Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 1 (2025), pp 43-51
DOI: 10.61186/ijmsi.20.1.43

Mappings Preserving Sum of Products $ab + b \circ a^*$ on Factor Von Neumann Algebras

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. In this paper, we proved that a bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(ab+b\circ a^*) = \Phi(a)\Phi(b)+\Phi(b)\circ\Phi(a)^*$ (where $\circ$ is the special Jordan product on $\mathcal{A}$ and $\mathcal{B}$), for all elements $a, b \in \mathcal{A}$, if and only if Φ is a $*$-ring isomorphism. In particular, if the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are type I factors, then Φ is a unitary isomorphism or a conjugate unitary isomorphism.

Keywords: $*$-ring isomorphisms, Factor von Neumann algebras.

2010 Mathematics subject classification: 47B48, 46L10.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{S}$ be rings. We say that a mapping $\Phi : \mathcal{R} \to \mathcal{S}$ *preserves product* if $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathcal{R}$, and that *preserves Jordan product* if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$, for all elements $a, b \in \mathcal{R}$. We say that a mapping $\Phi : \mathcal{R} \to \mathcal{S}$ is *additive* if $\Phi(a + b) = \Phi(a) + \Phi(b)$, for all elements $a, b \in \mathcal{R}$ and that is a *ring isomorphism* if Φ is an additive bijection that preserves products. The question of when a mapping preserving product or Jordan product is additive has received a fair amount of attentions. The results

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Received 23 June 2020; Accepted 03 April 2021

have shown that the structures of these products are enough to determine the ring or algebraic structure (for example, see [1], [4], [5] and [6]).

Let $\mathcal{R}$ and $\mathcal{S}$ be $*$-rings. We say that a mapping $\Phi : \mathcal{R} \to \mathcal{S}$ *preserves involution* if $\Phi(a^*) = \Phi(a)^*$, for all elements $a \in \mathcal{R}$, and that Φ is a $*$*-ring isomorphism* if Φ is a ring isomorphism that preserves involution. As in the case of rings, a natural problem on $*$-rings is to know when mappings defined on them preserving some new type of product are $*$-ring isomorphisms. Recently, many mathematicians devoted themselves to study mappings preserving new products on rings with involution (for example, see [2] and [3]). In particular, Li et al. [2] studied the bijective mappings preserving the new product $ab+ba^*$, where a, b are elements of $\mathcal{R}$. They showed that such mappings on factor von Neumann algebras are $*$-ring isomorphisms.

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively, and $\mathcal{A}$ and $\mathcal{B}$ be two algebras over $\mathbb{C}$. If $a, b \in \mathcal{A}$, let $a \circ b = \frac{1}{2}(ab + ba)$. We call $\circ$ the *special Jordan product* on $\mathcal{A}$. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ preserves *special Jordan product* if $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$, for all elements $a, b \in \mathcal{A}$.

A *von Neumann algebra* $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$, which are denote in this paper by lowercase Latin letters, containing the identity operator $1_{\mathcal{A}}$, called *elements of* $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called *factor* if its center contains multiples of the identity operator only. It is well known that the factor von Neumann algebra $\mathcal{A}$ is prime, that is, if a and b are elements of $\mathcal{A}$ such that $a\mathcal{A}b = 0$, then either $a = 0$ or $b = 0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. We say that a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves sum of products* $ab + b \circ a^*$ if

$$\Phi(ab + b \circ a^*) = \Phi(a)\Phi(b) + \Phi(b) \circ \Phi(a)^*, \tag{1.1}$$

for all elements $a, b \in \mathcal{A}$.

Similarly to the research performed in [2], the aim of the present paper is to investigate when a bijective mapping preserving sum of products $ab + b \circ a^*$ on factor von Neumann algebras is a $*$-ring isomorphism. Our main result reads as follows.

Main Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ the identities of them, respectively. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(ab + b \circ a^*) = \Phi(a)\Phi(b) + \Phi(b) \circ \Phi(a)^*$, for all elements $a, b \in \mathcal{A}$, if and only if Φ is a $*$-ring isomorphism.*

2. The proof of main theorem

The proof of the Main Theorem is made by proving several lemmas. We begin with the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ a mapping that preserves sum of products $ab + b \circ a^*$. If $\Phi(c) = \Phi(a) + \Phi(b)$,* *for elements a, b, c of $\mathcal{A}$, then hold the following identities: (i) $\Phi(tc + c \circ t^*) = \Phi(ta + a \circ t^*) + \Phi(tb + b \circ t^*)$ and (ii) $\Phi(ct + t \circ c^*) = \Phi(at + t \circ a^*) + \Phi(bt + t \circ b^*)$, for all elements $t \in \mathcal{A}$.*

