_R M)$ is a graded semi 2-absorbing ideal of R, then N is a graded semi J_{gr}-2-absorbing submodule of M.*

Proof. Let $r_{g_1} \in h(R)$ and $m_{h_1} \in h(M)$ such that $r_{g_1}^2 m_{h_1} \in N$ and $r_{g_1}^2 \notin (N :_R M)$. Since M is a graded cyclic module, there exists $m_{h_2} \in h(M)$ such that $M = Rm_{h_2}$. So there exists $s_{g_2} \in h(R)$ such that $m_{h_1} = s_{g_2} m_{h_2}$, hence $r_{g_1}^2 s_{g_2} m_{h_2} \in N$ and then $r_{g_1}^2 s_{g_2} \in (N :_R Rm_{h_2})$. This yields that $r_{g_1} s_{g_2} \in (N :_R Rm_{h_2})$ as $(N :_R Rm_{h_2})$ is a graded semi 2-absorbing ideal of R, so we get $r_{g_1} m_{h_1} = r_{g_1} s_{g_2} m_{h_2} \in N \subseteq N + J_{gr}(M)$. Therefore, N is a graded semi J_{gr}-2-absorbing submodule of M. □

3. Graded weakly semi J_{gr}-2-absorbing submodules

Definition 3.1. Let R be a G-graded ring, M a graded R-module. A proper graded submodule N of M is said to be *a graded weakly semi J_{gr}-2-absorbing submodule of* M, if whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $0 \neq r_g^2 m_h \in N$, then either $r_g^2 \in (N + J_{gr}(M) :_R M)$ or $r_g m_h \in N + J_{gr}(M)$.

Definition 3.2. Let R be a G-graded ring. A proper graded ideal I of R is said to be a *graded semi J_{gr}-2-absorbing ideal of* R if I is a graded semi J_{gr}-2-absorbing submodule of R-module R.

A proper graded submodule N of a graded R-module M is said to be a graded weakly J_{gr}-2-absorbing submodule if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N + J_{gr}(M)$ or $s_h m_\lambda \in N + J_{gr}(M)$ or $r_g s_h \in (N + J_{gr}(M) :_R M)$ (see [7]).

It is easy to see that every graded weakly J_{gr}-2-absorbing submodule is a graded weakly semi J_{gr}-2-absorbing submodule. We can see from Example 2.2 that the converse is not true in general.

Recall from [16] that a graded ring R is called graded regular ring if for each $r_g \in h(R)$, there exists $s_h \in h(R)$ such that $r_g s_h r_g = r_g$. Also, a graded R-module M is said to be a graded regular module if for each $m_g \in h(M)$, $R/Ann(m_g)$ is a graded regular ring.

Theorem 3.3. *Let R be a G-graded ring, M a graded regular R-module and N a graded submodule of M. If N is a graded weakly semi J_{gr}-2-absorbing submodule of M, then N is a graded weakly semi 2-absorbing submodule of M.*

Proof. Let $r_{g_1} \in h(R)$ and $m_h \in h(M)$ such that $0 \neq r_{g_1}^2 m_h \in N$. Now, there exists $s_{g_2} \in h(R)$ such that $r_{g_1}^2 s_{g_2} + Ann(m_h) = r_{g_1} + Ann(m_h)$ as M is a graded regular module. Hence $(r_{g_1}^2 s_{g_2} - r_{g_1}) m_h = 0$, and then $r_{g_1} m_h = r_{g_1}^2 s_{g_2} m_h \in N$. Therefore, N is a graded weakly semi-2-absorbing submodule of M. □

Theorem 3.4. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If $N + J_{gr}(M)$ is a graded weakly semi J_{gr}-2-absorbing submodule of M, then N is a graded weakly semi J_{gr}-2-absorbing submodule of M.*

Proof. Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $0 \neq r_g^2 m_h \in N \subseteq N + J_{gr}(M)$. Hence, either $r_g m_h \in N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$ as $N + J_{gr}(M)$ is a graded weakly semi J_{gr}-2-absorbing submodule of M. □

Theorem 3.5. *Let R be a G-graded ring, M a graded R -module and N, K be two graded submodules of M such that $N \subsetneq K$. If N is a graded weakly semi J_{gr}-2-absorbing submodule of M and $J_{gr}(M) \subseteq J_{gr}(K)$, then N is a graded weakly semi J_{gr}-2-absorbing submodule of K.*

Proof. Let $r_g \in h(R)$ and $k_h \in K \cap h(M)$ such that $0 \neq r_g^2 k_h \in N$, then either $r_g k_h \in N + J_{gr}(M)$ or $r_g^2 \in (N + J_{gr}(M) :_R M)$ as N is a graded weakly semi J_{gr}-2-absorbing submodule of M. But $J_{gr}(M) \subseteq J_{gr}(K)$, so we get either $r_g k_h \in N + J_{gr}(K)$ or $r_g^2 \in (N + J_{gr}(K) :_R M) \subseteq (N + J_{gr}(K) :_R K)$. Therefore, N is a graded weakly semi J_{gr}-2-absorbing submodule of K. □

The following example shows that the intersection of graded weakly semi J_{gr}-2-absorbing submodules is not necessarily a graded weakly semi J_{gr}-2-absorbing submodule.

Example 3.6. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 2\mathbb{Z}$ and $K = 25\mathbb{Z}$ of M. N and K are graded semi weakly J_{gr}-2-absorbing submodules of$M$. But $N \cap K = 50\mathbb{Z}$ is not a graded weakly semi J_{gr}-2-absorbing submodule since $0 \neq 5^2 \cdot 2 = 50 \in 50\mathbb{Z}$ and neither $5 \cdot 2 = 10 \in 50\mathbb{Z}$ nor $5^2 = 25 \in 50\mathbb{Z}$.