Proof. For any element $t \in \mathcal{A}$ we have

$$\begin{aligned}
\Phi(tc + c \circ t^*) &= \Phi(t)\Phi(c) + \Phi(c) \circ \Phi(t)^* \\
&= \Phi(t)(\Phi(a) + \Phi(b)) + (\Phi(a) + \Phi(b)) \circ \Phi(t)^* \\
&= \Phi(t)\Phi(a) + \Phi(a) \circ \Phi(t)^* + \Phi(t)\Phi(b) + \Phi(b) \circ \Phi(t)^* \\
&= \Phi(ta + a \circ t^*) + \Phi(tb + b \circ t^*).
\end{aligned}$$

Similarly, we prove (ii). □

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $1_{\mathcal{A}}$ the identity of $\mathcal{A}$. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ that preserves sum of products $ab + b \circ a^*$ is additive.*

We shall organize the proof of Lemma 2.2 in a series of properties, based on the techniques used by Li et al. [2]. We begin with the following property.

Property 2.3. $\Phi(0) = 0$.

Proof. From the surjectivity of Φ there exists an element $a \in \mathcal{A}$ such that $\Phi(a) = 0$. Thus,

$$\Phi(0) = \Phi(0a + a \circ 0^*) = \Phi(0)\Phi(a) + \Phi(a) \circ \Phi(0)^* = \Phi(0)0 + 0 \circ \Phi(0)^* = 0.$$

□

Let p_1 be a non-trivial projection of $\mathcal{A}$ and write $p_2 = 1_{\mathcal{A}} - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function*.

Property 2.4. *For any elements $a_{11} \in \mathcal{A}_{11}$, $a_{22} \in \mathcal{A}_{22}$, $b_{12} \in \mathcal{A}_{12}$ and $b_{21} \in \mathcal{A}_{21}$ hold: (i) $\Phi(a_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$, (ii) $\Phi(a_{11} + b_{21}) = \Phi(a_{11}) + \Phi(b_{21})$, (iii) $\Phi(a_{22} + b_{12}) = \Phi(a_{22}) + \Phi(b_{12})$ and (iv) $\Phi(a_{22} + b_{21}) = \Phi(a_{22}) + \Phi(b_{21})$.*

Proof. According to the hypothesis on Φ choose $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ such that $\Phi(c) = \Phi(a_{11}) + \Phi(b_{12})$. Hence, by Lemma 2.1(i) we have

$$\Phi(p_2 c + c \circ p_2^*) = \Phi(p_2 a_{11} + a_{11} \circ p_2^*) + \Phi(p_2 b_{12} + b_{12} \circ p_2^*) = \Phi(\frac{1}{2} b_{12}).$$

This shows that $p_2 c + c \circ p_2^* = \frac{1}{2} b_{12}$ which implies that $c_{12} = b_{12}$ and $c_{21} = c_{22} = 0$. Therefore, $\Phi(c_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$. It follows that, for any element $d_{12} \in \mathcal{A}_{12}$ we have

$$\Phi(d_{12}(c_{11} + b_{12}) + (c_{11} + b_{12}) \circ d_{12}^*) = \Phi(d_{12} a_{11} + a_{11} \circ d_{12}^*)$$

$$+\Phi(d_{12}b_{12}+b_{12}\circ d_{12}^*)$$

which leads to

$$\Phi(\frac{1}{2}d_{12}^*c_{11}+b_{12}\circ d_{12}^*)=\Phi(\frac{1}{2}d_{12}^*a_{11})+\Phi(b_{12}\circ d_{12}^*).$$

This implies that

$$\begin{aligned}&\Phi(p_1(\frac{1}{2}d_{12}^*c_{11}+b_{12}\circ d_{12}^*)+(\frac{1}{2}d_{12}^*c_{11}+b_{12}\circ d_{12}^*)\circ p_1^*)\\&\quad=\Phi(p_1(\frac{1}{2}d_{12}^*a_{11})+(\frac{1}{2}d_{12}^*a_{11})\circ p_1^*)+\Phi(p_1(b_{12}\circ d_{12}^*)+(b_{12}\circ d_{12}^*)\circ p_1^*)\end{aligned}$$

which yields that $\Phi(b_{12}d_{12}^*+\frac{1}{2^2}d_{12}^*c_{11})=\Phi(\frac{1}{2^2}d_{12}^*a_{11})+\Phi(b_{12}d_{12}^*)$. As consequence, we obtain

$$\begin{aligned}&\Phi(p_2(b_{12}d_{12}^*+\frac{1}{2^2}d_{12}^*c_{11})+(b_{12}d_{12}^*+\frac{1}{2^2}d_{12}^*c_{11})\circ p_2^*)\\&\quad=\Phi(p_2(\frac{1}{2^2}d_{12}^*a_{11})+(\frac{1}{2^2}d_{12}^*a_{11})\circ p_2^*)+\Phi(p_2(b_{12}d_{12}^*)+(b_{12}d_{12}^*)\circ p_2^*).\end{aligned}$$

This leads to $\Phi(\frac{3}{2^3}d_{12}^*c_{11})=\Phi(\frac{3}{2^3}d_{12}^*a_{11})$ which results that $c_{11}=a_{11}$.

Similarly, we prove (ii), (iii) and (iv). □

Property 2.5. *For any elements $a_{12}\in\mathcal{A}_{12}$ and $b_{21}\in\mathcal{A}_{21}$ holds $\Phi(a_{12}+b_{21})=\Phi(a_{12})+\Phi(b_{21})$.*

Proof. According to the hypothesis on Φ choose $c=c_{11}+c_{12}+c_{21}+c_{22}\in\mathcal{A}$ such that $\Phi(c)=\Phi(a_{12})+\Phi(b_{21})$. Hence, for any element $d_{12}\in\mathcal{A}_{12}$ we have

$$\begin{aligned}\Phi(d_{12}c+c\circ d_{12}^*)&=\Phi(d_{12}a_{12}+a_{12}\circ d_{12}^*)+\Phi(d_{12}b_{21}+b_{21}\circ d_{12}^*)\\&=\Phi(a_{12}\circ d_{12}^*)+\Phi(d_{12}b_{21})\end{aligned}$$

which yields that

$$\begin{aligned}&\Phi(p_2(d_{12}c+c\circ d_{12}^*)+(d_{12}c+c\circ d_{12}^*)\circ p_2^*)\\&\quad=\Phi(p_2(a_{12}\circ d_{12}^*)+(a_{12}\circ d_{12}^*)\circ p_2^*)+\Phi(p_2(d_{12}b_{21})+(d_{12}b_{21})\circ p_2^*)\\&\quad=\Phi(d_{12}^*a_{12}).\end{aligned}$$

It follows that $p_2(d_{12}c+c\circ d_{12}^*)+(d_{12}c+c\circ d_{12}^*)\circ p_2^*=d_{12}^*a_{12}$ which implies that $c_{11}=0$, $c_{12}=a_{12}$ and $c_{22}=0$. Thus $\Phi(a_{12}+c_{21})=\Phi(a_{12})+\Phi(b_{21})$. As result, we obtain

$$\begin{aligned}&\Phi(d_{21}(a_{12}+c_{21})+(a_{12}+c_{21})\circ d_{21}^*)\\&\quad=\Phi(d_{21}a_{12}+a_{12}\circ d_{21}^*)+\Phi(d_{21}b_{21}+b_{21}\circ d_{21}^*)\\&\quad=\Phi(d_{21}a_{12})+\Phi(b_{21}\circ d_{21}^*)\end{aligned}$$

which leads to

$$\Phi(p_1(d_{21}a_{12}+c_{21}\circ d_{21}^*)+(d_{21}a_{12}+c_{21}\circ d_{21}^*)\circ p_1^*)$$

$$= \Phi(p_1(d_{21}a_{12}) + (d_{21}a_{12}) \circ p_1^*) + \Phi(p_1(b_{21} \circ d_{21}^*) + (b_{21} \circ d_{21}^*) \circ p_1^*)$$
$$= \Phi(d_{21}^* b_{21}).$$

Therefore, $\Phi(d_{21}^* c_{21}) = \Phi(d_{21}^* b_{21})$ which results that $c_{21} = b_{21}$. $\square$

Property 2.6. *For any elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$ and (ii) $\Phi(b_{12} + c_{21} + d_{22}) = \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.*