Theorem 3.7. *Let R be a G-graded ring, M a graded cyclic R-module and N a proper graded submodule of M. If $(N :_R M)$ is a graded weakly semi J_{gr}-2-absorbing ideal of R with $J_{gr}(R) \subseteq (N :_R M)$, then N is a graded weakly semi J_{gr}-2-absorbing submodule of M.*

Proof. Let $r_{g_1} \in h(R)$ and $m_{h_1} \in h(M)$ such that $r_{g_1}^2 m_{h_1} \in N$ and $r_{g_1}^2 \notin (N :_R M)$. Now, since M is a graded cyclic module, $M = Rm_{h_2}$ for some $m_{h_2} \in h(M)$. So there exists $s_{g_2} \in h(R)$ such that $m_{h_1} = s_{g_2} m_{h_2}$. Hence $r_{g_1}^2 s_{g_2} \in (N :_R m_{h_2}) = (N :_R M)$, then $r_{g_1} s_{g_2} \in (N :_R m_{h_2})$ as $(N :_R M)$ is a graded weakly semi J_{gr}-2-absorbing ideal of R and $J_{gr}(R) \subseteq (N :_R M)$, which yields that $r_{g_1} m_{h_1} = r_{g_1} s_{g_2} m_{h_2} \in N \subseteq N + J_{gr}(M)$. Therefore, N is a graded weakly semi J_{gr}-2-absorbing submodule of M. □

Theorem 3.8. *Let R be a G-graded ring, M, M' be two graded R-modules and $f : M \to M'$ be a graded homomorphism.*

(i) *If f is a graded epimorphism and N is a graded weakly semi J_{gr}-2-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded weakly semi J_{gr}-2-absorbing submodule of M'.*
(ii) *If $f : M \to M'$ is a graded isomorphism and N' is a graded weakly semi J_{gr}-2-absorbing submodule of M' with $ker(f) <<_g M$, then $f^{-1}(N')$ is a graded weakly semi J_{gr}-2-absorbing submodule of M.*

Proof. The proof is similar to that of Theorem 2.10 . □

Theorem 3.9. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded weakly semi J_{gr}-2-absorbing submodule of M, then $S^{-1}N$ is a graded weakly semi J_{gr}-2-absorbing submodule of $S^{-1}M$.*

Proof. The proof is similar to that of Theorem 2.11. □

Acknowledgments

The authors wish to thank sincerely the referees for their valuable comments and suggestions.

References

1. K. Al-Zoubi, Some Properties of Graded 2-prime Submodules, *Asian-Eur. J. Math.*, **8**(2), (2015), 1550016-1– 1550016-5.
2. K. Al-Zoubi, R. Abu-Dawwas, On Graded 2-absorbing and Weakly Graded 2-absorbing Submodules, *J. Math. Sci. Adv. Appl.*, **28**, (2014), 45-60.
3. K. Al-Zoubi, R. Abu-Dawwas, On Graded Quasi-prime Submodules, *Kyungpook Math. J.*, **55**(2), (2015), 259-266.
4. K. Al-Zoubi, I. Al-Ayyoub, M. Al-Dolat, On Graded 2-absorbing Compactly Packed Modules, *Adv. Stud. Contemp. Math. (Kyungshang)*, **28**(3), (2018), 479-486.
5. K. Al-Zoubi, M. Al-Azaizeh, Some Properties of Graded 2-absorbing and Graded Weakly 2-absorbing Submodules, *J. Nonlinear Sci. Appl.*, **12**(8), (2019), 503–508.
6. K. Al-Zoubi, A. Al-Qderat, Some Properties of Graded Comultiplication Modules, *Open Mathematics,* **15**, (2017), 187–192.
7. K. Al-Zoubi, S. Alghueiri, On Graded J_{gr}-2-absorbing and Graded Weakly J_{gr}-2-absorbing Submodules of Graded Modules Over Graded Commutative Rings, *Int. J. Math. Comput. Sci.*, **16**(4), (2021), 1169–1178.
8. K. Al-Zoubi, S. Alghueiri, On Graded J_{gr}-semiprime Submodules, *Italian Journal Pure and Applied Mathematics,* **46**, (2021), 361-369.
9. K. Al-Zoubi, M. Jaradat, R. Abu-Dawwas, On Graded Classical Prime and Graded Prime Submodules, *Bull. Iranian Math. Soc.,* **41**(1), (2015), 217-225.
10. R. Abu-Dawwas, K. Al-Zoubi, On Graded Weakly Classical Prime Submodules, *Iran. J. Math. Sci. Inform.* **12**(1), (2017), 153-161.
11. R. Abu-Dawwas, M. Bataineh, Graded r-Ideals, *Iran. J. Math. Sci. Inform.* **14**(2), (2019), 1-8.
12. S. E. Atani, On Graded Prime Submodules, *Chiang Mai J. Sci.*, **33**(1), (2006), 3-7.
13. R. Hazrat, *Graded Rings and Graded Grothendieck Groups*, CambridgeUniversity Press, Cambridge, 2016.
14. C. Nastasescu, F. Van Oystaeyen, *Graded and Filtered Rings and Modules*, Lecture notes in mathematics 758, Berlin-New York: Springer-Verlag, 1982.
15. C. Nastasescu, F. Van Oystaeyen, *Graded Ring Theory, Mathematical Library 28*, North Holand, Amsterdam, 1982.
16. C. Nastasescu, F. Van Oystaeyen, *Methods of Graded Rings*, LNM 1836. Berlin-Heidelberg: Springer-Verlag, 2004.
17. K. H. Oral, U. Tekir, A. G. Agargun, On Graded Prime and Primary Submodules, *Turk. J. Math.*, **35**, (2011), 159–167.