Proof. Choose $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$. Hence, by Property 2.5 we have

$$\Phi(p_2 f + f \circ p_2^*) = \Phi(p_2 a_{11} + a_{11} \circ p_2^*) + \Phi(p_2 b_{12} + b_{12} \circ p_2^*) + \Phi(p_2 c_{21} + c_{21} \circ p_2^*)$$
$$= \Phi(\frac{1}{2} b_{12}) + \Phi(\frac{3}{2} c_{21}) = \Phi(\frac{1}{2} b_{12} + \frac{3}{2} c_{21})$$

which implies that $p_2 f + f \circ p_2^* = \frac{1}{2} b_{12} + \frac{3}{2} c_{21}$. This shows that $f_{12} = b_{12}$, $f_{21} = c_{21}$ and $f_{22} = 0$. Therefore

$$\Phi(f_{11} + b_{12} + c_{21}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}).$$

It follows that, for any element $d_{12} \in \mathcal{A}_{12}$ we have

$$\Phi(d_{12}(f_{11} + b_{12} + c_{21}) + (f_{11} + b_{12} + c_{21}) \circ d_{12}^*)$$
$$= \Phi(d_{12} a_{11} + a_{11} \circ d_{12}^*) + \Phi(d_{12} b_{12} + b_{12} \circ d_{12}^*) + \Phi(d_{12} c_{21} + c_{21} \circ d_{12}^*)$$

which results that

$$\Phi(d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) = \Phi(\frac{1}{2} d_{12}^* a_{11}) + \Phi(b_{12} \circ d_{12}^*) + \Phi(d_{12} c_{21}).$$

As a consequence, we have

$$\Phi(p_2(d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) + (d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) \circ p_2^*)$$
$$= \Phi(p_2(\frac{1}{2} d_{12}^* a_{11}) + (\frac{1}{2} d_{12}^* a_{11}) \circ p_2^*) + \Phi(p_2(b_{12} \circ d_{12}^*) + (b_{12} \circ d_{12}^*) \circ p_2^*)$$
$$+ \Phi(p_2(d_{12} c_{21}) + (d_{12} c_{21}) \circ p_2^*)$$

which yields that

$$\Phi(p_2(d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) + (d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) \circ p_2^*)$$
$$= \Phi(\frac{3}{2^2} d_{12}^* a_{11}) + \Phi(d_{12}^* b_{12}) = \Phi(\frac{3}{2^2} d_{12}^* a_{11} + d_{12}^* b_{12}).$$

Thus

$$p_2(d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) + (d_{12} c_{21} + \frac{1}{2} d_{12}^* f_{11} + b_{12} \circ d_{12}^*) \circ p_2^*$$
$$= \frac{3}{2^2} d_{12}^* a_{11} + d_{12}^* b_{12}$$

which leads to $f_{11} = a_{11}$.

Similarly, we prove (ii). □

Property 2.7. *For any elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$ holds $\Phi(a_{11}+b_{12}+c_{21}+d_{22}) = \Phi(a_{11})+\Phi(b_{12})+\Phi(c_{21})+\Phi(d_{22})$.*

Proof. Choose $f = f_{11}+f_{12}+f_{21}+f_{22} \in \mathcal{A}$ such that $\Phi(f) = \Phi(a_{11})+\Phi(b_{12})+\Phi(c_{21})+\Phi(d_{22})$. By Property 2.6(i), we have

$$\Phi(p_1 f + f\circ p_1^*) = \Phi(p_1a_{11}+a_{11}\circ p_1^*)+\Phi(p_1b_{12}+b_{12}\circ p_1^*)+\Phi(p_1c_{21}+c_{21}\circ p_1^*)$$
$$+\Phi(p_1d_{22}+d_{22}\circ p_1^*) = \Phi(2a_{11})+\Phi(\frac{3}{2}b_{12})+\Phi(\frac{1}{2}c_{21}) = \Phi(2a_{11}+\frac{3}{2}b_{12}+\frac{1}{2}c_{21})$$

which implies that $p_1f+f\circ p_1^* = 2a_{11}+\frac{3}{2}b_{12}+\frac{1}{2}c_{21}$. This shows that $f_{11}=a_{11}$, $f_{12}=b_{12}$ and $f_{21}=c_{21}$. Hence

$$\Phi(p_2 f + f\circ p_2^*) = \Phi(p_2a_{11}+a_{11}\circ p_2^*)+\Phi(p_2b_{12}+b_{12}\circ p_2^*)+\Phi(p_2c_{21}+c_{21}\circ p_2^*)$$
$$+\Phi(p_2d_{22}+d_{22}\circ p_2^*) = \Phi(\frac{1}{2}b_{12})+\Phi(\frac{3}{2}c_{21})+\Phi(2d_{22}) = \Phi(\frac{1}{2}b_{12}+\frac{3}{2}c_{21}+2d_{22})$$

which yields that $p_2\circ f+fp_2^* = \frac{1}{2}b_{12}+\frac{3}{2}c_{21}+2d_{22}$. As consequence, we obtain $f_{22}=d_{22}$. □

Property 2.8. *For any elements $a_{12}, b_{12} \in \mathcal{A}_{12}$ and $a_{21}, b_{21} \in \mathcal{A}_{21}$ hold: (i) $\Phi(a_{12}+b_{12}) = \Phi(a_{12})+\Phi(b_{12})$ and (ii) $\Phi(a_{21}+b_{21}) = \Phi(a_{21})+\Phi(b_{21})$.*

Proof. First, we note that the following holds

$$(p_1+a_{12})(p_2+b_{12})+(p_2+b_{12})\circ(p_1+a_{12})^*$$
$$= a_{12}+\frac{3}{2}b_{12}+\frac{1}{2}a_{12}^*+\frac{1}{2}b_{12}a_{12}^*+\frac{1}{2}a_{12}^*b_{12}.$$

Hence, by Property 2.7 we have

$$\begin{aligned}
& \Phi(a_{12}+\frac{3}{2}b_{12})+\Phi(\frac{1}{2}a_{12}^*)+\Phi(\frac{1}{2}b_{12}a_{12}^*)+\Phi(\frac{1}{2}a_{12}^*b_{12}) \\
= \; & \Phi(a_{12}+\frac{3}{2}b_{12}+\frac{1}{2}a_{12}^*+\frac{1}{2}b_{12}a_{12}^*+\frac{1}{2}a_{12}^*b_{12}) \\
= \; & \Phi((p_1+a_{12})(p_2+b_{12})+(p_2+b_{12})\circ(p_1+a_{12})^*) \\
= \; & \Phi(p_1+a_{12})\Phi(p_2+b_{12})+\Phi(p_2+b_{12})\circ\Phi(p_1+a_{12})^* \\
= \; & \big(\Phi(p_1)+\Phi(a_{12})\big)\big(\Phi(p_2)+\Phi(b_{12})\big) \\
& +\big(\Phi(p_2)+\Phi(b_{12})\big)\circ\big(\Phi(p_1)^*+\Phi(a_{12})^*\big) \\
= \; & \Phi(p_1)\Phi(p_2)+\Phi(p_2)\circ\Phi(p_1)^*+\Phi(p_1)\Phi(b_{12})+\Phi(b_{12})\circ\Phi(p_1)^* \\
& +\Phi(a_{12})\Phi(p_2)+\Phi(p_2)\circ\Phi(a_{12})^*+\Phi(a_{12})\Phi(b_{12})+\Phi(b_{12})\circ\Phi(a_{12})^* \\
= \; & \Phi(p_1p_2+p_2\circ p_1^*)+\Phi(p_1b_{12}+b_{12}\circ p_1^*)+\Phi(a_{12}p_2+p_2\circ a_{12}^*) \\
& +\Phi(a_{12}b_{12}+b_{12}\circ a_{12}^*) \\
= \; & \Phi(\frac{3}{2}b_{12})+\Phi(a_{12}+\frac{1}{2}a_{12}^*)+\Phi(\frac{1}{2}b_{12}a_{12}^*+\frac{1}{2}a_{12}^*b_{12})
\end{aligned}$$

$$= \quad \Phi(a_{12}) + \Phi(\frac{3}{2}b_{12}) + \Phi(\frac{1}{2}a_{12}^*) + \Phi(\frac{1}{2}b_{12}a_{12}^*) + \Phi(\frac{1}{2}a_{12}^*b_{12})$$

which leads to $\Phi(a_{12} + \frac{3}{2}b_{12}) = \Phi(a_{12}) + \Phi(\frac{3}{2}b_{12})$. This allows us to conclude that $\Phi(a_{12} + b_{12}) = \Phi(a_{12}) + \Phi(b_{12})$.

Similarly, we prove (ii) using the following identity

$$(p_2 + a_{21})(p_1 + b_{21}) + (p_1 + b_{21}) \circ (p_2 + a_{21})^* \\ = a_{21} + \frac{3}{2}b_{21} + \frac{1}{2}a_{21}^* + \frac{1}{2}b_{21}a_{21}^* + \frac{1}{2}a_{21}^*b_{21}.$$

□

Property 2.9. *For any elements $a_{11}, b_{11} \in \mathcal{A}_{11}$ and $a_{22}, b_{22} \in \mathcal{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{11}) = \Phi(a_{11}) + \Phi(b_{11})$ and (ii) $\Phi(a_{22} + b_{22}) = \Phi(a_{22}) + \Phi(b_{22})$.*

Proof. According to the hypothesis on Φ choose $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathcal{A}$ such that $\Phi(c) = \Phi(a_{11}) + \Phi(b_{11})$. Hence, for any element $d_{12} \in \mathcal{A}$ we have

$$\Phi(d_{12}c + c \circ d_{12}^*) = \Phi(d_{12}a_{11} + a_{11} \circ d_{12}^*) + \Phi(d_{12}b_{11} + b_{11} \circ d_{12}^*) \\ = \Phi(\frac{1}{2}d_{12}^*a_{11}) + \Phi(\frac{1}{2}d_{12}^*b_{11}) = \Phi(\frac{1}{2}d_{12}^*a_{11} + \frac{1}{2}d_{12}^*b_{11}),$$

by Property 2.8(i). As consequence, we obtain $d_{12}c + c \circ d_{12}^* = \frac{1}{2}d_{12}^*(a_{11} + b_{11})$ which implies $c_{11} = a_{11} + b_{11}$ and $c_{12} = c_{21} = c_{22} = 0$.

Similarly, we prove (ii). □

Property 2.10. *Φ is an additive mapping.*

Proof. The result is an immediate consequence of Properties 2.7, 2.8 and 2.9. □

Lemma 2.11. *Let $\mathcal{A}$ be an arbitrary factor von Neumann algebra with the identity operator $1_{\mathcal{A}}$ and an element $a \in \mathcal{A}$. If $\mathcal{A}$ satisfies the condition $ab + b \circ a^* = 0$, for all elements $b \in \mathcal{A}$, then $a \in i\mathbb{R}1_{\mathcal{A}}$ (i is the imaginary number unit).*

Proof. Replacing b with $1_{\mathcal{A}}$ in the equation above we get $a^* = -a$. It follows that $ab - ba = 0$, for all elements $b \in \mathcal{A}$, which results that a belongs to the center of $\mathcal{A}$. As $\mathcal{A}$ is a factor, then we conclude that $a \in i\mathbb{R}1_{\mathcal{A}}$. □

In the remainder of this paper, all lemmas satisfy the conditions of the Main Theorem.

Lemma 2.12. $\Phi(i\mathbb{R}1_{\mathcal{A}}) = i\mathbb{R}1_{\mathcal{B}}$ *and* $\Phi(\mathbb{C}1_{\mathcal{A}}) = \mathbb{C}1_{\mathcal{B}}$.

Proof. For any elements $\lambda \in \mathbb{R}$ and $a \in \mathcal{A}$ we have $0 = \Phi((i\lambda 1_{\mathcal{A}})a + (i\lambda 1_{\mathcal{A}})^*a) = \Phi((i\lambda 1_{\mathcal{A}})a + a \circ (i\lambda 1_{\mathcal{A}})^*) = \Phi(i\lambda 1_{\mathcal{A}})\Phi(a) + \Phi(a) \circ \Phi(i\lambda 1_{\mathcal{A}})^*$ which implies that $\Phi(i\lambda 1_{\mathcal{A}}) \in i\mathbb{R}1_{\mathcal{B}}$, by Lemma 2.11. Since λ is an arbitrary element, it follows easily that $\Phi(i\mathbb{R}1_{\mathcal{A}}) \subseteq i\mathbb{R}1_{\mathcal{B}}$. Note that $i\mathbb{R}1_{\mathcal{B}} \subseteq \Phi(i\mathbb{R}1_{\mathcal{A}})$, since Φ^{-1} has the same properties as Φ. Thus $\Phi(i\mathbb{R}1_{\mathcal{A}}) = i\mathbb{R}1_{\mathcal{B}}$. Next, for any element $a \in \mathcal{A}$ such that $a^* = -a$, we have $0 = \Phi((i1_{\mathcal{A}})a + (i1_{\mathcal{A}})a^*) = \Phi(a(i1_{\mathcal{A}}) + (i1_{\mathcal{A}}) \circ a^*) = \Phi(a)\Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}}) \circ \Phi(a)^*$ which implies that $\Phi(a) = -\Phi(a)^*$. Since Φ^{-1} has the same properties as Φ, then we can conclude that $a = -a^*$ if and only if $\Phi(a) = -\Phi(a)^*$. Hence, for any elements $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$ such that $a^* = -a$, we have $0 = \Phi((\lambda 1_{\mathcal{A}})a + (\lambda 1_{\mathcal{A}})a^*) = \Phi(a(\lambda 1_{\mathcal{A}}) + (\lambda 1_{\mathcal{A}}) \circ a^*) = \Phi(a)\Phi(\lambda 1_{\mathcal{A}}) + \Phi(\lambda 1_{\mathcal{A}}) \circ \Phi(a)^*$ which shows that $\Phi(a)\Phi(\lambda 1_{\mathcal{A}}) = \Phi(\lambda 1_{\mathcal{A}})\Phi(a)$. It follows that $b\Phi(\lambda 1_{\mathcal{A}}) = \Phi(\lambda 1_{\mathcal{A}})b$, for an arbitrary element $b \in \mathcal{B}$ such that $b^* = -b$. As a consequence of this we have $b\Phi(\lambda 1_{\mathcal{A}}) = \Phi(\lambda 1_{\mathcal{A}})b$, for any elements $\lambda \in \mathbb{C}$ and $b \in \mathcal{B}$, since all element $b \in \mathcal{B}$ can be write as $b = b_1 + ib_2$, where $b_1 = \frac{b-b^*}{2}$ and $b_2 = \frac{b+b^*}{2i}$. This shows that $\Phi(\mathbb{C}1_{\mathcal{A}}) \subseteq \mathbb{C}1_{\mathcal{B}}$. By a similar argument we also get $\mathbb{C}1_{\mathcal{B}} \subseteq \Phi(\mathbb{C}1_{\mathcal{A}})$. Consequently, $\Phi(\mathbb{C}1_{\mathcal{A}}) = \mathbb{C}1_{\mathcal{B}}$. $\square$

Lemma 2.13. $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ *and there is a non-zero real number* μ *satisfying* $\mu^2 = 1$ *such that* $\Phi(ia) = i\mu\Phi(a)$, *for all elements* $a \in \mathcal{A}$.

Proof. First of all, fix a non-trivial projection p. By Lemma 2.12, there is a non-zero element λ of $\mathbb{C}$ such that $\Phi(1_{\mathcal{A}}) = \lambda 1_{\mathcal{B}}$ which implies that $2\Phi(p) = \Phi(2p) = \Phi(p1_{\mathcal{A}} + 1_{\mathcal{A}} \circ p^*) = \Phi(p)\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}}) \circ \Phi(p)^* = \lambda(\Phi(p) + \Phi(p)^*)$. This allows us to write $\Phi(p) = \lambda q$, where $q = \frac{1}{2}(\Phi(p) + \Phi(p)^*)$. Now, $2\Phi(p) = \Phi(2p) = \Phi(1_{\mathcal{A}}p + p \circ 1_{\mathcal{A}}^*) = \Phi(1_{\mathcal{A}})\Phi(p) + \Phi(p) \circ \Phi(1_{\mathcal{A}})^* = (\lambda + \overline{\lambda})\Phi(p)$ which results that $\lambda + \overline{\lambda} = 2$. Also, $2\Phi(p) = \Phi(2p) = \Phi(pp + p \circ p^*) = \Phi(p)\Phi(p) + \Phi(p) \circ \Phi(p)^* = \Phi(p)\Phi(p) + \Phi(p)\Phi(p)^* = (\lambda^2 + \lambda\overline{\lambda})q^2 = 2\lambda q^2$. This show that q is a projection. Let a be an element of $\mathcal{A}$ such that $s = pap^{\perp}$ is a non-zero element. Then $\frac{3}{2}\Phi(s) = \Phi(\frac{3}{2}s) = \Phi(ps + s \circ p^*) = \Phi(p)\Phi(s) + \Phi(s) \circ \Phi(p)^* = \lambda q\Phi(s) + \overline{\lambda}\Phi(s) \circ q$ which implies $q\Phi(s)q = 0$, $q^{\perp}\Phi(s)q^{\perp} = 0$, $(1-\lambda)q\Phi(s)q^{\perp} = 0$ and $(3-\overline{\lambda})q^{\perp}\Phi(s)q = 0$. If $\lambda \neq 1$ and 3, then $q\Phi(s)q^{\perp} = 0$ and $q^{\perp}\Phi(s)q = 0$ which results that $\Phi(s) = 0$ yielding in $s = 0$ which is a contradiction. Thus $\lambda = 1$ and $q^{\perp}\Phi(s)q = 0$ which implies that $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and $\Phi(s) = q\Phi(s)q^{\perp}$.

Let μ be a non-zero real number such that $\Phi(i1_{\mathcal{A}}) = i\mu 1_{\mathcal{B}}$. Then $2\Phi(ip) = \Phi(p(i1_{\mathcal{A}}) + (i1_{\mathcal{A}}) \circ p^*) = \Phi(p)\Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}}) \circ \Phi(p)^* = 2i\mu q$ which implies that $\Phi(ip) = i\mu q$. This results that $\frac{1}{2}\Phi(is) = \Phi(\frac{1}{2}is) = \Phi((ip)s + s \circ (ip)^*) = \Phi(ip)\Phi(s) + \Phi(s) \circ \Phi(ip)^* = (i\mu q)\Phi(s) + \Phi(s) \circ (i\mu q)^* = \frac{1}{2}i\mu\Phi(s)$. Thus, $\Phi(is) = i\mu\Phi(s)$. It follows that $-\frac{1}{2}\Phi(s) = \Phi(-\frac{1}{2}s) = \Phi((ip)(is) + (is) \circ (ip)^*) = \Phi(ip)\Phi(is) + \Phi(is) \circ \Phi(ip)^* = (i\mu q)(i\mu\Phi(s)) + (i\mu\Phi(s)) \circ (i\mu q)^* = -\frac{1}{2}\mu^2\Phi(s)$ which yields $\mu^2 = 1$.

Let t be an element of $\mathcal{A}$ satisfying $t = t^*$. Then $2\Phi(t) = \Phi(2t) = \Phi(t1_{\mathcal{A}} + 1_{\mathcal{A}} \circ t^*) = \Phi(t)\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}}) \circ \Phi(t)^* = \Phi(t) + \Phi(t)^*$ which implies that $\Phi(t) = \Phi(t)^*$. This results that $2\Phi(it) = \Phi(2it) = \Phi(t(i1_{\mathcal{A}}) + (i1_{\mathcal{A}}) \circ t^*) = \Phi(t)\Phi(i1_{\mathcal{A}}) + \Phi(i1_{\mathcal{A}}) \circ \Phi(t)^* = \Phi(t)(i\mu 1_{\mathcal{B}}) + (i\mu 1_{\mathcal{B}}) \circ \Phi(t)^*$ which yields $\Phi(it) = i\mu\Phi(t)$. It follows that, for any element $a \in \mathcal{A}$, write $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$. Then $\Phi(ia) = \Phi(ia_1 - a_2) = \Phi(ia_1) - \Phi(a_2) = i\mu\Phi(a_1) + i\mu\Phi(ia_2) = i\mu(\Phi(a_1) + \Phi(ia_2)) = i\mu\Phi(a_1 + ia_2) = i\mu\Phi(a)$. $\square$

Lemma 2.14. *$\Phi : \mathcal{A} \to \mathcal{B}$ is a $*$-ring isomorphism.*

Proof. For any elements $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\Phi(ab - b \circ a^*) &= \Phi((ia)(-ib) + (-ib) \circ (ia)^*) \\ &= \Phi(ia)\Phi(-ib) + \Phi(-ib) \circ \Phi(ia)^* = (i\mu\Phi(a))(-i\mu\Phi(b)) \\ &\quad + (-i\mu\Phi(b)) \circ (i\mu\Phi(a))^* = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*. \quad (2.1)\end{aligned}$$

From (1.1) and (2.1), we obtain $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathcal{A}$. This shows that Φ is a ring isomorphism. Moreover, $\Phi(a)+\Phi(a^*) = \Phi(a+a^*) = \Phi(a1_{\mathcal{A}} + 1_{\mathcal{A}} \circ a^*) = \Phi(a)\Phi(1_{\mathcal{A}}) + \Phi(1_{\mathcal{A}}) \circ \Phi(a)^* = \Phi(a)1_{\mathcal{B}} + 1_{\mathcal{B}} \circ \Phi(a)^* = \Phi(a) + \Phi(a)^*$, for all elements $a \in \mathcal{A}$. This allows us to conclude that Φ is a $*$-ring isomorphism.

The Theorem is proved. □

Corollary 2.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be two type I factor von Neumann algebras acting on complex Hilbert spaces H and K, respectively. Then every bijective mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi(ab + b \circ a^*) = \Phi(a)\Phi(b) + \Phi(b) \circ \Phi(a)^*$, for all elements $a, b \in \mathcal{A}$, if and only if there exists a unitary or conjugate unitary operator $U : H \to K$ such that $\Phi(a) = UaU^*$, for all elements $a \in \mathcal{A}$.*

Proof. The result is a direct consequence of the Main Theorem and the fact that every ring isomorphism between type I factor von Neumann algebras is spatial. □

Acknowledgments

The authors are highly grateful to editor and referees for their valuable comments and suggestions for improving the paper.

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